COMPOSITION SERIES OF THE SOLVABLE ABELIAN FACTOR GROUP SOURCE OF EQUATION OF ALL POLYNOMIAL EQUATIONS

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ABSTRACT

This work penciled down the Composition Series of Factor Abelian Group over the source of all polynomial equations gleaned through the nth roots of unity regular gons on a unit circle, a circle of radius one and centered at zero. To get the composition series, the third isomorphism theorem has to be passed through. But, the third isomorphism theorem itself gleaned via the first which is a deduction of the naturally existing canonical map. The solution of the source atom of the equation of all equation of polynomials are solvable by the intertwine of the Euler's Formula and the De Moivre's Theorem which after the inter-math, they become within the domain of complex analysis. For the source root of the equations, there is a recursive set of homomorphisms and ontoness of the mappings geneting the sequential terms in the composition series.

Keywords: Cauchy Sequence, Rouche Theorem, Class Equation, Aleph Naught

INTRODUCTION

One of the two greatest achievement of group theory inventing is solving equations. The other one is 'counting'. The $x^{nm} + a$ is solvable by radicals since $x^{nm} - a$ is. Hence $x^{nm} \pm a$ is solvable by composition series despite $S_{nm}, nm \geq 5$ – the symmetric groups of length 5 and above – are not soluble, not even by the radical. When $a = 1$, all the solvable groups are multiplicative and are P-Factor Groups. They all lie on the unit circle $|z| \leq 1$. Since they are P-Factor Groups, they have normal P-Sylow Subgroups. The normality comes from the Index Theorem. Because they all have index 2 in their P-Groups, they are the maximal proper normal P-Sylow Subgroups and their factor groups are abelian accounting to the solubility of $x^{nm} \pm 1$ by composition series. We combine the classical Euler Formula and the De Moivre Theorem to present the solvability of $x^{nm} \pm a$. The P-Groups over $x^{2nm} \pm a$ and $x^{2nm+1} \pm a$ are multiplicative. $x^{2nm} \pm a$ are subsequences of $x^{nm} \pm a$ and it converges to the limit point of the $x^{nm} \pm a$.

Group is the father, father as in personification, of other algebraic structures viz: Ring (additive abelian group equipped with an associative multiplicative binary operation that is both left and right distributive over the addition), vector space(group acting on a set), module(ring acting on a set), field(both additive and multiplicative abelian groups), semigroup (a generalization of group, Christopher (2009), that was born 87 years after group) and other consequences of group all under the universal algebra Stanley and Sankappanavar (1981).. Group has analyzed very many regular shapes (symmetry groups) and numbers are partly regular shapes. Symmetry groups are embedded in the symmetric groups. We shall present the bottom line of construction of regular shapes using $x^n - a$ in the anal of this article. The p-groups of the aforementioned multiplicative solvable groups all lie on the unit circle $|z| = 1$. There are uncountable infinitely many number of complex numbers on $|z| = 1$ as there are uncountable infinitely many number of real numbers in $(0, 1) \subseteq \mathbb{R}$ (Alechenu *et al.,* 2021).

Whenever there is a subgroup of a finite group, then the order of the subgroup divides the order of the group. This is the statement of Lagrange Theorem, first stated by Lagrange himself in 1771 almost 58 years before the definition of group by Galois. 30 years after the statement, Petro Abati proved the theorem (Richard , 2001). Today, like many other

decoded theorems, there are numerous alternative proofs of Lagrange Theorem. This brings us to: Proving a theorem is easier than stating it. The converse of the theorem is: Whenever there is a group of a finite order n, is there a subgroup of an order a divisor of n? The answer is yes if the group is abelian of a prime power order. Cauchy stated and proved that the group needs not to be abelian so far it is finite of prime order. The group is forced to equal its centre, the abelian subgroup of every group.

That is, Cauchy proved that every P-Group has a P-Subgroup; but his proof contained egregious error. Today, there is a simple proof of Cauchy Theorem via the Class Equation, Heinstein (1975). Sylow (John and Robertson (2017)) narrowly states that there are $P^m r$ -Group where r is a positive integer that does not divide P. It has the largest P-Subgroup, the P-Sylow Subgroup that forms a single conjugacy class of all the other P-non Sylow Subgroups. Sylow further gave $\equiv 1 \bmod p$, the formula that could find the number n of distinct P-Sylow Subgroups . That is, $\frac{n-1}{p} = m \in \mathbb{Z}$. That is, $1 + mp\backslash p^m r$.

Every P-Group is nilpotent and every nilpotent group is solvable. Every group has a composition series which may be trivial or non-trivial (interesting). The terms of the composition series of the group over $x^n - 1$ are P-Groups having P-Sylow Subgroups of Index 2. Hence, the factor group is abelian due to blending the following theorems Vasistha and Vasistha (2006) and Heinstein (1975).

Theorem 1.1. If H is a subgroup of index 2 in G, then H is a normal subgroup of G and $G/_H$ is a cyclic group of order 2.

Proof. Since the index of H in G is 2, there are only two right cosets of H in G. One of them is H and the other must be Hg, where g is an element of G that is not appearing in H for if $g \in H, g^{-1}hg \in H, \forall h \in H$. So let $g \notin H$ and let $g^{-1}hg \notin H$. Then $g^{-1}hg \in Hg$, the only other coset of G. But $h_1 g \in Hg$ for some $h_1 \in H$.That is $g^{-1}hg = h_1 g \in Hg$ which implies $g^{-1}h = h_1$ which implies $g = hh_1^{-1} \in H$. This contradicts $g \notin H$. Hence, $g^{-1}hg \in H, \forall g \in G$ and $h \in H$. H is a nomal subgroup of G.

$G/_H$ is a cyclic group of order 2 because of the Lagrange Theorem.

Theorem 1.2. If G is a group of prime order, then it is cyclic.

Proof. Let $|G| = p$, a prime number. Then every element of G has order 1 or p by Lagrange Theorem. But, the only element of order 1 is the identity. Therefore, all the other elements have order p and there is at least one because $|G| \geq 2$, the smallest

prime. Thus, every non-identity element of G generates G. Hence, G is cyclic.

Theorem 1.3. Every cyclic group is abelian.

Proof. Let g^m and g^n be any two elements of the cyclic group $G = \{g^i : i > 0 \in \mathbb{Z}, g^{k-1} = g^1, k > 1\}$. Then $g^m. g^n = g^{m+n} = g^{n+m} = g^n. g^m$. This is from the fact that $n + m = m + n, \forall m, n \in \mathbb{Z}$. Hence G is abelian. Hence, $G/_H$ is an abelian group.

Preliminaries

Galois sought to solve a problem that had stymied mathematicians for centuries. Methods for solving linear and quadratic equations were known thousands of years ago. In the 16th century, Italian mathematician developed formulas involving only the operations of addition, subtraction, multiplication, division and extraction of roots (radicals). The general formulas for the general cubics $f(x) = x^3 + ax^2 + bx + c$ and the quartics $y = x^4 + ax^3 + bx^2 + cx + d = 0$ are a little more complicated.

A polynomial over $y = F[x]$, a field, is solvable by radicals if we can obtain all its zeros if it is $F[x] = 0$ by the use of any mighty formula. In other words, each zero of the polynomial can be written as an expression involving elements of $F[x]$ combined by the operations of addition, subtraction, multiplication, division and extraction of roots. We say that a group G is solvable if G has a series of subgroups $\{e\} = F_0 H_0 \subset F_1 H_1 \subset F_2 H_2 \subset \cdots \subset F_k H_k = FG$, where for each $0 \leq i \leq k, F_i H_i$ is normal in $F_{i+1} H_{i+1}$ and $F_{i+1}H_{i+1}/_{F_i H_i}$ is abelian.

Solvable groups have been investigated for over seventy years. Feit and Thompson (1963) proved a long standing conjecture of Burnside (1997) that every group of odd order is solvable. Burnside had shown this result to be true for groups of order less than 40,000. The proof of Feit-Thompson Proof extends to over 250 pages of deep mathematics.

Theorem 2.1 (Splitting Field of $x^n - a$). Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^n - a$ over F, then the Galois Group $Gal(E/_F)$ is solvable.

Proof. We first handle the case where F contains a primitive nth root of unity w. Let b be a zero of $x^n - a$ in E. Then the zeros of $x^n - a$ are $, wb, w^2 b, \ldots, w^{n-1} b$. Therefore, $E = F(b)$. Let $Gal(E/_F)$ is abelian. Then it is solvable. To see this, observe that any automorphism in $Gal(E/_F)$ is completely determined by its action on b. Since b is a zero of $x^n - a$; any element of $Gal(E/_F)$ sends b to another zero of $x^n - a$. That is, any element of $Gal(E/_F)$

takes b to $w^i b$ for some i. Let $\emptyset$ and σ be two elements of $\text{Gal}\left(E/F\right)$. Then, since $w \in F$, $\emptyset$ and σ fixes w and $\emptyset(b) = w^j b$ and $\emptyset(b) = w^k b$ for some j and k. Thus, $(\sigma(\emptyset))(b) = \sigma\left(\emptyset(b)\right) = \sigma\left(w^j b\right) = \sigma\left(w^j\right)\sigma(b) = w^j w^k b = w^{j+k} b$ and $(\emptyset(\sigma))(b) = (\emptyset\left(\sigma(b)\right) = \emptyset(w^k b) = \emptyset(w^k) s\emptyset(b) = w^k w^j b = w^{k+j} b$. So that $\sigma\emptyset$ and $\emptyset\sigma$ agree on b and fix the elements of F. This shows that $\sigma\emptyset = \emptyset\sigma$. Therefore, $\text{Gal}\left(E/F\right)$ is abelian. Now suppose that F does not contain a primitive nth root of unity. Let w be a primitive nth root of unity and let $b \neq 0$. Since wb is also a zero of $x^n - a$, both w and wb belong to E_1. Therefore, $w = \frac{wb}{b} \in E$. Thus, $F(w)$ is contained in E, and F(w) is the splitting field of $x^n - 1$ over F. Analogously, for any automorphism $\emptyset$ and σ in $\text{Gal}\left(F(w)/F\right)$, we have $\emptyset(w) = w^j$ for some j and $\sigma(w) = w^k$ for some k. Then, $(\sigma(\emptyset))(w) = (\sigma\left(\emptyset(w)\right) = \sigma\left(w^j\right) = (\sigma(w))^j = (w^k)^j = (w^j)^k = \emptyset(w)^k = \emptyset(w^k) = \emptyset\left(\sigma(w)\right) = (\emptyset\sigma)(w)$. Some elements of $\text{Gal}\left(F(w)/F\right)$ are completely determined by their action on w, this shows that $\text{Gal}\left(F(w)/F\right)$ is abelian. Because E is the splitting field of $x^n - a$ over F(w) and F(w) contains a primitive nth root of unity, $Gal\left(E/F(w)\right)$ is abelian, and the series $\{e\} \subseteq Gal\left(E/F(w)\right) \subseteq Gal\left(E/F\right)$ is a normal series. Since both $Gal\left(E/F(w)\right)$ and

$$\left.Gal\left(E/F\right)\right/Gal\left(E/F(w)\right) \simeq Gal\left(F(w)/F\right) \quad \text{are}$$

abelian; $Gal\left(E/F\right)$ is solvable. See Gallian (2013) for the theorem and this accompanied proof.

Theorem 2.2 (Splitting Field of $x^{nm} - a$). Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^{nm} - a$ over F, then the Galois Group $\text{Gal}\left(E/F\right)$ is solvable.

Proof. The proof follows analogously to that of Theorem 2.1 by replacing $x^n - a$ with $x^{nm} - a$.

We now take the following theorems that we will use in section 3.

Theorem 2.3 (Eisentein Criterion). Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \in \mathbb{Z}[x]$. If there exist some prime p in $\mathbb{Z}$ such that $p \nmid a_n, p\backslash a_i$ where $i = 0,1,\ldots,n-1$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. Refer to Heinstein [7] for the proof.

Theorem 2.4 (Rouche). Let $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C. Then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Proof. Refer to Seymour *et al.* [10] for the proof. But, we need the complex analytic concept because our solutions do not always rely of real number roots or their functions, or their functions of their functions. And, what is happening on real-line should happen within the complex plane because real-line of also part of the complex plane. Consider $g(x) = 3x^5 - 15x + 5$. By Einstein Criterion, $g(x)$ is irreducible over $\mathbb{Q}$. Since $g(x)$ is continuous and $g(-2) = -61$ and $g(-1) = 17$, we know that g(x) has a real zero between -2 and -1. A similar analysis shows that g(x) also has real zeros between 0 and 1 and between 1 and 2. Each of these real zeros has multiplicity 1. $G(x)$ has no more than three zeros, because of the Rouche Theorem. So far, $g(x)$ have no real zeros, $g'(x)$ would have to have three real zeros, and it does not. Thus, the other two zeros are non real complex numbers, say, $a + bi$ and $a - bi$.

Lemma 2.5. If $N \trianglelefteq G$ and both N and G/N are solvable groups, then G is a solvable group.

Proof. Refer to Gallian [6] for the proof.

Theorem 2.6. Every p-Group G_P is solvable.

Proof. We will induct $|G|$, with the case $|G| = 1$ being trivial. Assume that the result is true for all p-Groups of order less than $|G|$. Since G is a nontrivial group, it contains a nontrivial centre $Z(G)$. If $Z(G) = p$, then G is abelian and therefore, it is solvable. If $Z(G) \neq p$, then both $Z(G)$ and $G_P/Z(G)$ are p-groups of order less than $|G|$. By the induction hypothesis, both $Z(G)$ and $G_P/Z(G)$ are solvable. The result follows immediately from Lemma 2.5.

Theorem 2.7. Every pq-Factor Group G_{Pq} is solvable.

Proof. We will induct $|G|$, with the case $|G| = 1$ being trivial. Assume that the result is true for all pq-Factor Groups of order less than $|G|$. Since G is a nontrivial group, it contains a nontrivial centre $Z(G)$. If $Z(G) = pq$, then G is abelian and therefore, it is solvable. If $Z(G) \neq pq$, then both $Z(G)$ and $G_{Pq}/Z(G)$ are pq-groups of order less than $|G|$. By the induction hypothesis, both $Z(G)$ and $G_{Pq}/Z(G)$ are solvable. The result follows immediately from Theorem 2.6.

3.0. Results and Discussion

This section presents the: Composition Series of the Solvable Factor Group over $x^{nm} \pm a$, Construction of Regular Shapes Using $x^{nm} - a$, Discussion on $P-$Sylow Subgroups of $x^{pqm} - a, x^{pqrm} \pm a, x^{p^2qm} \pm a$.

3.1. Composition Series of the Solvable Factor Group over $x^{nm} \pm a$

Euler Formula is given by $e^{in\theta} = cosn\theta + i \sin n\theta$ and De Moivre Theorem states that $z^m = r^m e^{im\theta} = r^m(\cos m\theta + i \sin m\theta)$. When $m = \frac{1}{n}$, we have: $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{\theta}{n} + i \, sin\frac{\theta}{n}\right)$. For nth root of unity, $r^{\frac{1}{n}} = 1$, i.e. $|z| = 1$, the unit circle. Thus $\theta = 2\pi k$ and $(-1)^{\frac{1}{n}} = cos\frac{2\pi k}{n} + i \, sin\frac{2\pi k}{n}$. For $x^{nm} - a$, we have $\sqrt{a}\left(cos\frac{2\pi k}{nm} + i \, sin\frac{2\pi k}{nm}\right)$ or $\sqrt{a}w$, where $k = 0, 1, 2, \ldots, nm - 1$ or $k = 1, 2, \ldots, nm$. When $n = 1$ and $a = 1$; we have: $x = 1$. The group G over this is the group containing only the identity, the trivial group $\{1\}$ since $G = (G, *)$ when * is known.

When $n = 2$ and $a = 1$; we have: $x^{2 \times 1} - 1 = (x - 1)(x + 1) \Rightarrow x = 1 \, or \, x = -1$. The group over this is $\{1, -1\}$, a group isomorphic to the kernel group of absolute values and isomorphic to parity group $\{even, odd\}$. It is also a P-group since 2 is the oddest (even) prime.

When $n = 3$ and $a = 1$; we have: $e^0 = 1, e^{120} = cos \, 120 + i \sin 120 = -0.5 + 0.9i, e^{240} = cos \, 240 + i \sin 240 = -0.5 - 0.9i$. Thus, the multiplicative group is: $\{1, -0.5 + 0.9i, -0.5 - 0.9i\}$.

When $n = 4$ and $a = 1$, we have: $(x^2 - 1)(x^2 + 1)$. The group is $\{1, -1, i, -i\}$ which is a P-Group of the P-Sylow Subgroup $\{1, -1\}$ with index 2. Thus $\{-1, 1\}$ is normal in $\{1, -1, i, -i\}$. The factor group is $\{\{1, -1\}, \{i, -i\}\}$ and the canonical map is $\{1, -1, i, -i\} \rightarrow \{\{1, -1\}, \{i, -i\}\}$. Hence, it is abelian because of Theorem 1.1, 1.2 and 1.3.

When $n = 5$ and $a = 1$; we have: $e^0 = 1, e^{72} = cos \, 72 + i \sin 72 = 0.3 + i, e^{144} = cos \, 144 + i \sin 144 = -0.8 + 0.6i, \; e^{216} = cos216 + i \sin 216 = -0.8 - 0.6i, e^{299} = cos \, 299 + i \sin 288 = 0.3 - i$. Thus, the multiplicative group is: $\{1, 0.3 + i, -0.8 + 0.6i, -0.8 - 0.6i, 0.3 - i\}$.

When $n = 6$ and $a = 1$; we have: $e^0 = cos \, 0 + i \sin 0 = 1, e^{60} = cos60 + i \sin 60 = 0.5 + 0.9i, e^{120} = cos \, 120 + i \, sin120 = -0.5 + 0.9i, e^{180} = cos180 + i \sin 180 = -1, e^{240} = cos240 + i \sin 240 = -0.5 - 0.9i, e^{300} = cos300 + isin300 = 0.5 - 0.9i$. The group with this underlying set has order $2p$. Hence we take the following theorem.

Theorem 3.2. If $|G| = 2pq$, pq an odd prime; then G has one and only one subgroup of order pq and either G has exactly pq subgroups of order 2 or it has exactly one subgroup of order 2.

The subgroup of order 3 is $\{1, -0.5 + 0.9i, -0.5 - 0.9i\}$. It is a normal subgroup because of the Index Theorem. It is a P-Sylow Subgroup. The subgroup of order two is $\{1, -1\}$. It is normal because of the theorem that says any subgroup of an abelian group is normal. It is a P-Subgroup. $\{1, -0.5 + 0.9i, -0.5 - 0.9i\}$ is the conjugacy class of $\{1, -1\}$.

When $n = 7$ and $a = 1$; we have the following group: $\{e^0 = cos0 + i \, sin0 = 1, e^{51.4} = cos51.4 + isin51.4 = 0.6 + 0.8i, e^{102.9} = cos102.9 + isin102.9 = -0.2 + i, e^{154.3} = cos154.3 + i \, sin154.3 = -0.9 + 0.4i, e^{205.7} = cos205.7 + isin205.7 = -0.9 - 0.4i, e^{257.1} = cos257.1 + i \, sin257.1 = -0.2 - i, e^{308.6} = cos308.6 + isin308.6 = 0.6 - 0.8i\}$

When $n = 8$ and $a = 1$, we have the following group: $\{e^0 = cos0 + isin0 = 1, e^{45} = cos45 + isin45 = 0.7 + 0.7i, e^{90} = cos90 + isin90 = i, e^{135} = cos135 + isin135 = -0.7 + 0.7i, e^{180} = cos180 + isin180 = -1, e^{225} = cos225 + isin225 = -0.7 - 0.7i, e^{270} = cos270 + isin270 = -i, e^{315} = cos315 + isin315 = 0.7 - 0.7i\}$. This is a P-Group because it is of order $2^3 \times 1$, it is of the form $p^m r$. It has at least a P-Sylow Subgroup $\{1, -1, i, -i\}$. The index theorem applies. This 8-Group is the identity of the coset classes of the Factor 16-Group by 8-Normal Subgroup. The canonical map is natural.

Theorem 3.3. This works in general.

Proof. We prove this by the Induction Hypothesis. Case I: When $a = 1$. For $n = 1$, we got $e^0 = cos0 + i \, sin0$. Assume that it is true for $n = k$. Then: $e^{ki\theta} = cosk\theta + isnk\theta$. Now $e^{ki\theta}e^{i\theta} = (cosk\theta + isink\theta)(cos\theta + isin\theta) = cosk\theta cos\theta + icosk\theta sin\theta + isink\theta cos\theta - sink\theta sin\theta = [cosk\theta cos\theta - sink\theta sin\theta] + i[cosk\theta sin\theta + sink\theta cos\theta] = cos(k + 1)\theta + isin(k + 1)\theta = e^{(k+1)i\theta}$. Whence, it is true for $n = k + 1$.

Case II: When $a \neq 1$, for $n = 1$, we have: $ae^0 = a(cos0 + i \, sin0) = a$. Assume that it is true for $n = k$. Then: $a^k e^{ki\theta} = a^k(cosk\theta + isnk\theta)$. Now $a^k e^{ki\theta} ae^{i\theta} = a^k a(cosk\theta + isink\theta)(cos\theta + isin\theta) = a^k a(cosk\theta cos\theta + icosk\theta sin\theta + isink\theta cos\theta - sink\theta sin\theta) = a^{k+1}\{cosk\theta cos\theta - sink\theta sin\theta + i[cosk\theta sin\theta + sink\theta cos\theta]\} = a^{k+1}[cos(k + 1)\theta + isin(k + 1)\theta] = a^{k+1}e^{(k+1)i\theta}$. Hence, it is true for $n = k + 1$

Let $z = ae^{i\theta} = a(cos\theta + isin\theta).$ Then $z^n = a^n e^{in\theta} = a^n(cosn\theta + isinn\theta)$, for all n, by the induction hypothesis. Since $z^n = a^n e^{in\theta} = a^n(cosn\theta + isinn\theta), \forall n,$ $z^{\frac{1}{m}} = a^{\frac{1}{m}}e^{i\frac{\theta}{m}} = a^{\frac{1}{m}}\left(cos\frac{\theta}{m} + isin\frac{\theta}{m}\right)$, where $m = \frac{1}{n}$.

Remark 3.4. Do not go to nth degree and conclude the proof mathematics theorem. Induction is embedding an assertion in an extended set of natural numbers that includes aleph naught in this regards.

3.5. Construction of Regular Shapes Using $x^n - a$

We are aware that group theory that gleaned 85 years before the introduction of ring and 87 years before the introduction of semigroup has seemingly and apparently no number analysis: Semigroup analyzes natural numbers, ring analyzes integers, field analyzes rational and real numbers, vector space analyzes complex numbers. Group, the mother of algebraic structures, analyzes any regular shapes which includes numbers. Any regular shape is constructible using $x^n - a$ (Alechenu *et al.*, 2021):

When n = 1, e^0 . When $n = 2$, .

When $n = 3$, e^0 . When $n = 4$, .

When n = 5, . When $n = 6$, .
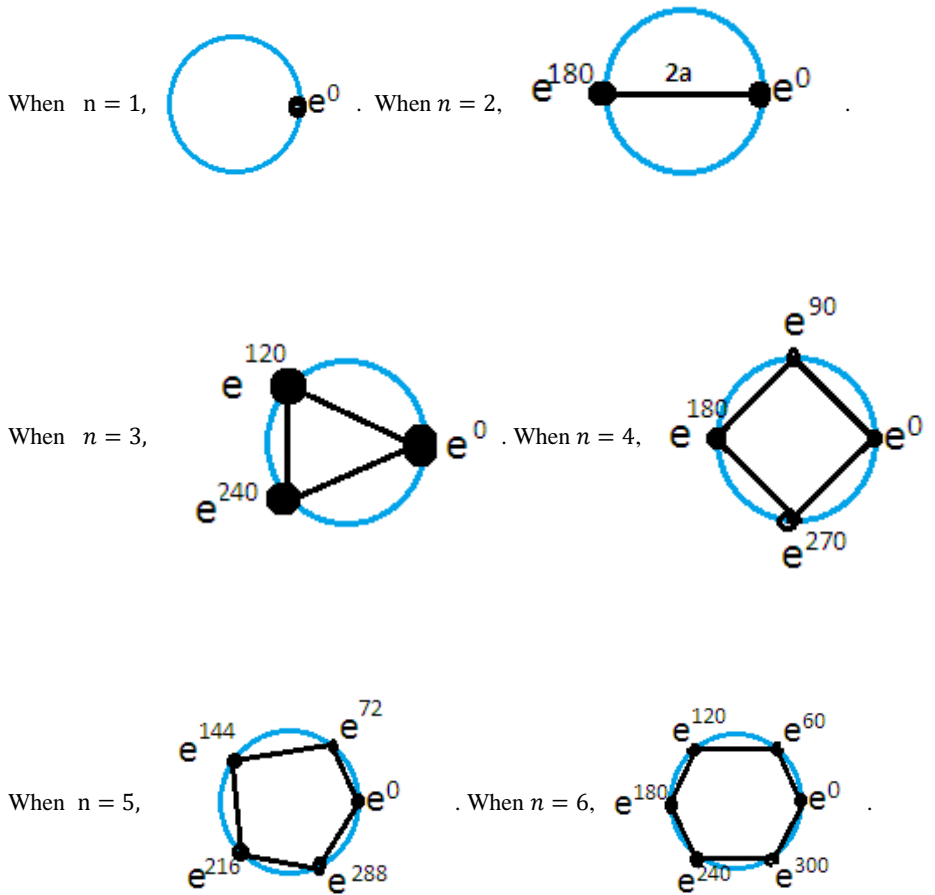
When n = k,
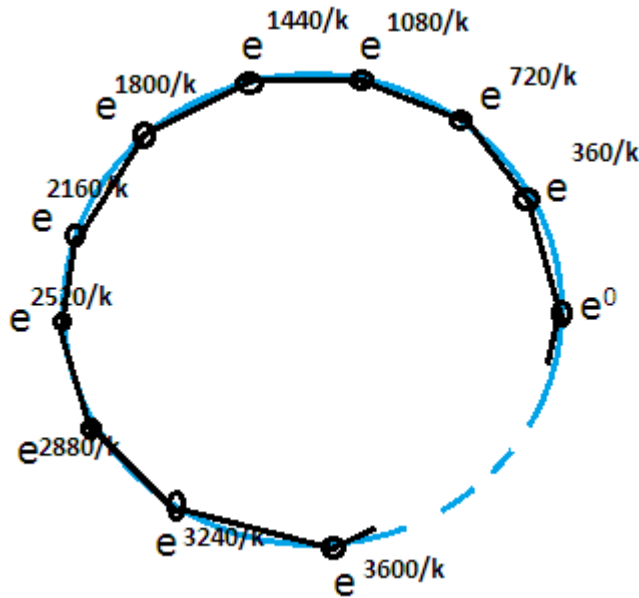
The P −Sylow Subgroups of the Factor Group over $x^{pq} − a, x^{pqr} ± a, x^{p^2q} ± a$ when $m = 1$

Any group over $x^p − a$ having order p is a normal subgroup of the group over $x^{pq} − a$. Any integer p is either a prime or a composite. If it is a prime, it is a P-Group as well as a P-Sylow Subgroup. If it is a composite, it has a P-Sylow Subgroup because of the following theorems (Heinstein, 1975):

Theorem 3.7. If $|G| = pq$, where p,q are distinct primes such that $q \not\equiv 1 mod p$; then G has a normal Sylow Subgroup.

Proof. The number of n distinct sylow p-subgroups of G is a divisor of q and $n \equiv 1 \ mod \ p$. Since q is prime, n is either 1 or q. Since $q \not\equiv 1 mod p$, $n = 1$. That is, G has a unique sylow p-sylow subgroup. It is normal in G.

Corollary 3.8. If $|G| = pq$ where p,q are distinct primes, then G has a proper normal subgroup.

Proof. We may assume without loss of generality that $p > q$. Then $q − 1$ cannot be divisible by p, and so by Theorem 3.7, G has a normal sylow p-subgroup.

The group over $x^{nm} − a$ may be of small or big order. It duo two composition series $x^{2nm} − a$ and $x^{2nm+1} − a$. The groups over $x^{2nm} − a$ are forming p-Groups and normal p-sylow subgroups of index 2. Thus, they are nilpotent and every nilpotent group is solvable. Besides, we presented the theorem that states that every p-group is solvable. The groups over

$x^{2nm+1} − a$ are solvable because of the Burnside Lemma.

Now since $x^{nm} − 1$ works out well, then $x^{nm} − a$ works: The group over this is the group containing only the identity, the trivial group $\{1\}$ since $G = (G,*)$ when * is known. When $= 2$, we have: $\{\sqrt{a}, −\sqrt{a}\}$. When $n = 3$, we have: $\sqrt[3]{a}\{1, −0.5 + 0.9i, −0.5 − 0.9i \}$. When $= 4$, we have: $\{\sqrt[4]{a}, −1\sqrt[4]{a}, i\sqrt[4]{a}, −i\sqrt[4]{a}\}$. When n = 5 ; we have: $\sqrt[5]{a}\{1, 0.3 + i, −0.8 + 0.6i, −0.8 − 0.6i, 0.3 − i\}$.
When $n = 6$; we have: $\sqrt[6]{a}\{e^0 = \cos 0 + i \sin 0 = 1, e^{60} = cos60 + i \sin 60 = 0.5 + 0.9i, e^{120} = \cos 120 + i sin120 = −0.5 + 0.9i, e^{180} = cos180 + i \sin 180 = −1, e^{240} = cos240 + i \sin 240 = −0.5 − 0.9i, e^{300} = cos300 + isin300 = 0.5 − 0.9i\}$. When $n = 7$, we have the following group: $\sqrt[7]{a}\{e^0 = cos0 + i \ sin0 = 1, e^{51.4} = cos51.4 + isin51.4 = 0.6 + 0.8i, e^{102.9} = cos102.9 + isin102.9 = −0.2 + i, e^{154.3} = cos154.3 + i \ sin154.3 = −0.9 + 0.4i, e^{205.7} = cos205.7 + isin205.7 = −0.9 − 0.4i, e^{257.1} = cos257.1 + i \ sin257.1 = −0.2 − i, e^{308.6} = cos308.6 + isin308.6 = 0.6 − 0.8i\}$.
When $n = 8$,we have the following group: $\sqrt[8]{a}\{e^0 = cos0 + isin0 = 1, e^{45} = cos45 + isin45 = 0.7 + 0.7i, e^{90} = cos90 + isin90 = i, e^{135} = cos135 + isin135 = −0.7 + 0.7i, e^{180} = cos180 + isin180 = −1, e^{225} = cos225 + isin225 =$

$-0.7 - 0.7i, e^{270} = cos270 + isin270 = -i, e^{315} = cos315 + isin315 = 0.7 - 0.7i\}$.

If $x^{nm} - a$ works out well, nothing can deny $x^{nm} - (-a) = x^{nm} + a$ to work out. This is the reason for the prior assumption. $x^{2nm} \pm a$ and $x^{2nm+1} \pm a$ are partitions of $x^{nm} \pm a$ under modulo 2. $x^{2nm} \pm a$ are subsequences of $x^{nm} \pm a$ that converges to the limit point of $x^n \pm a$ due to the following theorem.

Theorem 3.9. A subsequence converges to the limit point of its super sequence.

Proof. Let x^{2nm} be a subsequences of x^{nm}. Since every Cauchy Sequence Converges, $|x^{nm} - x^{2nm}| < \frac{\varepsilon}{2}, \forall n \geq N. x^n$ converges l means: Given $\varepsilon > 0, \exists N > 0$(N is no matter how large) such that $|x^{nm} - l| < \frac{\varepsilon}{2}, \forall n \geq N$. Hence, we have $|x^{nm} - x^{2nm}| < \frac{\varepsilon}{2}$ and $|x^{nm} - l| < \frac{\varepsilon}{2}, \forall n \geq N$. From Triangular Inequality,

$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} > |x^{nm} - x^{2nm}| + |x^n - l| > |x^{2nm} - x^{nm} + x^{nm} - l|, \forall n \geq N$. That is $x^{2nm} - l| < \varepsilon, \forall n \geq N$. Hence, x^{2nm} converges to l, the limit point of x^{nm}.

Theorem 3.10. If $|G| = p^2 q$, where p and q are distinct primes, then G has either a normal sylow p−subgroup or a normal sylow q−subgroup and G is not simple.

Proof. Let n_p and n_q be respectively the number of sylow p−subgroups and the number of sylow q−subgroups of G. Suppose on the contrary, $n_p > 1$ and $n_q > 1$. By sylow 3rd theorem, n_p divides q which is a prime. Hence, $n_p = q$. Also $n_p \equiv 1 \mod p$ implies $q > p$. Again, by sylow 3rd theorem, n_q divides p^2, so n_q is either p or p^2. Any element of order q in G generates a subgroup of order q which is a sylow q- subgroup of G. Any two distinct sylow q−subgroup of G of order q intersect in 1 and so there are in G, $n_q(q - 1)$ distinct elements of order q. Hence, if $n_q = p^2$, there are in G just $p^2 - q - p^2(q - 1) = p^2$ element which are not of order q. Since no element of a sylow p- subgroup of G has order q and since $|P| = p^2$. P must be a unique sylow p−subgroup of G in contradiction to $n_q > 1$. Therefore, $n_q = p$. Since, $n_q \equiv 1 \mod q, p > q$. This is a contradiction.

Theorem 3.11. If $|G| = pqr$, where p, q, r are distinct primes, then G is not simple.

Proof. Let $p > q > r$. Suppose on the contrary, G is simple. Let n_p, n_q and n_r be respectively the number of sylow p subgroups, sylow q subgroups and sylow r subgroups of G. Then $n_p > 1, n_q > 1$ and $n_r > 1$. Any two distinct sylow p−subgroup of G intersect in 1. Hence, n_p sylow p subgroup of G contains $n_p(p - 1)$ distinct element of order p. Similarly, n_q sylow q−subgroups of G contains $n_p(p - 1)$ distinct elements of order p. n_q sylow q−subgroups of G contains $n_q (q - 1)$ distinct elements of order q and n_r sylow r subgroup of G contains $n_r(r - 1)$ distinct elements of order r. Therefore,

$$|G| = pqr \geq 1 + n_p(p - 1) + n_q(q - 1) + n_r(r - 1)$$

By the 3rd sylow theorem, n_p divides qr and $n_q \equiv 1 \mod p$. Since $n_p > 1$ and $p > q > r$, it follows that $n_p = qr$. Also, n_q divides pq. So that $n_r \geq q$. Thus,

$$pqr \geq 1 + qr(p - 1) + p(q - 1) + q(r - 1)$$
$$pqr \geq 1 + pqr - qr + pq - p - qr - q \text{ or}$$
$$0 \geq 1 + pq - q - p$$
$$0 \geq$$

$(p - 1)(q - 1)$, the contradiction.

The proofs are gotten from for example (Heinstein, 1975).

CONCLUSION

There are infinitely many elements on the circle$\{a + ib: a^2 + b^2 = 1\}$, and $\{a + ib: a^2 + b^2 = 1\} \equiv |z| = 1$ as there are infinitely many elements of $(0, 1) \subseteq \mathbb{R}$. Every group, including the symmetry group, is embedded in a symmetric group and every symmetric group is constructible with at least$x^n - a$. To this end, the existence of composition series of every solvable group are the consequences of the construction of any regular shape using$x^n - a$.

Galois in 1831 first introduced group in an attempt to group some one-one functions and in an attempt to prove Abel Proposition: There is no general formula that solves the linear quintic equations (Buya, 2017)) and above. Galois proved Abel right using the group theory. The definition of group was followed by Cayley Theorem due to Cayley (1821-1895). The Cayley Theorem was followed by the present definition of group by Henrish Weber and Walter von Dyck in 1852. The aim of Henrish Weber and Walter von Dick is that any algebraic structure that endured the present-day definition of group is embeddable in the Galois Group, now mostly called symmetric group.

The solvability of the group $x^n - a$ or the factor $x^{nm} - a$ were studied by converting them to sequences as $x_n - a$ and $x_{nm} - a$ respectively to ease the steps of getting the composition series. This series is not a sum or products of the sequence transformation but an injective chain of groups or factor groups that are abelian and like train chain, each chain is linked with a subsequent one.

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