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DEVELOPMENT OF BIVARIATE INVERTED NADARAJAH HAGHIGHI DISTRIBUTIONS: COPULA APPROACH

***Najmuddeen Muhammad Sani, Hussaini Garba Dikko, Yakubu Aliyu, Tasi'u Musa and Abubakar Usman**

Department of Statistics, Ahmadu Bello University, Zaria, Nigeria.

*Corresponding authors' email: najmuddeenmsani@gmail.com

ABSTRACT

Bivariate lifetime models are crucial in reliability analysis and survival research, necessitating flexible marginal distributions and dependence structures to accurately depict real-world data. This paper introduces a family of five-parameter bivariate distributions derived from the Inverted Nadarajah–Haghighi distribution by the use of copula functions, motivated by the inadequacies of current bivariate models in representing varied dependence structures. The Farlie–Gumbel–Morgenstern (FGM) and Plackett copulas are utilized to model the dependent structure.The primary objective of this work is to develop these new bivariate models, investigate their statistical properties, and assess the efficiency of parameter estimation methods. Parameters are estimated using Maximum Likelihood Estimation (MLE) and the Inference Function for Margins (IFM) approach, and the efficiency of the two methods is compared. The results indicate that MLE provides more efficient estimation of the copula parameter for both the FGM and Plackett copulas.To illustrate the applicability of the proposed models, two real data sets are analyzed. The findings show that the Bivariate Inverted Nadarajah–Haghighi distribution based on the Plackett copula offers a better fit than the corresponding model based on the FGM copula. Further comparison with the Bivariate Generalized Exponential Distribution reveals that while the latter performs better under the FGM copula, the proposed model under the Plackett copula outperforms it, yielding lower AIC and BIC values. These results demonstrate the flexibility and practical relevance of the proposed models for analyzing dependent lifetime data.

Keywords: Inverted Nadarajah-Haghighi Distribution, Bivariate Models, Copula Function, Inference Function for Margins

INTRODUCTION

Copulas have been widely used to determine the joint distribution functions between two or more variables in various areas. Lately, areas such as flood, drought and storm event also emerged in applying copula to calculate the joint behaviour between variables (Ariff *et al* (2012), Kao and Govindaraju (2010), Renard and Lang (2007), Requena *et al* (2013)). Traditional standard bivariate model assume that the joint distribution should come from the same family of distribution. However, the model assumptions could cause defected result in the study due to the dependence structure of variables, while a copula model does not require these kinds of assumptions (Genest and Favre, 2007).

According to Nelsen (1999), a copula is a function that connects a multivariate distribution function with its univariate marginal distribution function by making use of dependence measures among correlated random variables. Univariate marginal distribution can be defined independently from the joint behaviour of the variables in the copulas. Therefore, the dependence structure of the random variables in a copula can be modeled depending on the family of the marginal distributions.

Many researchers have shown that existing statistical distributions are not the most appropriate model that adequately describes real life data such as those obtained from experimental investigations. As such, developing a new distribution that will, to some extent, address this problem is necessary. Researchers have generalized the exponential distribution in order to add flexibility to the distribution. For instance, Gupta and Kundu (1999) generalizes the exponential distribution to the exponentiated exponential distribution, Nadarajah and Haghighi (2006) to the Beta-exponential distribution, Nadarajah and Haghighi (2011) to the Nadarajah-Haghighi distribution which allows increasing, decreasing and constant hazard rate. But most of these distributions are not flexible enough to model real life data sets which exhibit decreasing and upside-down bathtub hazard rate shapes. As such, Tahir *et al* (2018) introduced the inverted Nadarajah-Haghighi distribution which is more flexible and capable of modeling real data sets that exhibit decreasing and upside-down bathtub hazard rate shapes, but this distribution still failed to address dependence between random variables and cannot model bivariate survival data. As a result, we intend to propose a new distribution called the Bivariate Inverted Nadarajah- Haghighi Distribution that could effectively modeled bivariate survival data in different situations including censored data where two lifetimes are observed for the same individual.

The univariate Inverted Nadarajah Haghighi distribution has the distribution function, probability density function and the hazard rate function given by equation (1), (2) and (3) respectively:

$$F(t;\alpha,\beta) = e^{(1-(1+\alpha t^{-1})^{\beta})} \quad (1)$$

$$f(t;\alpha,\beta) = \alpha\beta t^{-2}(1+\alpha t^{-1})^{\beta-1}e^{(1-(1+\alpha t^{-1})^{\beta})} \quad (2)$$

$$h(t;\alpha,\beta) = \alpha\beta t^{-2}(1+\alpha t^{-1})^{\beta-1}[e^{\{-(1-(1+\alpha t^{-1})^{\beta})-1\}}] \quad (3)$$

Where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter and $t > 0$

A bivariate distribution function with uniform marginal distributions is known as a copula function. Sklar (1959) coined the term "*copula*," which derives from the Latin verb *copulare*, which means "to join together."

Copula

Let X and Y be continuous random variables with bivariate distribution functions $H(x,y)$and respective marginal distribution functions $F(x)$ and $G(y)$. By performing the probability integral transformation on each variate [i.e., $U = F(X)$ and $V = G(Y)$] we obtain a new pair of variates U and

V, each with a uniform distribution on the interval $[0,1]$and whose joint distribution function, $C(u,v)$, has its mass confined to the unit square $[0,1]$. Then $C(u,v)$ is a copula function.

Copulas have been of interest to statisticians for two main reasons: firstly, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions, sometimes with a view to simulation.

Sklar Theorem

Let H be a joint distribution function with margins $F$and G. Then there exists a copula C such that for all x, in$\bar{R}$.

$$H(x,y) = C[F(x),G(y)] \quad (4)$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $RanF \times RanG$. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (4) is a joint distribution function with margins F and G

Inverted Nadarajah Haghighi Distribution

Tahir *et al.* (2018) introduce a new inverted model called the inverted Nadarajah–Haghighi distribution which exhibits decreasing and unimodal (right-skewed) density while the hazard rate shapes are decreasing and upside-down bathtub. The inverted (or inverse) distributions are sometimes very useful to explore additional properties which non-inverted distributions cannot (Tahir *et al.* (2018)).

Let $T = 1/Z$ be a random variable where Z follows Nadarajah-Haghighi Distribution, then T is said to follow Inverted Nadarajah-Haghighi Distribution denoted by $T \sim INH(\alpha,\beta)$ if the Cumulative Distribution Function (CDF), Probability Density Function (PDF) and the Hazard Rate Function (HRF) of X are respectively given by equation (5), (6) and (7)

$$F(t;\alpha,\beta) = e^{(1-(1+\alpha t^{-1})^{\beta})} \quad (5)$$

$$f(t;\alpha,\beta) = \alpha\beta t^{-2}(1+\alpha t^{-1})^{\beta-1}e^{(1-(1+\alpha t^{-1})^{\beta})} \quad (6)$$

$$h(t;\alpha,\beta) = \alpha\beta t^{-2}(1+\alpha t^{-1})^{\beta-1}[e^{\{-(1-(1+\alpha t^{-1})^{\beta})-1\}}] \quad (7)$$

Where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter

Farlie Gumbel Mogernstern Copula

The FGM family is one of the most popular parametric families of copulas discussed by Morgenstern in (1956), Gumbel in (1960) and Farlie in (1960). The expression of distribution function for FGM copula is:

$$C(u,v) = uv[1+\theta(1-u)(1-v)] \quad (8)$$

And the density function is given by:

$$c(u,v) = f(t_1)f(t_2)[1+\theta(1-2u)(1-2v)] \quad (9)$$

Where u and $v \in I$, and $\theta \in [-1,1]$ is a dependence parameter. If the dependence parameter θ equals zero, then the FGM copula corresponds the independence.

Although the FGM copula family is tractable mathematically, it does not model high dependences. The range of the dependence measures Kendall's tau τ and Spearman's rho ρ are $\tau \in [-0.222,0.222]$ and $\rho \in [-0.333,0.333]$ respectively.

Plackett Copula

It is proposed by Plackett (1965). Its distribution function is defined as:

$$C(u,v) = \frac{1+(\theta-1)(u+v)-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)} \quad (10)$$

And the density function is defined as:

$$c(u,v) = \frac{\theta[1+(u-2uv+v)(\theta-1)]}{\sqrt[3]{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}} \quad (11)$$

Where $\theta \in (0,\infty)$. The correlation measure Spearman's rho is$\rho = \frac{\theta+1}{\theta-1} - \frac{2\theta \log\theta}{(\theta-1)^2}$. There is no closed expression in θ for the correlation measure Kendall's tau.

Bivariate Inverted Nadarajah Haghighi (BIN-H) Distribution based on Farlie Gumbel Mogernstern Copula

Suppose the random variables T_1 and T_2 follow the Inverted Nadarajah Haghighi distribution each with distribution Function $F_1(t_1)$and $F_2(t_2)$ respectively, then the joint distribution function $F(t_1,t_2)$is defined as:

$$F(t_1,t_2) = C[F_1(t_1),F_2(t_2)]$$
$$=C(u,v)$$
$$= uv[1+\theta(1-u)(1-v)] \quad (12)$$

where; $u = F_1(t_1)$, $v = F_2(t_2)$ and the density function which is obtained by differentiating (12) partially with respect to $t_1$and $t_2$is given by:

$$f(t_1,t_2) = f(t_1)f(t_2)c[F_1(t_1),F_2(t_2)]$$
$$=f(t_1)f(t_2)[1+\theta(1-2u)(1-2v)] \quad (13)$$

The marginal distribution functions of the univariate Inverted Nadarajah Haghighi are:

$$u = F_1(t_1) = e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})} \quad (14)$$

$$v = F_2(t_2) = e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})} \quad (15)$$

The marginal density functions of the univariate Inverted Nadarajah Haghighi are:

$$f(t_1) = \alpha_1\beta_1 t_1^{-2}(1+\alpha_1 t_1^{-1})^{\beta_1-1}e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})} \quad (16)$$

$$f(t_2) = \alpha_2\beta_2 t_2^{-2}(1+\alpha_2 t_2^{-1})^{\beta_2-1}e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})} \quad (17)$$

Therefore, the joint CDF of the Bivariate Inverted Nadarajah Haghighi (BIN-H) distribution using the Farlie Gumbel Mogernstern Copula distribution is obtained by substituting (14) and (15) in (12) as follows:

$$F(t_1,t_2) = e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})+(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}[1+\theta(1-e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})})(1-e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})})] \quad (18)$$

While the joint density function is obtained by substituting (14) and (15) in (13) as:

$$f(t_1,t_2) = \alpha_1\beta_1 t_1^{-2}(1+\alpha_1 t_1^{-1})^{\beta_1-1}e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}.\alpha_2\beta_2 t_2^{-2}(1+\alpha_2 t_2^{-1})^{\beta_2-1}e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})} \times [1+\theta(1-2e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})})] \quad (19)$$

$$f(t_1,t_2) = [\alpha_1\beta_1 t_1^{-2}(1+\alpha_1 t_1^{-1})^{\beta_1-1}.\alpha_2\beta_2 t_2^{-2}(1+\alpha_2 t_2^{-1})^{\beta_2-1}]e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})+(1-(1+\alpha_2 t_2^{-1})^{\beta_2})} \times [1+\theta(1-2e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})})] \quad (20)$$

Re arranging (20) yields:

$$f(t_1,t_2) = [\alpha_1\beta_1 t_1^{-2}(1+\alpha_1 t_1^{-1})^{\beta_1-1}.\alpha_2\beta_2 t_2^{-2}(1+\alpha_2 t_2^{-1})^{\beta_2-1}]e^{2-(1-(1+\alpha_1 t_1^{-1})^{\beta_1})-(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}.\lambda \quad (21)$$

Where:

$$\lambda = [1+\theta(1-2e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})})]$$

$\alpha_1,\alpha_2 > 0$, $\beta_1,\beta_2 > 0$ are the scale and shape parameters respectively. $-1 < \theta < 1$ is the dependence parameter. It is important to note that, when the dependence parameter takes value zero, the model in (21) reduces to:

$$f(t_1,t_2) = [\alpha_1\beta_1 t_1^{-2}(1+\alpha_1 t_1^{-1})^{\beta_1-1}.\alpha_2\beta_2 t_2^{-2}(1+\alpha_2 t_2^{-1})^{\beta_2-1}]e^{2-(1-(1+\alpha_1 t_1^{-1})^{\beta_1})-(1-(1+\alpha_2 t_2^{-1})^{\beta_2})} \quad (22)$$

Bivariate Inverted Nadarajah Haghighi (BIN-H) Distribution based on Plackett Copula.
The second copula function used in this work is the Plackett copula proposed by Plackett (1965). Its cumulative distribution function and probability density function were given in equations (10) and (11) respectively.
Hence the cumulative distribution function of the BIN-H distribution based on Plackett Copula function is obtained by substituting (10) and (11) in (8) which yields:

$$F(t_1,t_2) = \frac{1+(\theta-1)\left(e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}+e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\right)-\sqrt{\left[1+(\theta-1)\left(e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}+e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\right)\right]^2-4e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\theta(\theta-1)}}{2(\theta-1)} \quad (23)$$

While the density function is obtained by substituting (10) and (11) in (9) which yields:

$$f(t_1,t_2) = \frac{\theta\left[1+\left(e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}-2e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}+e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\right)(\theta-1)\right]}{\sqrt{\left(\left[1+(\theta-1)\left(e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}+e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\right)\right]^2-4e^{(1-(1+\alpha_1 t_1^{-1})^{\beta_1})}e^{(1-(1+\alpha_2 t_2^{-1})^{\beta_2})}\theta(\theta-1)\right)^3}} \quad (24)$$

Where $\alpha_i, \beta_i > 0$, $i = 1,2$ are scale and shape parameters respectively and $\theta > 0$ is the copula parameter.

Parameter Estimation of the BIN-H Distribution
There are a number of methods for finding the estimates of model parameters. In this work, two different estimation methods were used to estimate the parameters of the proposed bivariate distributions. The first method is the maximum likelihood estimation (MLE) procedure and the second one is the Inference Function for Margins estimation method. These two estimation procedures will be used to estimate the parameters of both the BIN-H distributions based on the FGM and Plackett Copula functions.

Parameter Estimation of the Bivariate Inverted Nadarajah Haghighi (BIN-H) Distribution based on FGM Copula
Here, the parameters of the Bivariate Inverted Nadarajah Haghighi (BIN-H) Distribution based on FGM Copula were estimated using the methods of Maximum Likelihood Estimation and Inference Function for Margins techniques.
Parameter Estimation based on Maximum Likelihood Estimation Method
Let$(T_{11}, T_{12}), (T_{12}, T_{22}) \dots (T_{1n}, T_{2n})$ be a random sample from a bivariate distribution with vector of parameterΘ, then the likelihood function of the bivariate distribution is defined as:

$$L(\Theta) = \prod_{i=1}^{n} f(t_{1i}, t_{2i}) \quad (25)$$

Taking the natural logarithm of (7) gives the log-likelihood function of the bivariate distribution as:

$$l(\Theta) = \sum_{i=1}^{n} log(f(t_{1i}, t_{2i})) \quad (26)$$

Hence for any copula function, the log likelihood function in (15) can be written as:

$$l(\Theta) = ln\, L(\Theta) = \sum_{j=1}^{n}\left(ln\, f_1(t_{1j}) + ln\, f_2(t_{2j}) + ln\, C(F_1(t_{1j}), F_2(t_{1j}))\right) \quad (27)$$

Where $f_1(t_1)$ and $f_2(t_2)$ are the marginal probability density functions associated with the lifetimes T_1 and T_2 respectively. $F_1(t_1)$ and $F_2(t_2)$ are the marginal cumulative distribution functions associated with the life times T_1 and T_2 respectively. And $C(F_1(t_{1j}), F_2(t_{1j}))$ is a copula function.
Based on this, the log likelihood function for the BIN-H distribution based on the Farlie Gumbel Mogernstern Copula is therefore obtained by substituting equation (16) to give:

$$l(\Theta) = \sum_{j=1}^{n} ln\, \alpha_1 \beta_1 t_{1j}^{-2}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1}e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + \sum_{j=1}^{n} ln\, \alpha_2 \beta_2 t_{2j}^{-2}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1}e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + \sum_{j=1}^{n} ln[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})] \quad (28)$$

To obtain the estimates of all the parameters, we differentiate (28) with respect to each parameter separately.
Base on this, differentiating (28) with respect to α_1 we have:

$$\frac{\partial\, ln\, L}{\partial \alpha_1} = \sum_{j=1}^{n}\left(\frac{1}{\alpha_1} + \frac{(\beta_1-1)t_{1j}^{-1}}{1+\alpha_1 t_{1j}^{-1}} - \beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1} + \frac{\theta\left[\left(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right)\left(2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}.\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1}\right)\right]}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right) = 0$$

$$\frac{n}{\alpha_1} + (\beta_1-1)\sum_{j=1}^{n}\frac{t_{1j}^{-1}}{1+\alpha_1 t_{1j}^{-1}} - \beta_1\sum_{j=1}^{n} t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1} + \theta\sum_{j=1}^{n}\left(\frac{\left(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right)\left(2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}.\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1}\right)}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right) = 0 \quad (29)$$

also, differentiating equation (28) with respect to α_2, we have:

$$\frac{\partial\, ln\, L}{\partial \alpha_2} = \sum_{j=1}^{n}\left(\frac{1}{\alpha_2} + \frac{(\beta_2-1)t_{2j}^{-1}}{1+\alpha_2 t_{2j}^{-1}} - \beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1} + \frac{\theta\left[\left(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\right)\left(2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}.\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1}\right)\right]}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right) = 0$$

$$\frac{n}{\alpha_2} + (\beta_2-1)\sum_{j=1}^{n}\frac{t_{1j}^{-1}}{1+\alpha_2 t_{2j}^{-1}} - \beta_2\sum_{j=1}^{n} t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1} + \theta\sum_{j=1}^{n}\left(\frac{\left(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\right)\left(2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}.\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1}\right)}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right) = 0 \quad (30)$$

also differentiating equation (28) with respect toβ_1:

$$\frac{\partial \ln L}{\partial \beta_1}=\sum_{j=1}^{n}\left(\frac{1}{\beta_1}+\ln(1+\alpha_1 t_{1j}^{-1})-(1+\alpha_1 t_{1j}^{-1})^{\beta_1}\ln(-(1+\alpha_1 t_{1j}^{-1}))+\frac{\theta\left[(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\ln(1-(1+\alpha_1 t_{1j}^{-1}))\right]}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right)=0$$

So that,

$$\frac{n}{\beta_1}+\sum_{j=1}^{n}\ln(1+\alpha_1 t_{1j}^{-1})-\sum_{j=1}^{n}(1+\alpha_1 t_{1j}^{-1})^{\beta_1}\ln(-(1+\alpha_1 t_{1j}^{-1}))+\theta\sum_{j=1}^{n}\left(\frac{(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\ln(1-(1+\alpha_1 t_{1j}^{-1}))}{1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})}\right)=0 \quad (31)$$

Furthermore, differentiating equation (28) with respect to β_2:

$$\frac{\partial \ln L}{\partial \beta_2}=\sum_{j=1}^{n}\left(\frac{1}{\beta_2}+\ln(1+\alpha_2 t_{2j}^{-1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2}\ln(-(1+\alpha_2 t_{2j}^{-1}))+\frac{\theta\left[(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\ln(1-(1+\alpha_2 t_{2j}^{-1}))\right]}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}\right)=0$$

So that,

$$\frac{n}{\beta_2}+\sum_{j=1}^{n}\ln(1+\alpha_2 t_{2j}^{-1})-\sum_{j=1}^{n}(1+\alpha_2 t_{2j}^{-1})^{\beta_2}\ln(-(1+\alpha_2 t_{2j}^{-1}))+\theta\sum_{j=1}^{n}\left(\frac{(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\ln(1-(1+\alpha_2 t_{2j}^{-1}))}{1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})}\right)=0 \quad (32)$$

and finally, differentiating equation (28) with respect to the copula parameter gives:

$$\frac{\partial \ln L}{\partial \theta}=\frac{(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})}{[1+\theta(1-2e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})})(1-2e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})]}=0 \quad (33)$$

There exist no analytical solutions for the estimates of the parameters; as such they are handled numerically through statistical software.

Parameter Estimation based on Inference Function for Margins

This is a parametric method with two-step of estimation. We start by maximizing the log-likelihood function of each marginal density functions $f_1(t_1)$and $f_2(t_2)$ to obtain the estimates of the marginal distribution functions $\hat{F}_1(t)$and$\hat{F}_2(t)$.

The second step is estimating the copula parameter by maximizing the log-likelihood function of the copula density using the Maximum Likelihood estimates of the marginal $\hat{F}_1(t)$and $\hat{F}_2(t)$from first step. This is done as follows:

$\hat{\theta}=argmax\sum_{j=1}^{n}\ln c(\tilde{F}_1(t_{1j}),\tilde{F}_2(t_{2j}))$

Based on this method, consider the log likelihood of the marginal distributions of the BIN-H distribution as follows:

$$\ln L_{T_1}=\sum_{j=1}^{n}\ln f_1(t_{1j}) \quad (34)$$

$$\ln L_{T_2}=\sum_{j=1}^{n}\ln f_2(t_{2j}) \quad (35)$$

For a BIN-H distribution based on FGM copula, the parameters of each marginal distribution will be estimated separately using MLE. Thus (29) and (30) becomes:

$$\ln L_{T_i}(\alpha_i,\beta_i)=\sum_{j=1}^{n}\ln f_i(t_{ij}) \qquad i=1,2 \quad (36)$$

Substituting for $f_i(t_{ij})$, we have:

$$\ln L_{T_i}(\alpha_i,\beta_i)=\sum_{j=1}^{n}\ln\left[\alpha_i\beta_i t_{ij}^{-2}(1+\alpha_i t_{ij}^{-1})^{\beta_i-1}e^{(1-(1+\alpha_i t_{ij}^{-1})^{\beta_i})}\right] \quad (37)$$

$$\ln L_{T_i}(\alpha_i,\beta_i)=\sum_{j=1}^{n}\left[\ln\alpha_i\beta_i+\ln t_{ij}^{-2}+(\beta_i-1)\ln(1+\alpha_i t_{ij}^{-1})+(1-(1+\alpha_i t_{ij}^{-1})^{\beta_i})\right] \quad (38)$$

$$\ln L_{T_i}(\alpha_i,\beta_i)=n\ln\alpha_i\beta_i+\sum_{j=1}^{n}\ln t_{ij}^{-2}+(\beta_i-1)\sum_{j=1}^{n}\ln(1+\alpha_i t_{ij}^{-1})+\sum_{j=1}^{n}(1-(1+\alpha_i t_{ij}^{-1})^{\beta_i}) \quad (39)$$

Differentiating equation (39) with respect to α_i we have:

$$\frac{\partial \ln L_{T_i}}{\partial \alpha_i}=\frac{n}{\alpha_i}+(\beta_i-1)\sum_{j=1}^{n}\frac{t_{ij}^{-1}}{(1+\alpha_i t_{ij}^{-1})}+\beta_i t_{ij}^{-1}\sum_{j=1}^{n}(1+\alpha_i t_{ij}^{-1})^{(\beta_i-1)}=0 \quad (40)$$

The fixed point solution of (40) will provide the MLE of α_i, say $\hat{\alpha}_i$.

Also differentiating equation (39) with respect to β_i we have:

$$\frac{\partial \ln L_{T_i}}{\partial \beta_i}=\frac{n}{\beta_i}+\sum_{j=1}^{n}\ln(1+\alpha_i t_{ij}^{-1})-\sum_{j=1}^{n}(1+\alpha_i t_{ij}^{-1})^{\beta_i}\ln(1+\alpha_i t_{ij}^{-1})=0 \quad (41)$$

The fixed point solution of (41) will provide the MLE of β_i, say $\hat{\beta}_i$.

The second step is estimating the copula density using the marginal estimates $\hat{F}_1(t_1)$and $\hat{F}_2(t_2)$from the first step as follows:

$$\ln L_c=\sum_{j=1}^{n}\ln c(\hat{F}_1(t_{1j}),\hat{F}_2(t_{2j})) \quad (42)$$

$$\ln L_\theta=\sum_{j=1}^{n}\ln\left[1+\theta(1-2\hat{F}_1(t_{1j}))(1-2\hat{F}_2(t_{2j}))\right] \quad (43)$$

$$\ln L_\theta=\sum_{j=1}^{n}\ln\left[1+\theta(1-2e^{(1-(1+\hat{\alpha}_1 t_{1j}^{-1})^{\hat{\beta}_1})})(1-2e^{(1-(1+\hat{\alpha}_2 t_{2j}^{-1})^{\hat{\beta}_2})})\right] \quad (44)$$

Therefore, taking the derivative of (39) partially with respect to the copula parameter, we have:

$$\frac{\partial \ln L_c}{\partial \theta}=\sum_{j=1}^{n}\frac{(1-2e^{(1-(1+\hat{\alpha}_1 t_{1j}^{-1})^{\hat{\beta}_1})})(1-2e^{(1-(1+\hat{\alpha}_2 t_{2j}^{-1})^{\hat{\beta}_2})}))}{1+\theta\left((1-2e^{(1-(1+\hat{\alpha}_1 t_{1j}^{-1})^{\hat{\beta}_1})}))(1-2e^{(1-(1+\hat{\alpha}_2 t_{2j}^{-1})^{\hat{\beta}_2})})\right)}=0 \quad (45)$$

There exist no analytical solutions for the estimates of the parameters; as such they are handled numerically through statistical software.

Parameter Estimation of the Bivariate Inverted Nadarajah Haghighi (BINH) Distribution based on Plackett Copula Function

Here, the parameters of the Bivariate Inverted Nadarajah Haghighi (BIN-H) Distribution based on Plackett Copula are also estimated using the Maximum Likelihood Estimation and Inference Function for Margins methods.

Parameter Estimation based on Maximum Likelihood Estimation Method

The log likelihood function of a BIN-H distribution based on the Plackett Copula is defined as:

$$ln\, L = \sum_{j=1}^{n}\left(ln\, f_1\,(t_{1j}) + ln\, f_2\,(t_{2j}) + ln\, C\,(F_1(t_{1j}), F_2(t_{2j}))\right) \quad (46)$$

$$ln\, L = \sum_{j=1}^{n}\Big(ln\, \alpha_1\, \beta_1 t_{1j}^{-2}(1+\alpha_1 t_{1j}^{-1})^{\beta_1-1} e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + ln\, \alpha_2\, \beta_2 t_{2j}^{-2}(1+\alpha_2 t_{2j}^{-1})^{\beta_2-1} e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + ln\left(\frac{\theta\left[1+(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right]}{\left(\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right)^{\frac{3}{2}}}\right)\Big)$$

by further simplifying, we have:

$$ln\, L = \sum_{j=1}^{n}\Big(ln\, \alpha_1\, \beta_1 t_{1j}^{-2} + (\beta_1 - 1)\, ln(1 + \alpha_1 t_{1j}^{-1}) + (1 - (1+\alpha_1 t_{1j}^{-1})^{\beta_1}) + ln\, \alpha_2\, \beta_2 t_{2j}^{-2} + (\beta_2 - 1)\, ln(1 + \alpha_2 t_{2j}^{-1}) + (1 - (1+\alpha_2 t_{2j}^{-1})^{\beta_2}) + ln\, \theta\left[1 + (e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right] - \frac{3}{2} ln\left(\left[1 + (\theta - 1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta - 1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right) \quad (47)$$

Maximizing the log-likelihood function in (47) overα_1, we have:

$$\frac{\partial\, ln\, L}{\partial \alpha_1} = \sum_{j=1}^{n}\Big(\frac{1}{\alpha_1} + \frac{(\beta_1-1)t_{1j}^{-1}}{(1+\alpha_1 t_{1j}^{-1})} + \beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{(\beta_1-1)} + \frac{\theta(\theta-1)\left[-\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{(\beta_1-1)}e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + 2\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{(\beta_1-1)}e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1}-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right]}{\theta\left[1+(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right]} - \frac{3}{2}\left[\frac{2\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right](1-\theta)\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{(\beta_1-1)}e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + 4\theta(1-\theta)\beta_1 t_{1j}^{-1}(1+\alpha_1 t_{1j}^{-1})^{(\beta_1-1)}e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1}-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}{\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}\right] = 0 \quad (48)$$

Also, maximizing the log-likelihood function in (47) overα_2, we have:

$$\frac{\partial\, ln\, L}{\partial \alpha_2} = \sum_{j=1}^{n}\Big(\frac{1}{\alpha_2} + \frac{(\beta_2-1)t_{2j}^{-1}}{(1+\alpha_2 t_{2j}^{-1})} + \beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{(\beta_2-1)} + \frac{\theta(\theta-1)\left[-\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{(\beta_2-1)}e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + 2\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{(\beta_2-1)}e^{(2-(1+\alpha_2 t_{2j}^{-1})^{\beta_2}-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\right]}{\theta\left[1+(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right]} - \frac{3}{2}\left[\frac{2\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right](1-\theta)\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{(\beta_2-1)}e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + 4\theta(1-\theta)\beta_2 t_{2j}^{-1}(1+\alpha_2 t_{2j}^{-1})^{(\beta_2-1)}e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1}-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}{\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}\right] = 0 \quad (49)$$

Furthermore maximizing the log-likelihood function in (47) overβ_1, we have:

$$\frac{\partial\, ln\, L}{\partial \beta_1} = \sum_{j=1}^{n}\Big[\frac{1}{\beta_1} + ln(1 + \alpha_1 t_{1j}^{-1}) - (1+\alpha_1 t_{1j}^{-1})^{\beta_1}\, ln(-(1+\alpha_1 t_{1j}^{-1})) + \frac{\theta\left[-(1+\alpha_1 t_{1j}^{-1})^{\beta_1}\, ln(-(1+\alpha_1 t_{1j}^{-1}))e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2(1+\alpha_1 t_{1j}^{-1})^{\beta_1}\, ln(-(1+\alpha_1 t_{1j}^{-1}))e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right](\theta-1)}{\theta\left[1 + (e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right]} - \frac{3}{2}\left[\frac{2\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right](\theta-1)(-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})}\, ln(-(1+\alpha_1 t_{1j}^{-1})) - 4\theta(\theta-1)(-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})\, ln(-(1+\alpha_1 t_{1j}^{-1}))e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}{\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}\right] = 0 \quad (50)$$

Also maximizing the log-likelihood function in (47) overβ_2, we have:

$$\frac{\partial\, ln\, L}{\partial \beta_2} = \sum_{j=1}^{n}\Big[\frac{1}{\beta_2} + ln(1 + \alpha_2 t_{2j}^{-1}) - (1+\alpha_2 t_{2j}^{-1})^{\beta_2}\, ln(-(1+\alpha_2 t_{2j}^{-1})) + \frac{\theta\left[-(1+\alpha_2 t_{2j}^{-1})^{\beta_2}\, ln(-(1+\alpha_2 t_{2j}^{-1}))e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} - 2(1+\alpha_2 t_{2j}^{-1})^{\beta_2}\, ln(-(1+\alpha_2 t_{2j}^{-1}))e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\right](\theta-1)}{\theta\left[1+(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})(\theta-1)\right]} - \frac{3}{2}\left[\frac{2\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right](\theta-1)(-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}\, ln(-(1+\alpha_2 t_{2j}^{-1})) - 4\theta(\theta-1)(-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})\, ln(-(1+\alpha_2 t_{2j}^{-1}))e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}{\left[1+(\theta-1)(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}}\right] = 0 \quad (51)$$

Finally, maximizing the log-likelihood function in (47) over the copula parameter θwe have:

$$\frac{\partial\, ln\, L}{\partial \theta} = \frac{[(1+A(\theta-1))+\theta A]}{\theta[1+A(\theta-1)]} - \frac{3}{2}\left[\frac{[2C(e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}) - 4(2\theta-1)e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})}]}{B}\right] = 0 \quad (52)$$

Where;

$$A = (e^{(1-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})} - 2e^{(2-(1+\alpha_1 t_{1j}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})} + e^{(1-(1+\alpha_2 t_{2j}^{-1})^{\beta_2})})$$

$B = \left[1 + (\theta - 1)(e^{(1-(1+\alpha_1 t_{1j}{}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}{}^{-1})^{\beta_2})})\right]^2 - 4\theta(\theta - 1)e^{(2-(1+\alpha_1 t_{1j}{}^{-1})^{\beta_1})-(1+\alpha_2 t_{2j}{}^{-1})^{\beta_2})}$

$C = 1 + (\theta - 1)(e^{(1-(1+\alpha_1 t_{1j}{}^{-1})^{\beta_1})} + e^{(1-(1+\alpha_2 t_{2j}{}^{-1})^{\beta_2})})$

There exist no analytical solutions for the estimates of the parameters; as such they are handled numerically through statistical software.

Parameter Estimation based on Inference Function for Margins

Following the same procedure as in the previous section and using the Maximum Likelihood estimates of the marginal distribution functions obtained in equations (34) and (35), the copula parameter is estimated as follows:

$ln\, L_c = \sum_{j=1}^{n} ln\, c\,(\hat{F}_1(t_{1j}), \hat{F}_2(t_{2j}))$

$$ln\, L_\theta = \sum_{j=1}^{n} ln \left[\frac{\theta\left[1+\left(e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})} - 2e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})}e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})} + e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\right)(\theta-1)\right]}{\sqrt{\left(\left[1+(\theta-1)\left(e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})} + e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\right)\right]^2 - 4e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})}e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\theta(\theta-1)\right)^3}} \right] \quad (53)$$

Simplifying further we have:

$$ln\, L_\theta = \sum_{j=1}^{n} \left[ln\,\theta + ln[\,1 + \left(e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})} - 2e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})}e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})} + e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\right)(\theta - 1)]\right]$$

$$-\frac{3}{2} ln[\left[1 + (\theta - 1)\left(e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})} + e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\right)\right]^2 - 4e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})}e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}\theta(\theta - 1)] \quad (54)$$

Differentiating the log likelihood function in equation (3.53) partially with respect to the copula parameter we obtained:

$$\frac{\partial\, ln\, L}{\partial\theta} = \frac{[(1+\hat{A}(\theta-1))+\theta\hat{A}]}{\theta[1+\hat{A}(\theta-1)]} - \frac{3}{2}\left[\frac{[2\hat{C}(e^{(1-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})} + e^{(1-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}) - 4(2\theta-1)e^{(2-(1+\hat{\alpha}_1 t_1{}^{-1})^{\hat{\beta}_1})-(1+\hat{\alpha}_2 t_2{}^{-1})^{\hat{\beta}_2})}]}{\hat{B}}\right] \quad (55)$$

Where;

$\hat{A} = (e^{(1-(1+\hat{\alpha}_1 t_{1j}{}^{-1})^{\hat{\beta}_1})} - 2e^{(2-(1+\hat{\alpha}_1 t_{1j}{}^{-1})^{\hat{\beta}_1})-(1+\hat{\alpha}_2 t_{2j}{}^{-1})^{\hat{\beta}_2})} + e^{(1-(1+\hat{\alpha}_2 t_{2j}{}^{-1})^{\hat{\beta}_2})})$

$\hat{B} = \left[1 + (\theta - 1)(e^{(1-(1+\hat{\alpha}_1 t_{1j}{}^{-1})^{\hat{\beta}_1})} + e^{(1-(1+\hat{\alpha}_2 t_{2j}{}^{-1})^{\hat{\beta}_2})})\right]^2 - 4\theta(\theta - 1)e^{(2-(1+\hat{\alpha}_1 t_{1j}{}^{-1})^{\hat{\beta}_1})-(1+\hat{\alpha}_2 t_{2j}{}^{-1})^{\hat{\beta}_2})}$

$\hat{C} = 1 + (\theta - 1)(e^{(1-(1+\hat{\alpha}_1 t_{1j}{}^{-1})^{\hat{\beta}_1})} + e^{(1-(1+\hat{\alpha}_2 t_{2j}{}^{-1})^{\hat{\beta}_2})})$

RESULTS AND DISCUSSION

Two real data sets were analyzed in order to demonstrate the applicability of the proposed models. The first data set is the infections in kidney patients' data and the second data is the soccer data. Details about each of the data set were given in the next sections.

Infections in kidney patients

The first data set is the infections in kidney patients' data from McGilchrist and Aisbett (1991) which was previously analyzed by Achcar et al. (2015), Mirhosseini et al. (2015), Elaal and Jarwan (2017), Usman and Aliyu (2022) and Aliyu and Usman (2023). The recurrence times to infection at point of insertion of catheter using portable dialysis equipment for thirty-eight (38) kidney patients were recorded. For each patient, two such recurrence times were given with each row of the table corresponding to one patient as shown in table 1.

Table 1: Infections in Kidney

Patients	T_1	T_2
1	8	16
2	23	13
3	22	28
4	447	318
5	30	12
6	24	245
7	7	9
8	511	30
9	53	196
10	15	154
11	7	333
12	141	8
13	96	38
14	149	70
15	536	25
16	17	4
17	185	117
18	292	114
19	22	159

Patients	T_1	T_2
20	15	108
21	152	362
22	402	24
23	13	66
24	39	46
25	12	40
26	113	201
27	132	156
28	34	30
29	2	25
30	130	26
31	27	58
32	5	43
33	152	30
34	190	5
35	119	8
36	54	16
37	6	78
38	63	8

Table 2: Copula goodness-of-fit test, AIC, BIC and HQIC results for the proposed models - Kidney data

Copula	**p-value**	**Dependence parameter**	**AIC**	**BIC**	**HQIC**
FGM Copula	0.8950	-0.0412	864.4672	8262.3661	867.3804
Plackett Copula	0.0055	0.9876	852.8942	834.9953	855.8074

The resulting p-values of copula goodness-of-fit test were 0.8950 and 0.0055 for FGM and Plackett copula respectively, which confirms that only the FGM copula, having a p-value greater than 0.05, is suitable for the data set. while the resulting p-value of the Plackett copula shows that the Plackett copula, having a p-value less than 0.05, is not suitable for the data set. The estimated dependence parameter was -0.0412 for FGM copula and 0.9876 for Plackett copula. The results of AIC were 864.4672 for FGM and 852.8942 for Plackett. The results of BIC were 8262.3661 for FGM copula and 834.9953 for Plackett copula. The results of HQIC were 867.3804 for FGM copula and 855.8074 for Plackett copula. From the goodness of fit test, the AIC, BIC and HQIC results, it shows that the Plackett copula is fitting better than the FGM copula.

Table 3: The Estimates and the Corresponding Standard Errors (in Brackets) of Parameters of BIN-H Distribution Based on FGM and Plackett Copulas for the Kidney Data

		Estimates of Parameters				
Copula	**Estimation Methods**	α_1	α_2	β_1	β_2	θ
FGM	MLE	23.41273	12.64290	0.82368	1.31959	-0.02562
		(6.67406)	(6.11100)	(0.13919)	(0.41741)	(0.40877)
	IFM	0.83268	0.74833	22.68826	35.52005	-0.04121
		(0.09083)	(0.09289)	(2.91337)	(5.48186)	(0.31342)
Plackett	MLE	8.51825	1.65996	1.17124	0.65294	0.80094
		(2.58807)	(0.87012)	(0.27751)	(0.30901)	(0.02789)
	IFM	1.0089	1.0948	15.4209	16.9538	0.9876
		(0.1426)	(0.1475)	(3.5637)	(3.2701)	(0.3557)

From the result in table 3, the parameter with the least standard error is considered the best. Therefore, the efficient estimators of marginal parameters of the two models differ according to the parameters. It is also observed that the IFM estimates of scale parameters α_1 and α_2 of the two models under the FGM and Plackett copula functions are better than the corresponding ML estimates. Whereas, for the shape parameters $\beta_1$andβ_2, it is observed that the ML estimates are better than the corresponding IFM estimates in each of the two models. For copula parameterθ, the ML estimation method provided more efficient estimates compared to the IFM estimation method for the Plackett copula function. Whereas, for the FGM copula function, the IFM estimation provided an efficient estimate of the copula parameter than the ML estimation method.

Soccer Data

The second data set is the football (soccer) data from Meintanis (2007). Consider matches where (i) there was at least one goal scored by the home team and (ii) also at least one goal scored directly from a kick (foul kick, penalty kick or other kick) by any team. Let T_1 be the time (in minutes) of the first kick goal scored by any team, and T_2 be the time (in minutes) of the first goal of any type scored by the home team. Apparently, with this kind of nonnegative continuous data, all possibilities are open: for each match we may have$T_1 < T_2$,$T_1 > T_2$, and even $T_1 = T_2$. Table 4.4 presents such data for the group stage of the UEFA Champion's League for the years 2004–05 and 2005–2006.

Table 4: Soccer Data

2005-2006	X_1	X_2	2004-2005	X_1	X_2
Lyon-Real Madrid 3-0	26	20	Internazionale–Bremen 2–0	34	34
Milan-Fernerbahce 3-1	63	18	Real Madrid–Roma 4–2	53	39
Chelsea–Anderlecht 1–0	19	19	Man. United–Fenerbahce 6–2	54	7
Club Brugge–Juventus 1–2	66	85	Bayern–Ajax 4–0	51	28
Fenerbahce–PSV 3–0	40	40	Moscow–PSG 2–0	76	64
Internazionale–Rangers 1–0	49	49	Barcelona–Shakhtar 3–0	64	15
Panathinaikos–Bremen 2–1	8	8	Leverkusen–Roma 3–1	26	48
Ajax–Arsenal 1–2	69	71	Arsenal–Panathinaikos 1–1	16	16
Man. United–Benfica 2–1	39	39	Dynamo Kyiv–Real Madrid 2–2	44	13
Real Madrid–Rosenborg 4–1	82	48	Man. United–Sparta 4–1	25	14
Villarreal–Benfica 1–1	72	72	Bayern–M. Tel–Aviv 5–1	55	11
Juventus–Bayern 2–1	66	62	Bremen–Internazionale 1–1	49	49
Club Brugge–Rapid 3–2	25	9	Anderlecht–Valencia 1–2	24	24
Olympiacos–Lyon 1–4	41	3	Panathinaikos–PSV 4–1	44	30
Internazionale–Porto 2–1	16	75	Arsenal–Rosenborg 5–1	42	3
Schalke–PSV 3–0	18	18	Liverpool–Olympiacos 3–1	27	47
Barcelona–Bremen 3–1	22	14	M. Tel–Aviv–Juventus 1–1	28	28
Milan–Schalke 3–2	42	42			
Bremen–Panathinaikos 5–1	2	2			
Rapid–Juventus 1–3	36	52			

Table 5: Copula Goodness-Of-Fit Test, AIC, BIC and HQIC Results for The Proposed Models-Soccer Data

Copula	p-value	Dependence parameter	AIC	BIC	HQIC
FGM Copula	0.0002	0.2375	720.2906	717.5124	723.1302
Plackett Copula	0.0000	2.1191	269.2890	277.3436	272.1286

The resulting p-values of copula goodness-of-fit test were 0.0002 and 0.0000 for FGM and Plackett copula respectively, which confirms that both the FGM and Plackett copula functions are not suitable for the data set. The estimated dependence parameter was 0.2375 for FGM copula and 2.1191 for Plackett copula. The results of AIC were 720.2906 for FGM, 269.2890 for Plackett. The results of BIC were 717.5124 for FGM copula and 277.3436 for Plackett copula. The results of HQIC were 723.1302 for FGM copula and 272.1286 for Plackett copula. From the goodness of fit test, the AIC, BIC and HQIC results, it shows that the Plackett copula is fitting better than the FGM copula.

Table 6: The Estimates and the Corresponding Standard Errors (in Brackets) Of Parameters of BIN-H Distribution Based on FGM and Plackett Copulas for the Soccer Data Set

		Estimates of Parameters				
Copula	**Estimation Methods**	α_1	α_2	β_1	β_2	θ
FGM	MLE	16.3028 (2.6310)	11.7353 (4.9963)	1.0608 (0.1394)	0.9462 (0.2573)	0.2375 (0.6286)
	IFM	0.9222 (0.1139)	0.8847 (0.1100)	22.9934 (4.8103)	16.0332 (2.8657)	0.1355 (0.4414)
Plackett	MLE	26.9284 (6.3369)	0.4753 (0.1851)	1.3725 (0.1502)	2.0200 (0.7524)	2.1191 (0.2003)
	IFM	0.9719 (0.1129)	0.8399 (0.1502)	20.9531 (3.9978)	18.2647 (5.8415)	9.245 (3.425)

It is observed that the efficient estimators of marginal parameters of two models differ according to the parameters. It is also observed that the IFM estimates of scale parameters α_1 and α_2 of the two models under the FGM and Plackett copula functions, having the lower standard error; are better than the corresponding ML estimates. Whereas, for the shape parameters $\beta_1$andβ_2, it is observed that the ML estimates, having the lower standard error; are better than the corresponding IFM estimates in each of the two models. For copula parameterθ, the ML estimation method provided more efficient estimates compared to the IFM estimation method for the Plackett copula function. Whereas, for the FGM copula function, the IFM estimation provided an efficient estimate of the copula parameter than the ML estimation method.

Comparison between the performance of BIN-H and BGED

Here, the performance of the BIN-H distribution and that of the BGED is compared using both the kidney and soccer data set.

Table 7: Comparison Between the Performance of BIN-H and BGED Using the Kidney Data Set

Copula functions	Distributions	P-value	AIC	BIC
FGM	BIN-H	0.8950	864.4672	8262.3661
	BGED	0.7338	689.0881	696.0940
Plackett	BINH	0.0055	852.8942	834.9953
	BGED	0.7877	689.0495	696.0555

From Table 7, it can be seen that, based on the AIC and BIC values obtained, the BGED performed better than the BINH distribution in both the FGM and Plackett copula functions.

Table 8: Comparison Between the Performance of BINH and BGED Using the Soccer Data Set

Copula functions	Distributions	P-value	AIC	BIC
FGM	BIN-H	0.0002	720.2906	728.3452
	BGED	0.456	672.1324	680.1870
Plackett	BIN-H	0.0000	269.2890	277.3436
	BGED	0.0000	545.426	553.4806

From Table 8, under the Farlie Gumbel Morgernstern Copula, the BGED outperform the BIN-H distribution based on the AIC and BIC values. Whereas, for the Plackett copula function, the BIN-H distribution outperform the BGED having the lower AIC and BIC values.

CONCLUSION

This study was motivated by the need for flexible bivariate lifetime distribution capable of accommodating diverse dependence structures and marginal behaviors. Accordingly, a class of Bivariate Inverted Nadarajah–Haghighi (BIN-H) distributions was developed using the Farlie–Gumbel–Morgenstern and Plackett copula functions. The primary objective was to construct these models, estimate their parameters efficiently, and assess their performance using real data applications. Two parameter estimation methods, namely Maximum Likelihood Estimation (MLE) and Inference Functions for Margins (IFM), were employed and compared. The results indicate that the efficiency of the estimation methods for both marginal and copula parameters depends on the nature of the data and the underlying parameter values. However, based on the real data analyses considered in this study, MLE consistently provided more efficient estimates for the copula parameters under both the FGM and Plackett copulas. Furthermore, empirical results demonstrated that the BIN-H distribution based on the Plackett copula achieved a superior fit compared to its FGM-based counterpart. These models can be valuable in reliability analysis, survival studies, and related applied fields. Despite these contributions, the study is limited by the use of only two copula families and a restricted number of real data sets. Future research may extend this work by considering alternative copula functions, incorporating censoring mechanisms, exploring Bayesian estimation approaches, and applying the proposed models to a wider range of real-world data.

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