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ON THE PROPERTIES AND APPLICATIONS OF A NEW EXTENSION OF EXPONENTIATED RAYLEIGH DISTRIBUTION

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ABSTRACT
Statistical distributions already in existence are not the most appropriate model that adequately describes real-life data such as those obtained from experimental investigations. Therefore, there are needs to come up with their extended forms to give substitutive adaptable models. By adopting the method of Transformed-Transformer family of distributions, an extension of Exponentiated Rayleigh distribution titled Gompertz-Exponentiated Rayleigh (GOM-ER) distribution was proposed and proved to be valid. Some properties of the new distribution including random number generator, quartiles, distribution of smallest and largest order statistics, reliability function, hazard rate function, cumulative or integrated hazard function, odds function, non-central moments, moment generating function, mean, variance and entropy measures were derived. Using the methods of maximum likelihood and maximum product of spacing, the four unknown parameters were estimated. Shapes of the hazard function depicts that GOM-ER is a distribution that is strictly increasing while those of the PDF depicts that GOM-ER can be skewed or symmetrical. Two datasets were fitted to determine the flexibility of GOM-ER. Simulation study evaluates the consistency, accuracy and unbiasedness of the GOM-ER parameter estimates obtained from the two frequentist estimation methods adopted.

Keywords: Gompertz-Exponentiated Rayleigh; Probability Distribution; Order statistics; Entropy measures; Maximum Product of Spacing.

INTRODUCTION

Long time ago, probability distributions are known to be used by researchers to fit any given data adequately. As time goes by, data became large and tend to exhibit additional properties that are difficult to capture which leads to a problem in flexibility and improved inferences. These problems drew the attention of researchers in inferential statistics, after which they came up with an idea of generalizing and extending existing distributions with the aim of procuring distributions having additional tails features and different failure rates, and as a result makes them more flexible and capable of capturing real-world data in different areas.

Dating back to 19th century, several methods of defining probability distributions have been proposed , some of which include; ”Method of transformation” by (Johnson, 1949), ”Method of Generating Skewed Distributions” by (Azzalini, 1985), modified by (Azzalini, 1986), ”the Method of adding parameters” by (Mudholkar and Srivastava, 1993) and (Marshal and Olkin), ”Beta-Generated” by (Eugene et al., 2002) and (Jones, 2009), ”Kumaraswamy-Generated” by (Cordeiro and de Castro, 2011) and lastly the modern and most used method in the recent decade ”Transformed- Transformer(T-X)” by (Alzaatreh et al., 2013) modified to ”Exponentiated (T-X)” by (Alzaghal et al., 2013). These methods expand families of distributions for more flexibility and applications.

Adopting the method of ”Transformed- Transformer(T-X)”, a handful families of distributions namely “Weibull-G” by (Bourgiugnon et al., 2014) “Kumaraswamy-G” by (Cordeiro and De Castro, 2011)” “The generalized transmuted-G” by (Nofal et al., 2017), “Gompertz-G” by (Alizadeh et al., 2017), “The Inverse Lomax-G” by (Falgore and Doguwa., 2020) among many others have been developed.

Exponentiated Rayleigh (ER) also known as Generalized Rayleigh which is a special case of Exponentiated Weibull by (Mudholkar and Srivastava, 1993) was introduced by (Vodˇa, 1976). It have some properties of gamma with two parameters, (Weibull, 1939) and Generalized Exponential (Gupta and Kundu, 1999) distributions. To mention few studies on ER continuous distribution: Pathak and Chaturvedi, (2014) derived the reliability function; Kundu and Raqab, (2005) estimated the parameters using different frequentist approach while (Madi and Raqab, 2009) used Bayesian approach; Abd-Elfattah, (2011) studied the goodness of fit tests; centered on Unified Hybrid and generalized Type-II hybrid Censored Data, (Mahmoud and Ghazal, (2017) and (Ghazal and Hasaballah, 2017) respectively utilized the methods of maximum likelihood, Bayes and percentile bootstrap in estimating the unknown parameters.

Areas including communication theory, medical imaging science and engineering among others benefits from this distribution. More so, compared to other widely used classical distributions, lesser source of materials were found on ER distribution.

The Maximum Likelihood Estimation (MLE) is the most frequently used method of estimation because of its desirable properties. The method, however have it setbacks making the estimators fail sometimes. Maximum Product of Spacing Estimation (MPSE), introduced by (Cheng and Amin, 1983) is likely to serve as a competitor to MLE in cases where the estimates from MLE breaks down. The estimators obtained by maximizing the geometric mean of spacings between cumulative distribution function in close observations are consistent and as efficient as MLEs.

Ranneby (1984) noted that "The ML estimates perfectly the parameters of discrete distributions if the contribution to the likelihood function is bounded from above but not for compound continuous distributions". In addition, the consistency of MPS was studied and shown that it works in place of MLEs.

Compound continuous distributions by (Sen et al., 2019) and (Al-Mofleh and Afify, 2019) among many adopted MPS along with other frequentist methods to estimates the distribution's parameters. The two studies found MPS to be consistent and efficient.

Numerous modified distributions using Gompertz-G family have been established by a number of researchers, yet none was found on Gompertz-Exponentiated Rayleigh distribution. For this reason, we propose a four parameter lifetime distribution called Gompertz-Exponentiated Rayleigh (GOM-ER) distribution. The fact that few extensions of Exponentiated Rayleigh exist in literature also served as motivation for this study. More so, in addition to the method of Maximum likelihood frequently used in estimation of parameters, a competitive method known as MPS is explored.

MATERIALS AND METHOD

Gompertz-G family and Exponentiated Rayleigh distribution

Gompertz-G family

Alizaadeh et al., (2017) introduced a new generator of continuous distributions by adding two shape parameters to any given baseline distribution and was named Gompertz-G. The cumulative distribution function (CDF) of Gompertz-G is given as:

$$F(x) = \int_0^{-\log[1-G(x;\epsilon)]} \beta e^{\alpha x} e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)} dx$$
$$= 1 - e^{\frac{\beta}{\alpha}\{1-[1-G(x;\epsilon)]^{-\alpha}\}} \quad (1)$$

where $G(x;\epsilon)$ is the cumulative distribution of the baseline distribution depending on the parameter space ϵ and $\alpha > 0, \beta > 0$ are the additional positive shape parameters.

The probability density function (PDF) was obtained from (1) as:

$$f(x;\ \alpha,\beta,\epsilon) = \beta g(x;\epsilon)[1 - G(x;\epsilon)]^{-\alpha-1} e^{\frac{\beta}{\alpha}\{1-[1-G(x;\epsilon)]^{-\alpha}\}} \quad (2)$$

Exponentiated Rayleigh (ER) is a continuous distribution with scale and shape parameters studied by (Vodˇa, 1976). The CDF and PDF are respectively derived as:

$$G(x,\sigma,\delta) = \left[1 - e^{-(\sigma x)^2}\right]^{\delta} \quad (3)$$
$$g(x,\sigma,\delta) = 2\delta\sigma^2 x e^{-(\sigma x)^2}\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1} \quad (4)$$
$$x,\delta,\sigma > 0.$$

It is important to note that ER distribution satisfy the following conditions provided in (Alzaatreh et al., 2013) when $W[G(x)]$ was set $-\log[1 - G(x)]$.

(i) $W\ [G(x)]\ \in\ [a,b]$,

(ii) $W\ [G(x)]$ is differentiable and monotonically non decreasing, and

(iii) $W\ [G(x)]\ \to\ a$ as $x\ \to\ -\infty$ and $W\ [G(x)]\ \to\ b$ as $x\ \to\ \infty$

where $W[G(x)]$ is a function of the CDF of x.

Gompertz-Exponentiated Rayleigh (GOM-ER) distribution

To obtain the CDF and PDF of GOM-ER distribution, equations (3) and (4) are substituted respectively into equation (1) and equation (2). These give:

$$F(x;\alpha,\beta,\sigma,\delta) = 1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \quad (5)$$

and

$$f(x;\alpha,\beta,\sigma,\delta) = 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1-\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \quad (6)$$

where $\alpha,\beta,\delta\ >\ 0$ are shape parameters while $\sigma\ >\ 0$ is a scale parameter and $x\ >\ 0$.

Note that, if the exponentiated parameter, $\delta = 1$ and $\sigma = \sqrt{\frac{\theta}{2}}$, then equation (6) reduces to the PDF of Gompertz-Rayleigh by (Mohammed et al., 2020).

<u>Useful Representation</u>

The CDF and PDF of GOM-ER distribution can be represented in simpler form as follows:

By applying the power series expansion:

$$e^{x} = \sum_{b_1}^{\infty} \frac{x^{i}}{b_1!}$$

and generalized binomial series:

$$(1-m)^{b} = \sum_{b_2=1}^{\infty} \binom{b}{b_2} -1^{b_2} m^{b_2}$$

$$(1-m)^{-b} = \sum_{b_3=0}^{\infty} \binom{b+b_3-1}{b_3} m^{b_3}$$

where

$$|m| < 1 \text{ and } b_i > 0$$

gives the simplified densities as

$$F(x;\, \alpha,\beta,\sigma,\delta) = 1 - \sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty} \frac{1}{b_1!}\left(\frac{\beta}{\alpha}\right)^{b_1} \binom{b_1}{b_2} (-1)^{b_2+b_4} \binom{\alpha b_1 + b_3 - 1}{b_3} \binom{\delta b_3}{b_4} e^{-b_4(\sigma x)^2} \quad (7)$$

$$f(x;\, \alpha,\beta,\sigma,\delta) = 2\,\beta\,\delta\,\sigma^{2}\,x \sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty} \Psi_{b_1 b_2 b_3 b_4}\, e^{-(\sigma x)^2(1+b_4)} \quad (8)$$

where

$$\Psi_{b_1 b_2 b_3 b_4} = \left(\frac{\beta}{\alpha}\right)^{b_1} \binom{b_1}{b_2} (-1)^{b_2+b_4} \binom{\alpha(b_2+1)+b_3}{b_3} \binom{\delta(b_3+1)-1}{b_4}$$

Validity of the Gompertz-Exponentiated Rayleigh distribution

For a PDF f(x) to be valid, it is expected to satisfy

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

thus

$$\int_{-\infty}^{\infty} f(x)dx$$

$$= \int_{0}^{\infty} 2\,\beta\,\delta\,\sigma^{2}\,x\,e^{-(\sigma x)^2}\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1} \left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} dx$$

let

$$u = \frac{\beta}{\alpha}\left(1 - \left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)$$

then

$$\frac{du}{dx} = \frac{\beta}{\alpha}\,.\,\alpha\,.\left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha-1}.\,-\delta\,.\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}.\,-e^{-(\sigma x)^2}\,.\,-2(\sigma x).\,\sigma$$

Now

as $x \to 0,\ u \to 0$ and as $x \to \infty,\ u \to -\infty$

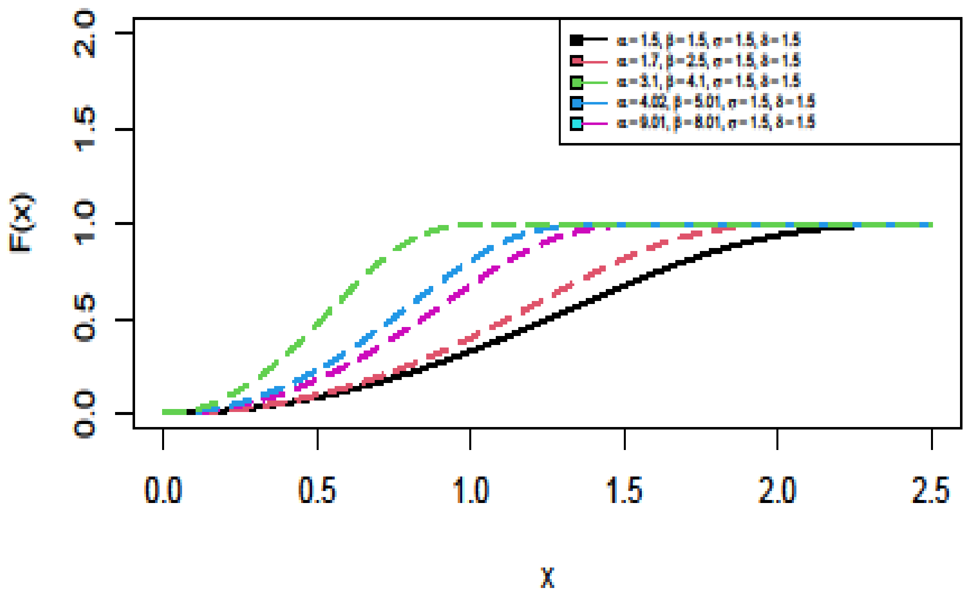
$$\int_{-\infty}^{0} e^{-u}\,du$$

$$= 1$$

therefore, GOM-ER is proved to be a valid probability distribution.

Possible shapes of GOM-ER

At different parameter values, the shapes of GOM-ER are depicted in the following figures.



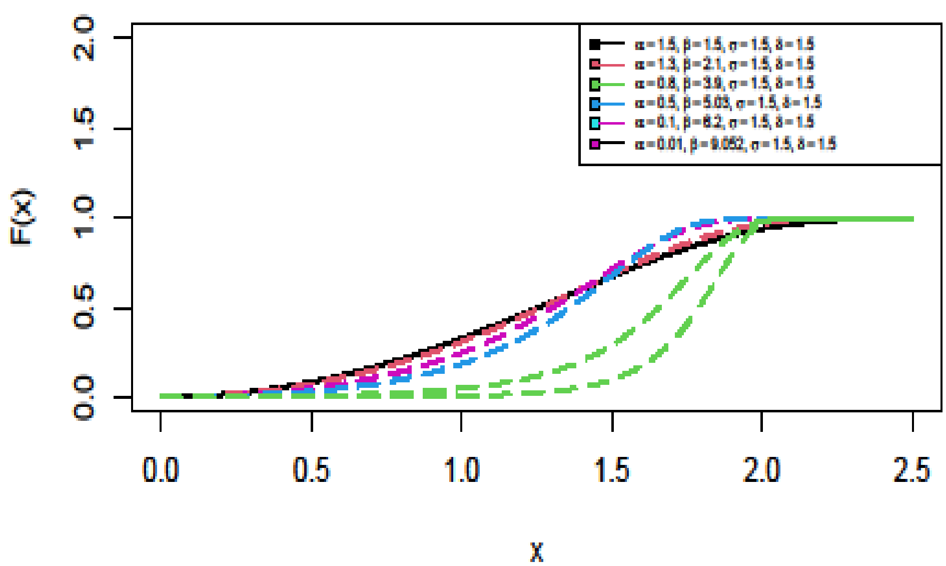


FIGURE 1: CDF OF GOM-ER

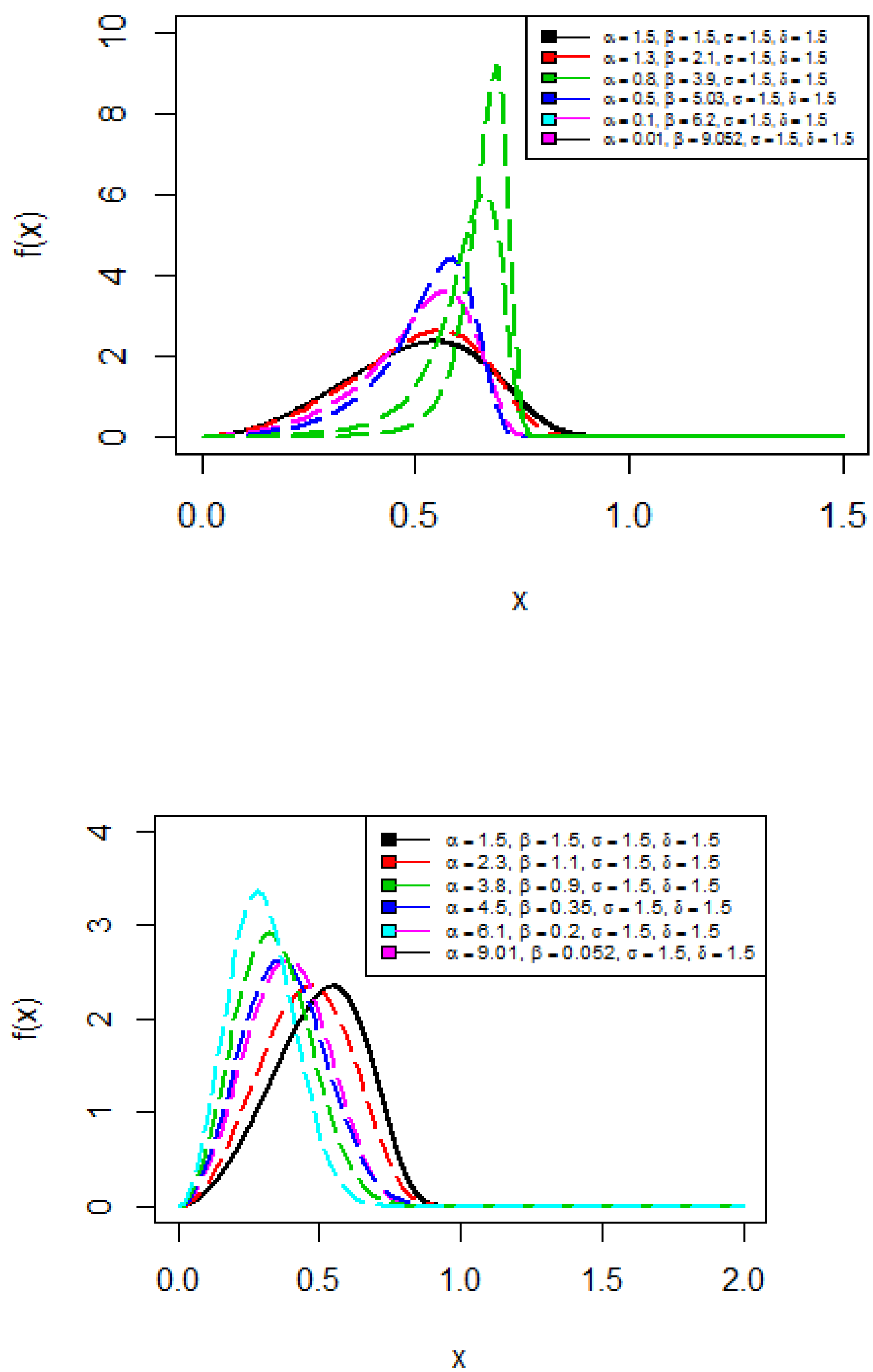
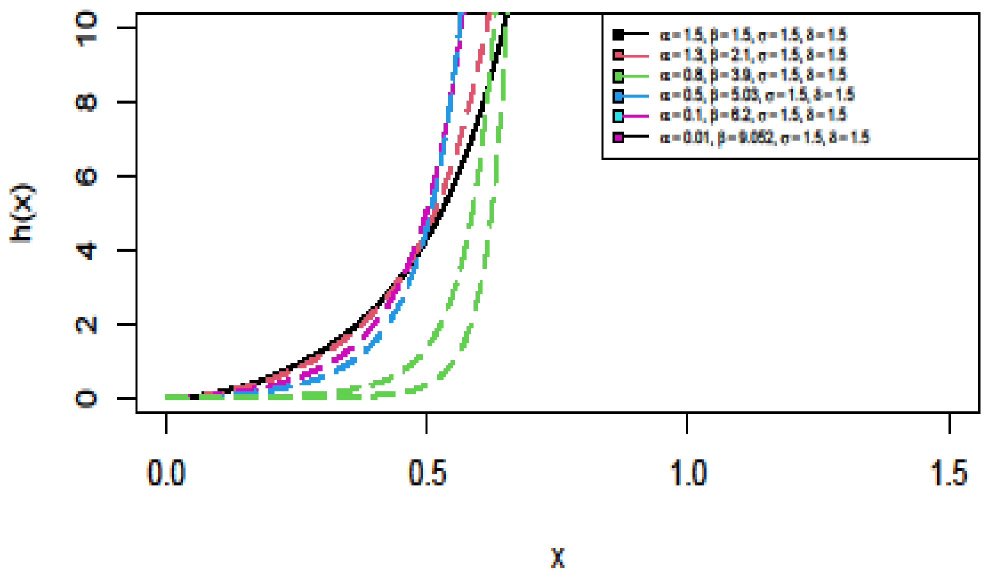


FIGURE 2: PDF OF GOM-ER



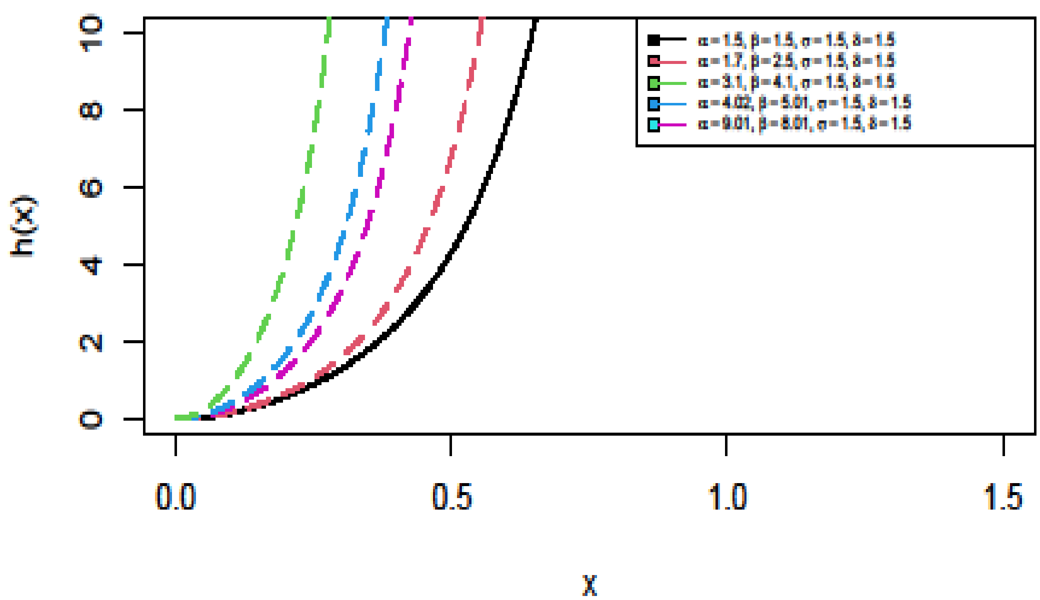


FIGURE 3: HAZARD FUNCTION OF GOM-ER

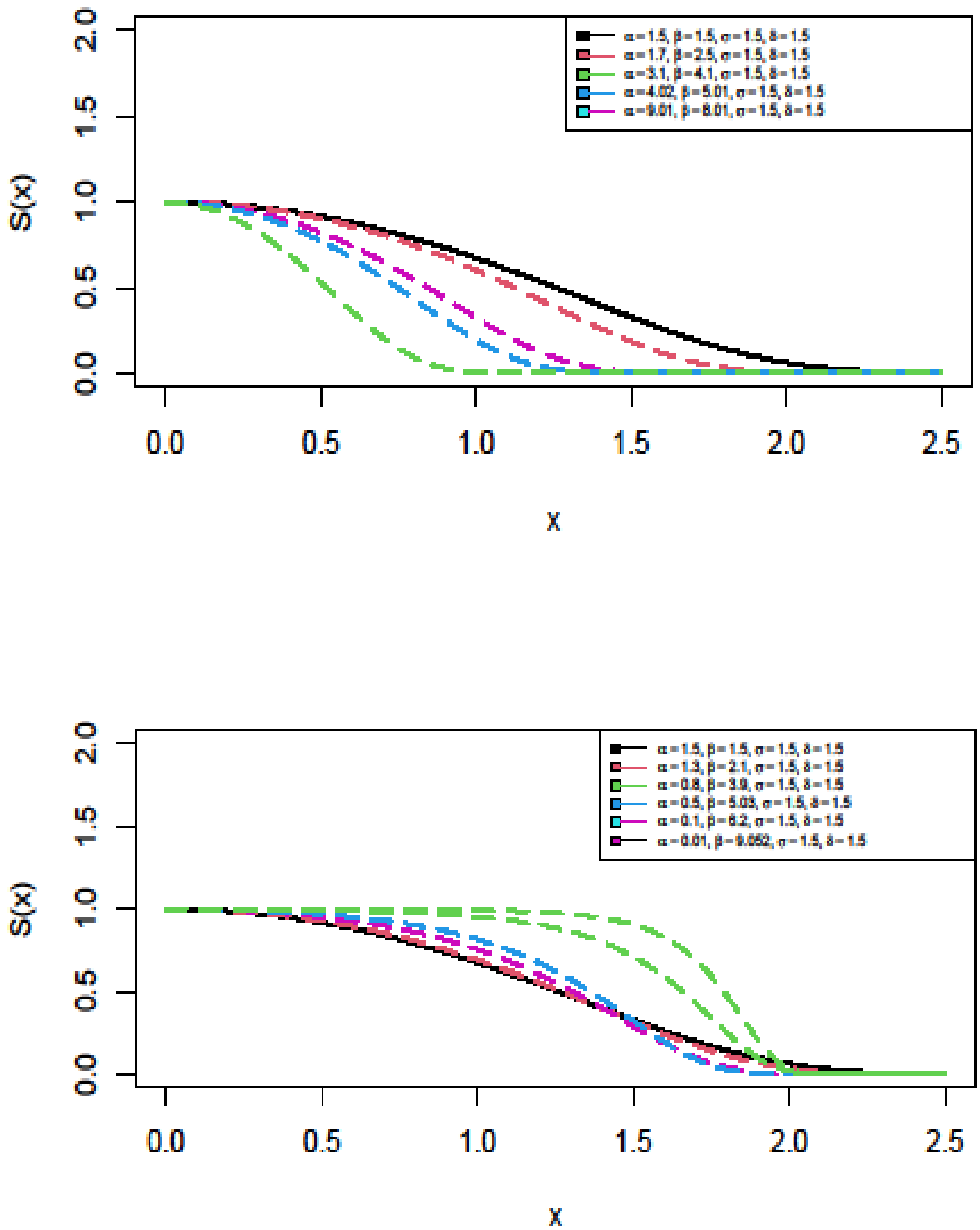


FIGURE 4: RELIABILITY FUNCTION OF GOM-ER

The plots showed that CDF of GOM-ER for different parameters values converges to one implying a valid probability distribution. The PDF is skewed and also symmetrical deducing it applications on different datasets. However, GOM-ER had hazard function that is strictly increasing for all parameter values.

Basic mathematical properties of GOM-ER distribution

Random number generator (RNG) and quartiles

Given any CDF, F(x) the RNG also referred to as quartile function Q (u) can be obtained using:

$$Q(u) = F^{-1}(u) \qquad 0 < u < 1$$

Hence the RNG was obtained as follows:
Let

$$U = F(x;\ \alpha, \beta, \sigma, \delta) \tag{9}$$

then

$$U = 1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \tag{10}$$

Solving for x results to

$$x = \frac{\left[-\log_e\left(1-\left\{1-\left[1-\frac{\alpha}{\beta}log_e(1-U)\right]^{-\frac{1}{\alpha}}\right\}^{\frac{1}{\delta}}\right)\right]^{\frac{1}{2}}}{\sigma} \tag{11}$$

therefore

$$Q(u) = \frac{\left[-\log_e\left(1-\left\{1-\left[1-\frac{\alpha}{\beta}log_e(1-U)\right]^{-\frac{1}{\alpha}}\right\}^{\frac{1}{\delta}}\right)\right]^{\frac{1}{2}}}{\sigma} \qquad 0 < u < 1 \tag{12}$$

Substituting u = 0.5, 0.50 and 0.75, we obtained the 1st, 2nd (median) and 3rd quartiles of the GOM-ER respectively as;

$$\text{1st quartile, } Q(0.25) = \frac{\left[-\log_e\left(1-\left\{1-\left[1-\frac{\alpha}{\beta}log_e(0.75)\right]^{-\frac{1}{\alpha}}\right\}^{\frac{1}{\delta}}\right)\right]^{\frac{1}{2}}}{\sigma} \tag{13}$$

$$\text{2nd quartile, } Q(0.50) = \frac{\left[-\log_e\left(1-\left\{1-\left[1-\frac{\alpha}{\beta}log_e(0.50)\right]^{-\frac{1}{\alpha}}\right\}^{\frac{1}{\delta}}\right)\right]^{\frac{1}{2}}}{\sigma} \tag{14}$$

$$\text{3rd quartile, } Q(0.75) = \frac{\left[-\log_e\left(1-\left\{1-\left[1-\frac{\alpha}{\beta}log_e(0.25)\right]^{-\frac{1}{\alpha}}\right\}^{\frac{1}{\delta}}\right)\right]^{\frac{1}{2}}}{\sigma} \tag{15}$$

Order statistic of the GOM-ER distribution

Given a distribution from whom a random sample of independent characteristics $X_1, X_2, \ldots, X_n$ was drawn. This sample can be represented in an ordered form as: notation $X_{(1,n)} \leq X_{(2,n)} \leq \cdots \leq X_{(n,n)})$ or $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. $X_{(1)}$ representing the 1^{st} order is considered the minimum , $X_{(2)})$ representing the 2^{nd} order, the second minimum while the n^{th} order statistics, $X_{(n)}$, is the maximum.

PDF of the k^{th} order statistics of GOM-ER distribution

Suppose $X_1, X_2, \ldots, X_n$ obtained from GOM-ER distribution is ordered $X_{(1)} \leq X_{(2)} \leq \ \ldots\ \leq X_{(n)}$ as the order statistic, then the PDF, $f_{(k,n)}(x)$, the kth order statistic is expressed as:

$$f_{(k,n)}(x) = \frac{n!}{(k-1)!\,(n-k)!}\, f(x) \times F(x)^{k-1} \times [1 - F(x)]^{n-k} \tag{16}$$

Where of F(x) and f(x) are the CDF and PDF of the GOM-ER distribution.
For easier simplifications, the binomial expansion of $[1 - F(x)]^{n-k}$ was used as:

$$[1 - F(x)]^{n-k} = \sum_{b_5}^{n-k} \binom{n-k}{b_5} (-1)^{b_5} [F(x)]^{b_5} \tag{17}$$

Substituting (17) and (16) yields

$$f_{(k,n)}(x) = \sum_{b_5}^{\infty} \frac{n!}{(k-1)!\,(n-k)!}\, f(x)\ .\ \binom{n-k}{b_5} (-1)^{b_5} [F(x)]^{b_5+k-1} \tag{18}$$

Again, substituting (5) and (6) into (18), gives the kth order statistic of the GOM-ER distribution as

$$f_{(k,n)}(x) = \sum_{b_5}^{\infty} \frac{n!\ (-1)^{b_5}}{(k-1)!\,(n-k-b_5)!\,b_5!}\ 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\left[1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right]^{b_5+k-1} \quad (19)$$

PDF of the smallest and largest order statistics

Substituting k=1 into (19) gives the PDF of minimum or 1st order statistic as:

$$f_{(1,n)}(x) = \sum_{b_5=0}^{\infty} \frac{n!\ (-1)^{b_5}}{(n-1-b_5)!\,b_5!}\ 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\left[1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right]^{b_5} \quad (20)$$

Similarly, the n^{th} order or the maximum order statistic was obtained by substituting k=n as

$$f_{(n,n)}(x) = \sum_{b_5}^{\infty} \frac{n!\ (-1)^{b_5}}{(n-1)!\,(-b_5)!\,b_5!}\ 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\left[1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right]^{b_5+n-1} \quad (21)$$

Reliability Analysis of GOM-ER distribution

Suppose a random variable X follows GOM-ER distribution with PDF, f(x) and CDF, F(x); the following properties were attained.

Reliability function

Having an event or a system, the probability that the event or system fail or die beyond a given time, say x is known as the reliability or survival function S(x). It is derived using the relation:

$$P(X > x) = S(x) \qquad x > 0$$

this implies

$$S(x) = 1 - F(x) = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \quad (22)$$

Hazard function (hf)

Unlike the survival function which is a probability, the hazard function is a conditional density expressed as the ratio of PDF and survival function. It is also referred to as failure rate and is derived as follows:

$$h(x) = \frac{f(x)}{S(x)}$$

$$= \frac{2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1-\ \left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}}{e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}}$$

$$= 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-\ e^{-(\sigma x)^2}\right]^{\delta-1} \quad (23)$$

Cumulative or integrated hazard function

This is a risk function and not a probability. The cumulative hazard function of GOM-ER is derived as follows:

$$H(x) = \int_0^t h(x)dx$$
$$= \int_0^t 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} dx$$
$$= 2\,\beta\,\delta\,\sigma^2 \int_0^t x\,e^{-(\sigma x)^2}\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} dx$$

Adopting integration by substitution

Let $P = 1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta}$ then $\frac{dP}{dx} = -\delta\,.\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}.-e^{-(\sigma x)^2}.-2\sigma^2 x$

Now, as $x \to 0,\ P \to 1$ and as $x \to t,\ P \to 1 - \left[1 - e^{-(\sigma t)^2}\right]^{\delta}$

$$= -\beta \int_1^{1-\left[1-e^{-(\sigma t)^2}\right]^{\delta}} P^{-\alpha-1} dP$$
$$= -\beta \left[\frac{P^{-\alpha}}{-\alpha}\right]_1^{\left[1-\left[1-e^{-(\sigma t)^2}\right]^{\delta}\right]}$$
$$= \frac{\beta}{\alpha}\left(\left\{1 - \left[1 - e^{-(\sigma t)^2}\right]^{\delta}\right\}^{-\alpha}\right) \qquad (24)$$

Odds function

This is the odds of the probability that the failure of a unit is bound to happen at a given time, say x, to the probability that it is bound to survive beyond that time. That is;

$$O(x) = \frac{F(x)}{S(x)}$$
$$= \frac{1 - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}}{e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}} \qquad (25)$$

Moment and Moment generating function

r^th non-central moment

This is an important property of any distribution and used in obtaining some measures comprising shapes, dispersion, central tendencies and so on.

Suppose a random variable X follows GOM-ER distribution, the r^th non-central moment, μ_r', can be obtained using the expression

$$\mu_r' = E(X^r)$$
$$= \int_{-\infty}^{\infty} x^r f(x)dx$$
$$= \int_{-\infty}^{\infty} x^r 2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1 - \left[1 - e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1} e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} dx$$

Recalling the useful representation

$$\mu_r' = \int_0^{\infty} x^r 2\,\beta\,\delta\,\sigma^2\,x \sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty} \Psi_{b_1b_2b_3b_4}\, e^{-(\sigma x)^2(1+b_4)}dx$$

$$= 2\,\beta\,\delta\,\sigma^2 \sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty} \Psi_{b_1b_2b_3b_4} \int_0^{\infty} x^{r+1}dx\, e^{-(\sigma x)^2(1+b_4)}dx$$

Where

$$\Psi_{b_1b_2b_3b_4} = \left(\frac{\beta}{\alpha}\right)^{b_1}\binom{b_1}{b_2}(-1)^{b_2+b_4}\binom{\alpha(b_2+1)+b_3}{b_3}\binom{\delta(b_3+1)-1}{b_4}$$

let $q = \sigma^2 x^2(1+b_4)$, then $\frac{dq}{dx} = 2\sigma^2 x\,(1+b_4)dx$ and $x = \frac{q^{\frac{1}{2}}}{\sigma\,(1+b_4)^{\frac{1}{2}}}$

therefore

$$\mu_r' = \frac{2\beta\delta\sigma^2}{\sigma}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty} \Psi_{b_1b_2b_3b_4} \int_0^{\infty}\left(\frac{q^{\frac{1}{2}}}{\sigma(1+b_4)^{\frac{1}{2}}}\right)^{r+1} e^{-q}\,.\,\frac{dm}{2\sigma^2.(1+b_4)x}$$

further substitutions and simplifications yield:

$$\beta\delta\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{\sigma^r(1+b_4)^{\frac{1}{2}+1}}\int_0^{\infty}q^{\frac{r}{2}}e^{-q}dq$$

hence

$$\mu_r' = \frac{\beta\delta}{\sigma^r}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^{\frac{1}{2}+1}}\,.\,\Gamma\left(1+\frac{r}{2}\right) \qquad (26)$$

MGF

Generally, the MGF of any random variable can be obtained using the relation:

$$M(\theta) = E(e^{\theta x})$$

since X is a continuous random variable with PDF f(x),

$$M(\theta) = \int_0^{\infty} e^{tx}f(x)\,\mathrm{dx}$$

or in simpler form

$$M(\theta) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f(x)\,dx \;; \qquad \text{since} \qquad e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

where μ_r' is the r[th] non-central moment

hence, the MGF of GOM-ER is given by:

$$\beta\delta\sum_{r=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\frac{t^r}{\sigma^r r!}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^{\frac{1}{2}+1}}\,.\,\Gamma\left(1+\frac{r}{2}\right) \qquad (27)$$

Mean and variance of GOM-ER

These properties are obtained from the r[th] non-central moment of GOM-ER distribution.

Mean

If in equation (26), r takes on value 1, the resulting equation gives the mean (1[th] moment) of GOM-ER distribution, given by:

$$\mu_1' = E(X)$$

$$= \frac{\beta\delta}{\sigma}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^{\frac{3}{2}}}\,.\,\Gamma\left(\frac{3}{2}\right) \qquad (28)$$

Variance

Using the relation

$$Var\,(X) = E(X^2) - [E(X)]^2$$

where $E(X^2)$ is the 2[nd] moment and obtained when r=2 in (26).

$$\mu_2' = E(X^2)$$

$$= \frac{\beta\delta}{\sigma^2}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^2}\,.\,\Gamma(2) \qquad (29)$$

but $\Gamma(2) = (2-1)! = 1$

thus

$$\mu_2' = \frac{\beta\delta}{\sigma^2}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^2} \qquad (30)$$

therefore the variance of GOM-ER is

$$Var\,(X) = \frac{\beta\delta}{\sigma^2}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^2} - \left[\frac{\beta\delta}{\sigma}\sum_{b_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_3=0}^{\infty}\sum_{b_4=0}^{\infty}\Psi_{b_1b_2b_3b_4}\frac{1}{(1+b_4)^{\frac{3}{2}}}\,.\,\Gamma\left(\frac{3}{2}\right)\right]^2 \qquad (31)$$

Entropy

The Renyi entropy (Renyi, 1961) is a measure of uncertainty defined as:

$$I_R(c) = \frac{1}{1-c}\log\int_0^{\infty}f^c(x)dx \qquad c > 0, c \neq 1 \qquad (32)$$

Suppose a random variable X follows the GOM-ER distribution, the degree of uncertainty can be derived as follows:

$$f^c(x;\alpha,\beta,\sigma,\delta) = (2\,\beta\,\delta\,\sigma^2)^c\,x^c\,e^{-c\,(\sigma x)^2}\left[1-\,e^{-(\sigma x)^2}\right]^{c(\delta-1)}\left\{1 - \left[1-\,e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-c(\alpha+1)}e^{\frac{c\beta}{\alpha}\left(1-\left\{1-\left[1-\,e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \qquad (33)$$

Using the expansions earlier

$$e^{\frac{c\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}=\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\frac{1}{d_1!}(-1)^{d_2}\binom{d_1}{d_2}\left(\frac{c\beta}{\alpha}\right)^{d_1}\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha d_2}$$

implying that

$$f^c(x;\alpha,\beta,\sigma,\delta)=(2\,\beta\,\delta\,\sigma^2)^c\,x^c\,e^{-c\,(\sigma x)^2}\left[1-e^{-(\sigma x)^2}\right]^{c(\delta-1)}\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\frac{1}{d_1!}(-1)^{d_2}\binom{d_1}{d_2}\left(\frac{c\beta}{\alpha}\right)^{d_1}\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha(d_2+c)+c)}$$

but

$$\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-(\alpha(d_2+c)+c)}=\sum_{d_3}^{\infty}\binom{(\alpha(d_2+c)+c)+d_3-1}{d_3}\left[1-e^{-(\sigma x)^2}\right]^{\delta d_3} \qquad (34)$$

hence

$$f^c(x;\alpha,\beta,\sigma,\delta)=(2\,\beta\,\delta\,\sigma^2)^c\,x^c\,e^{-c\,(\sigma x)^2}\times\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\sum_{d_3=0}^{\infty}\frac{1}{d_1!}(-1)^{d_2}\binom{d_1}{d_2}\left(\frac{c\beta}{\alpha}\right)^{d_1}\binom{(\alpha(d_2+c)+c)+d_3-1}{d_3}\left[1-e^{-(\sigma x)^2}\right]^{\delta(d_3+c)-c}$$

$$=(2\,\beta\,\delta\,\sigma^2)^c\,x^c\times\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\sum_{d_3=0}^{\infty}\sum_{d_4=0}^{\infty}\frac{1}{d_1!}(-1)^{d_2}\binom{d_1}{d_2}\left(\frac{c\beta}{\alpha}\right)^{d_1}\binom{(\alpha(d_2+c)+c)+d_3-1}{d_3}\binom{\delta(d_3+c)-c}{d_4}e^{-d_4(\sigma x)^2-c\,(\sigma x)^2}$$

$$=(2\,\beta\,\delta\,\sigma^2)^c\,x^c\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\sum_{d_3=0}^{\infty}\sum_{d_4=0}^{\infty}\mho_{(d_1,d_2,d_3,d_4)}\,e^{-(\sigma x)^2(d_4+c)}$$

where

$$\mho_{(d_1,d_2,d_3,d_4)}=\frac{1}{d_1!}(-1)^{d_2}\binom{d_1}{d_2}\left(\frac{c\beta}{\alpha}\right)^{d_1}\binom{(\alpha(d_2+c)+c)+d_3-1}{d_3}\binom{\delta(d_3+c)-c}{d_4}$$

$$I_R(C)=\frac{1}{1-c}\log\left[(2\,\beta\,\delta\,\sigma^2)^c\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\sum_{d_3=0}^{\infty}\sum_{d_4=0}^{\infty}\mho_{(d_1,d_2,d_3,d_4)}\int_0^{\infty}x^c\,e^{-(\sigma x)^2(d_4+c)}\,dx\right]$$

by letting $n=\sigma^2x^2(d_4+c)$ then $\frac{dn}{dx}=2\sigma^2x(d_4+c)$ and $x=\frac{n^{\frac{1}{2}}}{\sigma(d_4+c)^{\frac{1}{2}}}$

$$I_R(C)=\frac{1}{1-c}\log\left[\sum_{d_1,d_2,d_3,d_4=0}^{\infty}\mho_{(d_1,d_2,d_3,d_4)}\int_0^{\infty}\left[\frac{n^{\frac{1}{2}}}{\sigma(d_4+c)^{\frac{1}{2}}}\right]^{c-1}e^{-n}\,\frac{dn}{2\sigma^2x(d_4+c)}\right]$$

$$=\frac{1}{1-c}\log\left[\frac{2^{c-1}\beta^c\delta^c\sigma^{2c}}{\sigma^{\frac{c-1}{2}}}\right]\sum_{d_1,d_2,d_3,d_4=0}^{\infty}\mho_{(d_1,d_2,d_3,d_4)}\frac{1}{(d_4+c)^{\frac{c-3}{2}}}\int_0^{\infty}n^{\frac{c-1}{2}}\,e^{-n}\,dn$$

$$=\frac{1}{1-c}\log[2^{c-1}](\delta\beta)^c\sigma^{c-1}\sum_{d_1,d_2,d_3,d_4=0}^{\infty}\mho_{d_i}\frac{1}{(d_4+c)^{\frac{c-3}{2}}}\Gamma\left(\frac{1+c}{2}\right) \qquad (35)$$

Parameter Estimation

This section provides the estimates of the four unknown parameters (β, α, δ, σ) of GOM-ER distribution using the methods of MLE and MPS.

MLE

Assuming $X_1,X_2,\dots,X_n$ is a random sample of size n drawn from GOM-ER distribution.

$$f(x;\alpha,\beta,\sigma,\delta)=2\,\beta\,\delta\,\sigma^2\,x\,e^{-(\sigma x)^2}\left[1-e^{-(\sigma x)^2}\right]^{\delta-1}\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta-1}\right\}^{-\alpha-1}e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-(\sigma x)^2}\right]^{\delta}\right\}^{-\alpha}\right)}$$

Thus the likelihood function

$$L(X_1,X_2,\dots,X_n;\,\alpha,\beta,\sigma,\delta)=L(\alpha,\beta,\sigma,\delta)$$

$$=(2\,\beta\,\delta\,\sigma^2)^n\prod_{i=1}^{n}x_i\prod_{i=1}^{n}e^{-\sigma^2x_i^2}\prod_{i=1}^{n}\left(1-e^{-\sigma^2x_i^2}\right)^{\delta-1}\prod_{i=1}^{n}\left[1-\left(1-e^{-\sigma^2x_i^2}\right)^{\delta}\right]^{-(\alpha+1)}\prod_{i=1}^{n}e^{\frac{\beta}{\alpha}\left\{1-\left[1-\left(1-e^{-\sigma^2x_i^2}\right)^{\delta}\right]^{-\alpha}\right\}}$$

The corresponding log-likelihood function is obtained as:

$$\log L(\alpha,\beta,\sigma,\delta) = n\log(2\,\beta\,\delta\,\sigma^2) + \sum_{i=1}^{n}\log x_i - \sigma^2\sum_{i=1}^{n}x_i^2 + (\delta - 1)\sum_{i=1}^{n}\log\left(1 - e^{-\sigma^2 x_i^2}\right) - (\alpha+1)\sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right] + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left\{1 - \left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]^{-\alpha}\right\} \quad (36)$$

The estimates of the parameters $(\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\sigma})$, we differentiatelog $L(\alpha,\beta,\sigma,\delta)$ with respect to individual parameter and equate to zero. The resulting differentials are,

$$\frac{\partial \log L(\alpha,\beta,\sigma,\delta)}{\partial\beta} = \frac{n}{\beta} + \frac{1}{\alpha}\sum_{i=1}^{n}\left\{1 - \left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]^{-\alpha}\right\} \quad (37)$$

$$\frac{\partial \log L(\alpha,\beta,\sigma,\delta)}{\partial\alpha} = -\sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right] + \left\{\frac{\beta}{\alpha}\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]^{-\alpha}\log\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]\right\} - \frac{\beta}{\alpha^2}\sum_{i=1}^{n}\left\{1 - \left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]^{-\alpha}\right\} \quad (38)$$

$$\frac{\partial \log L(\alpha,\beta,\sigma,\delta)}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log\left(1 - e^{-\sigma^2 x_i^2}\right) + (\alpha+1)\sum_{i=1}^{n}\frac{\left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\log\left(1 - e^{-\sigma^2 x_i^2}\right)}{1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}} - \beta\sum_{i-1}^{n}\left\{\left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right]^{-(\alpha+1)}\log\left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta}\right\} \quad (39)$$

$$\frac{\partial \log L(\alpha,\beta,\sigma,\delta)}{\partial\sigma} = \frac{2n}{\sigma} - 2\sigma\sum_{i=1}^{n}x_i^2 + (\delta-1)\sum_{i=1}^{n}\frac{2\sigma x^2 e^{-\sigma^2 x_i^2}}{1 - e^{-\sigma^2 x_i^2}} + (\alpha+1)\sum_{i=1}^{n} -2\alpha\delta\sigma\sum_{i=1}^{n}x^2 e^{-\sigma^2 x_i^2}\left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta-1}\left[1 - \left(1 - e^{-\sigma^2 x_i^2}\right)^{\delta-1}\right]^{-(\alpha+1)} \quad (40)$$

The above equations are not in explicit form, hence, do not have exact solution. Therefore, the MLE can be obtained using some iterative methods such as Newton-Raphson to solve the equations analytically.

MPS

The maximum likelihood estimation is the most common and widely used estimation method but in cases such as that involving compound continuous distributions and large samples, the method might break down.

Cheng and Amin (1983) introduced the MPS method serving as an alternative to MLE method. Also, (Ranneby, 1989) independently studied the method as an approximation to Kullback-Leibler information and explained its consistency property.

If $X_1, X_2, \dots, X_n$ is a random sample from GOM-ER distribution having CDF $F(x,\epsilon)$ and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ represents the corresponding ordered sample. The spacing

$$D_i = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) \text{ for } i = 1, 2, \dots, n+1 \quad (41)$$

where

$$F\left(x_{(0)}\right) = 0 \text{ and } F\left(x_{(n+1)}\right) = 1$$

therefore

$$D_i = \left[1 - e^{\frac{\beta}{\alpha}\left(1 - \left\{1 - \left[1 - e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right] - \left[1 - e^{\frac{\beta}{\alpha}\left(1 - \left\{1 - \left[1 - e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right] \quad (42)$$

The parameter estimates are obtained by maximizing

$$P(\beta,\alpha,\delta,\sigma) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log_e D_i\,(\beta,\alpha,\delta,\sigma)$$

$$P(\beta,\alpha,\delta,\sigma) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log_e\left[e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} - e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)}\right] \quad (43)$$

The parameters estimates $\hat{\beta}_{MPS}, \hat{\alpha}_{MPS}, \hat{\delta}_{MPS}$ and $\hat{\sigma}_{MPS}$ can be found by differentiating P with respect to the individual parameters and solving the non-linear equations

$$\frac{\partial P(\beta,\alpha,\delta,\sigma)}{\partial\beta} = \frac{1}{n+1}\cdot\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\alpha,\delta,\sigma)}\cdot\left\{\Lambda_1\left[x_{(i-1)};\,\epsilon\right] - \left\{\Lambda_1\left[x_{(i)},;\epsilon\right]\right\}\right\} \quad (44)$$

$$\frac{\partial P(\beta,\alpha,\delta,\sigma)}{\partial\alpha} = \frac{1}{n+1}\cdot\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\alpha,\delta,\sigma)}\cdot\left\{\Lambda_2\left[x_{(i-1)};\epsilon\right] - \left\{\Lambda_1\left[x_{(i)};\epsilon\right]\right\}\right\} \quad (45)$$

$$\frac{\partial P(\beta,\alpha,\delta,\sigma)}{\partial\delta} = \frac{1}{n+1}\cdot\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\alpha,\delta,\sigma)}\cdot\left\{\Lambda_3\left[x_{(i-1)};\epsilon\right] - \left\{\Lambda_3\left[x_{(i)};\epsilon\right]\right\}\right\} \quad (46)$$

$$\frac{\partial P(\beta,\alpha,\delta,\sigma)}{\partial\sigma} = \frac{1}{n+1}\cdot\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\alpha,\delta,\sigma)}\cdot\left\{\Lambda_4\left[x_{(i-1)};\,\epsilon\right] - \left\{\Lambda_4\left[x_{(i)};\,\epsilon\right]\right\}\right\} \quad (47)$$

where

$$\Lambda_1\left[x_{(i-1)};\,\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times \frac{\beta}{\alpha}\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\log\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\} - \frac{\beta}{\alpha^2}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)$$

$$\Lambda_1\left[x_{(i)};\,\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times \frac{\beta}{\alpha}\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\log\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\} - \frac{\beta}{\alpha^2}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)$$

$$\Lambda_2\left[x_{(i-1)};\,\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times \frac{1}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)$$

$$\Lambda_2\left[x_{(i)};\,\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times \frac{1}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)$$

$$\Lambda_3\left[x_{(i-1)};\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times\left(-\beta\,.\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}.\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha-1}\log\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]\right)$$

$$\Lambda_3\left[x_{(i)};\,\epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times\left(-\beta\,.\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}.\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha-1}\log\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]\right)$$

$$\Lambda_4\left[x_{(i-1)};\ \epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times\left(-2\alpha\delta\sigma x_{(i-1)}^2 \,.\, e^{-\left(\sigma x_{(i-1)}\right)^2}.\left[1-\ e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta-1}\ .\ \left\{1-\left[1-\ e^{-\left(\sigma x_{(i-1)}\right)^2}\right]^{\delta}\right\}^{-\alpha-1}\right)$$

$$\Lambda_4\left[x_{(i)};\ \epsilon\right] = e^{\frac{\beta}{\alpha}\left(1-\left\{1-\left[1-e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha}\right)} \times\left(-2\alpha\delta\sigma x_{(i)}^2 \,.\, e^{-\left(\sigma x_{(i)}\right)^2}.\left[1-\ e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta-1}\ .\ \left\{1-\left[1-\ e^{-\left(\sigma x_{(i)}\right)^2}\right]^{\delta}\right\}^{-\alpha-1}\right)$$

The solutions of (44), (45), (46) and (47) are the MPS parameter estimates. However, like the MLE, the equations cannot be obtained analytically but rather with the use of numerical solutions.

Results and Discussion

Simulation study

In this subsection, Monte Carlo approach to simulation study was developed. The important objective of simulations was to determine the most efficient between ML and MPS methods for the GOM-ER distribution parameters. Using different parameters values and sample sizes (25-1000), the estimation methods were compared based on bias and root mean square error (RMSE) of the estimators.

$$Bias = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right)$$

$$RMSE = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right)}$$

Steps adopted are as follows:

1. Set the sample size and the vector of parameter values $\theta = (\alpha, \beta, \delta, \sigma)$
2. Generate sample of size n from GOM-ER $(\alpha, \beta, \delta, \sigma)$ using equation (12)
3. Using the values obtained above, obtain $\widehat{\alpha}, \widehat{\beta}, \widehat{\delta}\ and\ \widehat{\sigma}$ using MLE and MPSE.
4. In 1000 times, repeat steps (2) and (3).
5. Using θ and $\hat{\theta}$ compute Bias and RMSE.

Table 1: Means, Bias and RMSEs for the parameter estimates when $\alpha = 0.3, \beta = 1.5, \delta = 2.0, \sigma = 0.7$

N		MLE			MPSE		
		Means	Bias	RMSE	Means	Bias	RMSE
25	α	0.3378	0.0378	0.2298	0.4028	0.1028	0.2673
	β	1.5905	0.0905	0.3499	1.5104	0.0104	0.3722
	δ	2.1360	0.1360	0.5172	1.9177	-0.0823	0.5115
	σ	0.7058	0.0058	0.0579	0.6683	-0.0317	0.0732
100	α	0.3408	0.0408	0.1359	0.3510	0.0510	0.1508
	β	1.5401	0.0401	0.1778	1.5199	0.0199	0.1883
	δ	2.0995	0.0995	0.3259	1.9688	-0.0312	0.3604
	σ	0.6961	-0.0039	0.0348	0.6825	-0.0175	0.0431
250	α	0.3300	0.0300	0.0959	0.3294	0.0294	0.0992

	β	1.5383	0.0383	0.1289	1.5040	0.0040	0.1043
	δ	2.0624	0.0624	0.2397	1.9960	-0.0040	0.2290
	σ	0.6941	-0.0059	0.0242	0.6915	-0.0085	0.0262
450	α	0.3269	0.0269	0.0774	0.3208	0.0208	0.0765
	β	1.5332	0.0332	0.0972	1.5077	0.0077	0.0858
	δ	2.0480	0.0480	0.1694	1.9916	-0.0084	0.1617
	σ	0.6936	-0.0064	0.0196	0.6933	-0.0067	0.0214
1000	α	0.3203	0.0203	0.0534	0.3148	0.0148	0.0523
	β	1.5261	0.0261	0.0756	1.5059	0.0059	0.0708
	δ	2.0291	0.0291	0.1167	1.9980	-0.0020	0.1173
	σ	0.6943	-0.0057	0.0157	0.6954	-0.0046	0.0165

Table 2: Means, Bias and RMSEs for the parameter estimates when
$\alpha = 1.8, \beta = 5.9, \delta = 0.5, \sigma = 3.4$

n		MLE			MPSE		
		Means	Bias	RMSE	Means	Bias	RMSE
25	α	2.0437	0.2437	1.1881	2.1088	0.3088	1.1858
	β	6.0190	0.1190	1.2015	5.7158	-0.1842	1.2369
	δ	0.5369	0.0369	0.1107	0.4901	-0.0099	0.0903
	σ	3.7146	0.3146	0.9172	3.2374	-0.1626	0.7443
100	α	2.0437	0.2437	0.7466	2.0226	0.2226	0.7773
	β	6.0468	0.1468	0.7570	5.9087	0.0087	0.7112
	δ	0.5207	0.0207	0.0619	0.4978	-0.0022	0.0571
	σ	3.4374	0.0374	0.3734	3.2630	-0.1370	0.3688
250	α	1.9735	0.1735	0.5593	1.9165	0.1165	0.5831
	β	6.0378	0.1378	0.5399	5.9031	0.0031	0.5191
	δ	0.5122	0.0122	0.0436	0.4984	-0.0016	0.0432
	σ	3.3877	-0.0123	0.1884	3.3275	-0.0725	0.2199
450	α	1.9525	0.1525	0.4745	1.8928	0.0928	0.4494
	β	6.0278	0.1278	0.4405	5.9126	0.0126	0.3936
	δ	0.5100	0.0100	0.0362	0.5000	0.0000	0.0348
	σ	3.3752	-0.0248	0.1522	3.3489	-0.0511	0.1569
1000	α	1.9126	0.1126	0.3442	1.8715	0.0715	0.3441
	β	5.9827	0.0827	0.3181	5.9055	0.0055	0.2851
	δ	0.5069	0.0069	0.0274	0.5008	0.0008	0.0270
	σ	3.3772	-0.0228	0.0912	3.3686	-0.0314	0.0982

Table 3: Means, Bias and RMSEs for the parameter estimates when
$\alpha = 1.2, \beta = 3.5, \delta = 0.9, \sigma = 2.0$

		MLE			MPSE		
N		Means	Bias	RMSE	Means	Bias	RMSE
25	α	1.3915	0.1915	0.8480	1.4199	0.2199	0.8496
	β	3.5712	0.0712	0.9337	3.3904	-0.1096	0.8865
	δ	1.0010	0.1010	0.2754	0.8857	-0.0143	0.2277
	σ	2.1023	0.1023	0.3203	1.9278	-0.0722	0.3027
100	α	1.3656	0.1656	0.5274	1.3387	0.1387	0.5420
	β	3.5662	0.0662	0.5715	3.4566	-0.0434	0.5737
	δ	0.9467	0.0467	0.1483	0.8988	-0.0012	0.1354
	σ	2.0101	0.0101	0.1559	1.9638	-0.0362	0.1457
250	α	1.3229	0.1229	0.4104	1.2903	0.0903	0.4136
	β	3.5669	0.0669	0.3759	3.4885	-0.0115	0.3638
	δ	0.9280	0.0280	0.1009	0.9009	0.0009	0.0972
	σ	1.9939	-0.0061	0.0853	1.9775	-0.0225	0.0852
450	α	1.2905	0.0905	0.3025	1.2609	0.0609	0.3171
	β	3.5369	0.0369	0.2713	3.4767	-0.0233	0.2646
	δ	0.9201	0.0201	0.0765	0.9017	0.0017	0.0787
	σ	1.9952	-0.0048	0.0578	1.9882	-0.0118	0.0597
1000	α	1.2572	0.0572	0.2253	1.2329	0.0329	0.2194
	β	3.5282	0.0282	0.1818	3.4865	-0.0135	0.1742
	δ	0.9103	0.0103	0.0553	0.8997	-0.0003	0.0560
	σ	1.9932	-0.0068	0.0399	1.9931	-0.0069	0.0365

The results showed that both estimation methods were consistence as the sample size increases from 25 to 1000 since the RMSE decreases and the means converges to the actual values of the parameters. The consistency of MPSE justify the work of (Ranneby, 1984). Moreso, MPS estimators have the lower RMSEs and Bias closer to zero for most parameters notably for α and β when n $=$ 1000 demonstrating efficiency of the estimators.

Applications of GOM-ER

The advantage of GOM-ER over some related distributions having at least a parameter was portrayed by fitting two data sets. The comparison was done using the log-likelihood, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Corrected Akaike's Information Criteria (CAIC) and Hannan-Quinn Information Criteria (HQIC).

$$AIC = -(ll) + 2k$$

$$BIC = -(2 * ll) +) + (k * (\ln(n)))$$

$$CAIC = AIC + 2k(k+1)/(n-k-1)$$

$$HQIC = -(2 * ll) + (2 * k * (\ln(\ln(n)))$$

where ll is the log-likelihood, n is the sample size and k is the number of parameters to be fitted.

Using both data sets, measures of goodness of fit were compared with those of Gompertz Rayleigh (GomR), Kumaraswamy Exponentiated Rayleigh (KWER), Kumaraswamy Exponentiated Inverse Rayleigh (KEIR), Exponentiated Rayleigh (ER) and Rayleigh (R).

First data

Table 4: breaking stress of carbon fibers (in Gba)

0.39	0.81	0.85	0.98	1.08	1.12	1.17	1.18	1.22	1.25	1.36

1.41	1.47	1.57	1.57	1.59	1.59	1.61	1.61	1.69	1.69	1.71
1.73	1.8	1.84	1.84	1.87	1.89	1.92	2	2.03	2.03	2.05
2.12	2.17	2.17	2.17	2.35	2.38	2.41	2.43	2.48	2.48	2.5
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.76	2.77	2.79
2.81	2.81	2.82	2.83	2.85	2.87	2.88	2.93	2.95	2.96	2.97
2.97	3.09	3.11	3.11	3.15	3.15	3.19	3.19	3.22	3.22	3.27
3.28	3.31	3.31	3.33	3.39	3.39	3.51	3.56	3.6	3.65	3.68
3.68	3.68	3.7	3.75	4.2	4.38	4.42	4.7	4.9	4.91	5.08
5.56										

This data has already been used by (Mohammed, et al., 2020), (Bhat & Ahmad, 2020), (Yahaya & Ieren, 2017). It represents the breaking stress of carbon fibers of 50 mm length (GPa). The table below provide the descriptive statistics of the data

Table 5: Description statistics of breaking stress of carbon fibers (in Gba)

Variables	Description
Sample size	100
Maximum and Minimum value	5.56, 0.39
Mode	2.75
Kurtosis, Skewness	0.1049, 0.3682
Mean, Median, Variance	2.6214, 2.7, 1.02796

Table (6) presents each distribution with their maximum likelihood estimates and maximum product of spacing estimates while table (7) the distributions and their corresponding measures of comparison.

Table 6: Models parameters estimates

	MLE				MPSE			
Model	α	β	σ	δ	α	β	σ	δ
GOM-ER	2.8426	0.0195	1.6140	0.2737	0.7914	0.0436	1.6471	0.4423
GomR	1.6931	0.9097	3.0589	---	1.5923	0.6521	2.8951	----
KWER	1.6003	4.4361	0.2575	0.4253	4.5686	1.0345	0.3653	0.4079
ER	------	-------	1.8388	0.4253	-----	-------	1.6837	0.4126
KEIR	1.5687	1.1403	0.9527	2.3990	1.6822	1.1084	0.8695	2.4051
R	---	---	1.9861	----	----	----	2.0047	----

Table 7: Log-likelihood and information criteria

	MLE					MPSE				
Model	AIC	BIC	CAIC	HQIC	ll	AIC	BIC	CAIC	HQIC	ll
GOM-ER	274.66	264.24	274.24	270.44	-141.33	275.43	265.01	275.01	271.21	-141.72
GOmR	283.76	275.94	283.51	280.59	-144.88	284.23	276.42	283.98	281.07	-145.12
KWER	275.19	264.77	274.77	270.97	-141.59	275.76	265.34	275.34	271.54	-141.88
ER	279.19	273.98	279.06	277.08	-141.59	279.54	274.33	279.41	277.43	-141.77
KEIR	341.68	331.26	341.26	337.46	-174.84	341.72	331.30	341.30	337.50	-174.86
R	297.00	294.40	296.96	295.95	-149.50	297.04	294.43	297.00	295.98	-149.52

Second data

Table 8: Strength of 1.5cm glass fiber

0.55	0.74	0.77	0.81	0.84	1.24	0.93	1.04
1.11	1.13	1.3	1.25	1.27	1.28	1.29	1.48
1.36	1.39	1.42	1.48	1.51	1.49	1.49	1.5
1.5	1.55	1.52	1.53	1.54	1.55	1.61	1.58
1.59	1.6	1.61	1.63	1.61	1.61	1.62	1.62
1.67	1.64	1.66	1.66	1.66	1.7	1.68	1.68
1.69	1.7	1.78	1.73	1.76	1.76	1.77	1.89
1.81	1.82	1.84	1.84	2	2.01	2.24	

This data comprises 63 observations collected by employees of United Kingdom National Physical Laboratory of the strengths of 1.5 cm glass fibers has been used in earlier studies by (Bourguignon et al., 2014), (Oguntunde et al., 2014), (Falgore and Doguwa, 2020), (Eghwerido et al., 2020) among others. Table (9) depicts the descriptive statistics of the data.

Table 9: Descriptive statistics of strength of 1.5cm glass fiber

Variables	Description
Sample size	63
Maximum and Minimum	2.24 , 0.55
Mode	1.7

Kurtosis, Skewness	0.9238, -0.8999
Mean, Median, Variance	1.5068, 1.59, 0.1051

Table (10) gives each distribution with their maximum likelihood estimates and maximum product of spacing estimates while table (11) the distributions and their corresponding measures of comparison.

Table 10: Models parameters estimates

	MLE				MPS			
Model	α	β	σ	δ	α	β	σ	δ
GOM-ER	3.3424	4.1661	2.4712	0.5018	6.3660	6.6758	2.1602	0.3962
GomR	0.2810	3.7359	1.3921	----	0.3373	3.6672	1.4329	---
KWER	1.8336	3.4323	0.8726	0.9872	2.1675	1.4260	2.0683	0.8963
ER	-----	-----	5.4845	0.9869	------	------	4.7256	0.9551
KEIR	1.2644	5.7986	1.5318	2.3313	2.01649	4.6683	1.5881	1.2623
R	------	-----	1.0894	------	-----	------	1.0984	-------

Table 11: Log-likelihood and information criteria

	MLE					MPSE				
Model	AIC	BIC	CAIC	HQIC	ll	AIC	BIC	CAIC	HQIC	ll
GOM-ER	23.2798	14.7073	22.5901	19.9082	-15.6399	24.034	15.4615	23.3443	20.6624	-16.0170
GOmR	26.9978	20.5684	26.5910	24.4691	-16.4989	27.5026	21.0732	21.0732	24.9739	-16.7513
KWER	39.8574	31.2849	39.1677	36.4858	-23.9288	57.4938	48.9213	56.8041	54.1222	-32.7469
ER	43.8576	39.5713	43.6576	42.1718	-23.9288	44.3284	40.0421	44.1284	42.6426	-24.1642
KEIR	40.3284	31.7559	39.6387	36.9568	-33.6669	59.9792	51.4067	59.2895	56.6076	-33.9896
R	97.5818	95.4387	97.5162	96.7389	-49.7909	97.5984	95.4553	97.5328	96.7555	-49.7992

The distribution with lower information criteria and higher log likelihood is considered to better fit the data. For both parameter estimation methods, GOM-ER had the lowest AIC, BIC, CAIC, and HQIC compared to others as shown in table (7) and (11) with GomR and KWER also having lower values. However, values from MLE are lower than those from MPSE. These implies that although GomR and KWER are good models, the GOM-ER is a better model in fitting the two data sets. Furthermore, MLE having lower values of the measures used than MPSE are better considering GOM-ER distribution.

CONCLUSION

As aforementioned in the prior sections, little information has been established on generalization of ER distribution which has limitation of fitting data sets that are only tailed to the right. This study has been able to generalize the ER distribution using a generator in an earlier study.

The generated compound distribution has unimodal densities that are; positively skewed, leptokurtic and mesokurtic, hazard functions that are non-decreasing. Furthermore, as the value of x approach infinity, CDFs of the new distribution equal one, implying that the CDF and, by extension, the corresponding PDF is real.

Major statistical properties were studied and its applications was demonstrated using two real data sets to ascertained its flexibility over the sub models and related distributions. Upon application to these data sets and considering goodness-of-tests statistics, the proposed distribution provides better fit compared to Gompertz Rayleigh (GomR), Kumaraswamy Exponentiated Rayleigh (KWER), Kumaraswamy Exponentiated Inverse Rayleigh (KEIR), Exponentiated Rayleigh (ER) and Rayleigh (R).

The parameters were estimated using two frequentist approach, MLE and MPSE. Albeit application to two data sets portray the advantage of MLE over MPSE considering AIC and BIC, simulation study showed that the parameter estimates via both methods were consistence since as the sample size increases, the means converges to the actual values. However, the estimators $\hat{\alpha}_{MPS}$, $\hat{\beta}_{MPS}$ and $\hat{\delta}_{MPS}$ are more efficient than $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$ and $\hat{\delta}_{ML}$ at larger sample sizes.

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