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DERIVATION OF THE SINE EXPONENTIATED LOMAX DISTRIBUTION FOR MODELLING RIGHT SKEWED AND HEAVY TAILED DATA

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ABSTRACT
This study introduces the Sine-Exponentiated Lomax distribution, a three-parameter model designed for modeling heavy-tailed and right-skewed data. The probability density function, cumulative distribution function, and key mathematical properties including moments, quantile function, and entropy measures were derived. Parameters were estimated using maximum likelihood estimation, with simulation studies confirming estimator consistency and asymptotic normality across sample sizes from 50 to 2000 observations. The model's practical utility was demonstrated through four real-world applications: S&P 500 returns (finance), earthquake damage magnitudes (seismology), cancer remission times (biostatistics), and geyser eruption intervals (environmental science). In all cases, the S-EL distribution outperformed established models including the exponentiated Lomax, Weibull, and Burr distributions based on AIC, BIC, and other goodness-of-fit criteria. The distribution provides researchers with a robust, flexible tool for extreme-value modeling while maintaining mathematical coherence and computational practicality.

Keywords: Distribution, Heavy-tailed, Right-skewed, Sine-G & Sine-Exponentiated Lomax

INTRODUCTION

Statistical distributions form the fundamental framework for quantitative analysis across numerous disciplines, including finance, engineering, environmental science, and biomedical research. These mathematical functions describe the likelihood of different outcomes for random variables, enabling researchers to model real-world phenomena, understand data variability, and quantify uncertainty (Casella & Berger, 2021). The selection of an appropriate probability model represents a critical step in statistical analysis, as an ill-fitting distribution can lead to biased estimates, inaccurate predictions, and ultimately, erroneous conclusions. While classical models like the normal and exponential distributions remain foundational due to their simplicity, the increasing complexity of modern datasets has revealed their limitations, spurring the development of more flexible probability distributions capable of capturing complex data patterns.

The literature demonstrates significant advancements in distribution theory, particularly through various transformation methods. Kumar et al. (2015) introduced the sine-G family, applying trigonometric sine transformations to baseline cumulative distribution functions to create more flexible models. Subsequent research has expanded this approach, with Al-Babtain et al. (2020) developing the sine Lindley distribution and Haj Ahmad and Almetwally (2020) creating the sine Weibull-G family. Alternative methodologies include the odd Fréchet-G family (Haq & Elgarhy, 2018), exponentiated generalized class (Cordeiro et al., 2013), and Marshall-Olkin odd Lindley-G family (Bhatti et al., 2023). For heavy-tailed data specifically, researchers have developed various Lomax-based distributions, including the exponentiated Lomax (Adul-Moniem & Abdel-Hamed, 2012), Type II exponentiated half-logistic Lomax (Alshenawy et al., 2023), and Type II power Lomax (Ilic et al., 2023). Applications of these advanced distributions span multiple fields, including finance (Nadarajah & Zhang, 2023), environmental science (Haq & Yousof, 2022), insurance (Anjarwish et al., 2021), and public health (Tassaddiq et al., 2023; Reyad et al., 2022).

Despite these advancements, significant research gaps remain. Many existing heavy-tailed distributions either lack sufficient flexibility to capture the full spectrum of real-world data characteristics or suffer from excessive complexity that limits their practical application. The exponentiated Lomax distribution, while valuable for heavy-tailed phenomena, demonstrates limitations in modeling non-monotonic hazard rates and exhibits restricted capacity for capturing varying degrees of skewness and extreme tail behavior found in modern datasets. Furthermore, while trigonometric transformations have shown promise in enhancing distribution flexibility, their application to robust heavy-tailed baseline distributions remains underexplored. This gap is particularly notable given the demonstrated effectiveness of sine transformations in creating distributions with enhanced skewness and tail behavior (Kumar et al., 2015), yet most applications have focused on simpler baseline distributions rather than already flexible heavy-tailed models like the exponentiated Lomax.

This study aims to address these limitations by deriving the sine exponentiated Lomax (S-EL) distribution, which integrates the exponentiated Lomax baseline within the sine-G family framework. The specific objectives are: (i) to derive the mathematical properties of the proposed distribution, including its probability density function, cumulative distribution function, hazard function, moments, and quantile function; (ii) to estimate the parameters of the proposed distribution using the method of maximum likelihood; (iii) to examine the consistency of the estimated parameters through Monte Carlo simulation studies; and (iv) to demonstrate the practical utility of the proposed distribution through applications to real-life datasets from various domains. The proposed synthesis synergizes the heavy-tailed foundation of the Lomax distribution, the enhanced shape flexibility from the exponentiation parameter, and the exceptional adaptability of the sine transformation, potentially yielding superior capability for capturing nuanced characteristics of highly skewed and heavy-tailed data.

MATERIALS AND METHODS

Sine-G Family of Distribution

Kumar et al. (2015) defined the CDF and pdf of sine-G family of distribution as:

$$F_s(y;\xi)=\sin\left[\frac{\pi}{2}G(y;\xi)\right],\ y\in R \quad (1)$$

$$f_s(y;\xi)=\frac{\pi}{2}g(y;\xi)\cos\left[\frac{\pi}{2}G(y;\xi)\right],\ y\in R \quad (2)$$

Where $G(y;\xi)$ and $g(y;\xi)$ are the CDF and PDF of the baseline distribution with parameter vectors represented by ξ.

Exponentiated Lomax Distribution

Adul-Moniem and Abdel-Hamed (2012) defined the cumulative distribution function (CDF) and probability density function (PDF) of an exponentiated Lomax Distribution (ELD) respectively as:

$$G(y)=[1-(1+\lambda y)^{-\theta}]^{\alpha},\ y>0,\ \alpha,\theta \text{ and } \lambda>0 \quad (3)$$

$$g(y)=\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}(1+\lambda y)^{-(\theta+1)},\ y>0,\ \alpha,\theta \text{ and } \lambda>0 \quad (4)$$

where α, θ and λ are the power, shape and scale parameters respectively.

Proposed Sine-Exponentiated Lomax (S-EL) Distribution

The CDF and PDF of the proposed S-EL Distribution were obtained by inserting (3) and (4) in (1) and (2) and the results were presented in (5) and (6).

$$F_s(y;\alpha,\theta,\lambda)=\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right],\quad y\in R \quad (5)$$

$$f_s(y;\xi)=\frac{\pi}{2}\{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}(1+\lambda y)^{-(\theta+1)}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right],\ y\in R$$

$$f_s(y;\alpha,\theta,\lambda)=\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right],\ y\in R \quad (6)$$

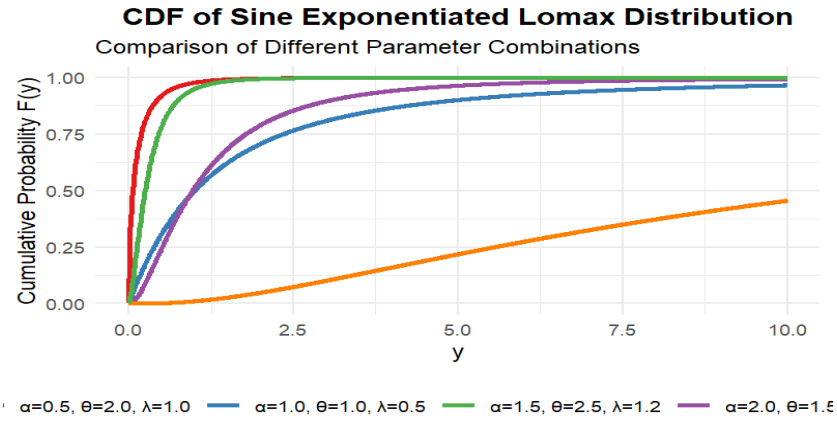

Figure 1: CDF Plot of S-EL Distribution with Different Parameter Combination

Based on the CDF plot comparing different parameter combinations for the S-EL distribution (Figure 1), the curves demonstrate how varying the power parameter (α), shape parameter (θ), and scale parameter (λ) collectively influence the model's behavior. The combination α=2.0, θ=1.5, λ=0.8 produces a CDF curve that rises steeply, indicating rapid probability accumulation and a concentration of mass at lower values, characteristic of a higher power parameter working in concert with a moderate shape parameter and a scale parameter that restricts dispersion. This curve can be compared directly with others, such as α=0.5, θ=2.0, λ=1.0, which would show a much slower ascent and heavier tails, highlighting how increased α and decreased λ both contribute to a tighter, more left-concentrated distribution while θ moderates the tail behavior.

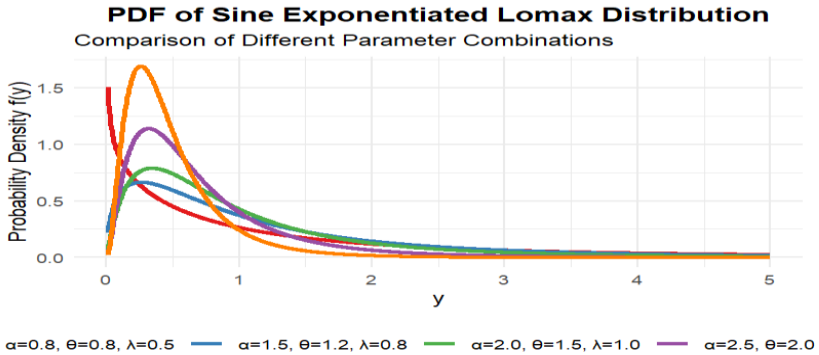

Figure 2: PDF Plot of S-EL Distribution with Different Parameter Combination

Based on the probability density function plot comparing different parameter combinations for the S-EL distribution (figure 4.5), the curves illustrate how varying the power parameter (α), shape parameter (θ), and scale parameter (λ) influence the model's density shape and tail behavior. The curve for α=2.5, θ=2.0, λ=1.2 shows a density that is more peaked and shifted slightly rightward compared to other combinations, indicating a tighter concentration of probability

mass with moderate scaling. In contrast, combinations with lower parameter values like $\alpha=0.8$, $\theta=0.8$, $\lambda=0.5$ produce a flatter, more spread-out density with a heavier tail. The progression of curves demonstrates that increasing the power and shape parameters generally leads to a higher peak and reduced tail weight, while the scale parameter controls the dispersion along the x-axis. The interaction of these parameters creates distinct distribution profiles, from leptokurtic shapes with rapid decay to more platykurtic forms with gradual tail decline.

Validity Check of S-EL Distribution

A random variable y is said to be continuous if there exist a function $f(y)$, called PDF of y, that satisfies the following properties:

i. $f(y)\geq 0$
ii. $\int_{-\infty}^{\infty} f(y)\,dy=1$
iii. $p(a<Y<b)=\int_a^b f(y)dy$

To establish the validity of the proposed distribution as a proper probability density function, we proceed by verifying that its integral from 0 to ∞ equals unity. That is:

$$\int_0^{\infty} f_s(y;\xi)\,dy=1 \qquad (7)$$

where $f_s(y;\xi)=\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]$, and

$\xi=(\alpha,\theta,\lambda)$ with $\alpha,\theta,\lambda>0$ and $y\geq 0$.

Let $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}$.

$\frac{du}{dy}=\alpha\theta\lambda\left[1-(1+\lambda y)^{-\theta}\right]^{\alpha-1}(1+\lambda y)^{-(\theta+1)}$.

$du=\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\,dy$

Therefore, $f_s(y;\xi)\,dy=\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left(\frac{\pi}{2}u\right)dy$

Thus, $f_s(y;\xi)\,dy=\frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right)du$

The next step is to change the limits of integration;

When $y=0$, $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}=[1-1]^{\alpha}=0$

When $y\to\infty$, $(1+\lambda y)^{-\theta}\to 0\Rightarrow u=[1-0]^{\alpha}=1$

Thus, as y goes from 0 to ∞, u goes from 0 to 1.

Therefore, the integral becomes:

$\int_0^{\infty} f_s(y;\xi)\,dy=\int_{u=0}^{u=1}\frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right)du$

Then, let $w=\frac{\pi}{2}u$, $dw=\frac{\pi}{2}du$, so:

$\int_0^1\frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right)du=\int_0^{\frac{\pi}{2}}\frac{\pi}{2}\cos(w)\,dw=[\sin(w)]_0^{\frac{\pi}{2}}=1$

Thus, the $\int_0^{\infty} f_s(y;\xi)\,dy=1$, this confirm that $\int_0^{\infty} f_s(y;\xi)\,dy$ is a valid probability density function.

Mathematical Properties of S-EL Distribution

Moment of S-EL Distribution

The r^{th} moment about the origin is defined as:

$$\mu_r'=E[Y^r]=\int_0^{\infty} y^r f_s(y;\alpha,\theta,\lambda)\,dy \qquad (8)$$

Where $f_s(y;\alpha,\theta,\lambda)$ is the probability density function (PDF) of the S-EL distribution. This definition aligns with the standard measure of moments in probability theory, which characterize the shape and properties of the distribution (Casella & Berger, 2002). Thus, given the PDF of S-EL distribution, the r^{th} is derived as follows:

$\mu_r'=\int_0^{\infty} y^r\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]dy$

Let $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}$, $du=\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\,dy$

Also, from substitution, $y=\frac{1}{\lambda}\left[\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]$.

Therefore, the r^{th} moment becomes:

$\mu_r'=\frac{\pi}{2\lambda^r}\int_0^1\left[\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]^r\cos\left(\frac{\pi}{2}u\right)du$

The term $\left[\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]^r$ was expanding using binomial theorem results to

$\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{k}{\theta}}$.

We further expand $\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{k}{\theta}}$ using the binomial series (for $\left|u^{\frac{1}{\alpha}}\right|<1$, which holds for $u\epsilon(0,1)$) as: $\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{k}{\theta}}=\sum_{j=0}^{\infty}\binom{-\frac{k}{\theta}}{j}(-1)^j u^{\frac{j}{\alpha}}=\sum_{j=0}^{\infty}\frac{\Gamma\left(j+\frac{k}{\theta}\right)}{\Gamma\left(\frac{k}{\theta}\right)j!}u^{\frac{j}{\alpha}}$.

We also expand $\cos\left(\frac{\pi}{2}u\right)$ as a series: $\cos\left(\frac{\pi}{2}u\right)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}u\right)^{2m}$

Then, the integral becomes fully separably expression. Evaluating

$\int_0^1 u^{\frac{j}{\alpha}}\cos\left(\frac{\pi}{2}u\right)du=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\int_0^1 u^{2m+\frac{j}{\alpha}}du=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\frac{1}{2m+\frac{j}{\alpha}+1}$,

the r^{th} moment is expressed as the infinite-series representation as:

$$\mu_r'=\frac{\pi}{2\lambda^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\sum_{j=0}^{\infty}\frac{\Gamma\left(j+\frac{k}{\theta}\right)}{\Gamma\left(\frac{k}{\theta}\right)j!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\frac{1}{2m+\frac{j}{\alpha}+1} \quad (9)$$

This provides the analytical form of the r^{th} raw moment for the S-EL distribution and serves as the basis for computing mean, variance, and higher-order moment-based characteristics.

Moment Generating Function S-EL Distribution

The moment generating function (MGF) is defined as:

$$M_Y(t)=E[e^{tY}]=\int_0^{\infty}e^{tY}f_s(y;\alpha,\theta,\lambda)\,dy \quad (10)$$

where $f_s(y;\alpha,\theta,\lambda)$ is the PDF of the S-EL distribution. This definition follows the standard probabilistic framework for generating moments of a distribution (Casella & Berger, 2002).

$$M_Y(t)=E[e^{tY}]=\int_0^{\infty}e^{tY}\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]dy$$

Using the transformation $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}$, $du=\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}dy$

The integral reduced to:

$$M_Y(t)=\frac{\pi}{2}\int_0^1\exp\left\{\frac{t}{\lambda}\left[\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]\right\}\cos\left(\frac{\pi}{2}u\right)du$$

This integral cannot be expressed in closed form but can be expanded using series expansions for practical computation as follows:

$$M_Y(t)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\lambda}\right)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{j=0}^{\infty}\frac{\Gamma\left(j+\frac{k}{\theta}\right)}{\Gamma\left(\frac{k}{\theta}\right)j!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\frac{1}{2m+\frac{j}{\alpha}+1}$$

Thus, MGF of S-EL distribution is given by:

$$M_Y(t)=\frac{\pi}{2}\int_0^1\exp\left\{\frac{t}{\lambda}\left[\left(1-u^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}-1\right]\right\}\cos\left(\frac{\pi}{2}u\right)du$$

with the infinite-series expansion above providing a tractable form for numerical evaluation.

Entropy of S-EL Distribution

The entropy of any random variable Y is a measure of indecisiveness, variability and details innate to the probable results of the variable. It is defined mathematically by Renyi (1961) entropy of order β (where $\beta>0$ and $\beta\neq1$) is defined as:

$$H_\beta=\frac{1}{1-\beta}\log\left(\int_0^{\infty}[f_s(y)]^{\beta}\,dy\right) \quad (11)$$

For the SE-L distribution, evaluation of the integral $I=\int_0^{\infty}[f_s(y)]^{\beta}\,dy$ requires a seris of transformation. Substituting $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}$ simplifies the integrand and allows the expression to be rewritten on the unit interval. A further change of variable $v=u^{\frac{1}{\alpha}}$ yields an integral involving powers of v and $(1-v)$, multiplied by $\left[\cos\left(\frac{\pi}{2}v^{\alpha}\right)\right]^{\beta}$. Although this expression has no closed form, the cosine term can be expanded as a power series, allowing the integral to be represented as a sum of Beta functions. Applying this expression gives:

$$I=\alpha\left(\frac{\pi}{2}\right)^{\beta}(\alpha\theta\lambda)^{\beta-1}\sum_{m=0}^{\infty}C_m\left(\frac{\pi}{2}\right)^{2m}B((\alpha-1)\beta+2\alpha m+1,\frac{(\beta-1)(\theta+1)}{\theta}+1) \quad (12)$$

Therefore, the Renyi entropy of order β for the sine exponential Lomax distribution is given by:

$$H_\beta=\frac{1}{1-\beta}\log\left[\alpha\left(\frac{\pi}{2}\right)^{\beta}(\alpha\theta\lambda)^{\beta-1}\sum_{m=0}^{\infty}C_m\left(\frac{\pi}{2}\right)^{2m}B((\alpha-1)\beta+2\alpha m+1,\frac{(\beta-1)(\theta+1)}{\theta}+1)\right] \quad (13)$$

Order Statistic of S-EL Distribution

Suppose $Y_1, Y_2, ..., Y_n$ are random sample from S-EL distribution with CDF $F(y)$ and PDF $f(y)$, the PDF of the k^{th} order statistic $Y_{(k)}$ is given by:

$$f_{Y_{(k)}}(y)=\frac{n!}{(k-1)!(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y) \quad (14)$$

This formula is a standard result in order statistics theory, describing the distribution of the k^{th} smallest value in the sample (David & Nagaraja, 2003). Where $F(y)$ and $f(y)$ are the CDF and PDF of the S-EL distribution and $k=1, 2, ..., n$.

After substituting the PDF and CDF, the k^{th} order statistic $Y_{(k)}$ becomes:

$$f_{Y_{(k)}}(y)=\frac{n!}{(k-1)!(n-k)!}\left[\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]\right]^{k-1}\left[1-\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]\right]^{n-k}\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]$$

where the transformation $u=[1-(1+\lambda y)^{-\theta}]^{\alpha}$ is used. The derivative satisfies:

$dy=\frac{du}{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}(1+\lambda y)^{-(\theta+1)}}$

Now, express all terms in $f_{Y_{(k)}}(y)$ in terms of u and simplifying, the PDF of the k^{th}order statistic of S-EL in terms of u is:

$$f_{Y_{(k)}}(u)=\frac{n!}{(k-1)!(n-k)!}\left[\sin\left(\frac{\pi}{2}u\right)\right]^{k-1}\left[1-\sin\left(\frac{\pi}{2}u\right)\right]^{n-k}\frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right),\ 0<u<1. \quad (15)$$

Survival Function of S-EL Distribution

The survival function of the $S(y)$ of a distribution is derived using the fundamental relationship $S(y)=1-F(y)$ (Klein & Moeschberger, 2003). Thus, the survival function of the S-EL distribution can be obtain using the relation:

$S(y)=1-F(y)$, where $F(y)$ is the CDF of the S-EL distribution.

$$S(y)=1-\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right] \quad (16)$$

This function can be used for reliability analysis, risk assessment, and other applications involving time-to-event data.

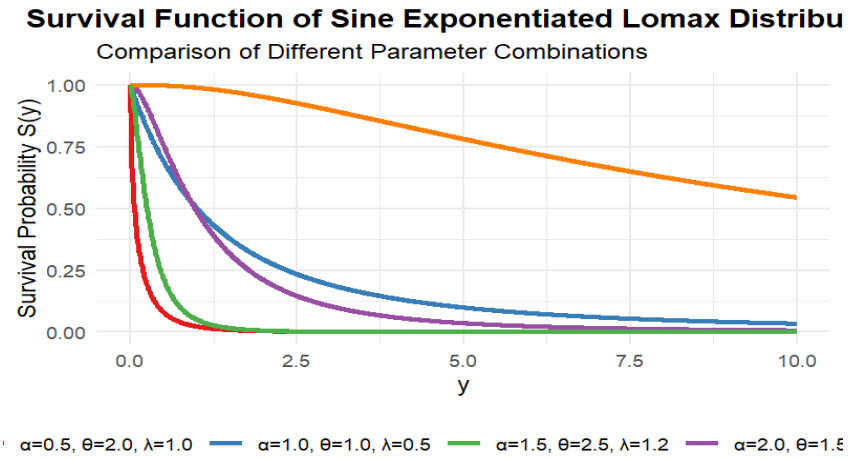

Figure 3: Survival Function of S-EL Distribution with Different Parameter Combinations

Based on the survival function plot (figure 3) comparing different parameter combinations for the S-EL distribution, the curves demonstrate how varying parameters influence the tail behavior and reliability of the model. The combination $\alpha=2.0$, $\theta=1.5$, $\lambda=0.8$ produces a survival curve that declines steeply, indicating a lower probability of survival at higher values and suggesting a distribution with relatively lighter tails and higher initial hazard. In contrast, combinations with lower power parameters and higher shape parameters, such as $\alpha=0.5$, $\theta=2.0$, $\lambda=1.0$, show a more gradual descent, reflecting heavier tails and a greater probability of exceeding larger values. The curves highlight the role of α in accelerating the failure rate, θ in modulating tail weight, and λ in scaling the spread, collectively shaping the survival characteristics from rapid decline to prolonged endurance.

Hazard Function of S-EL Distribution

The hazard function $h(y)$ of a distribution is obtain using the relation $h(y)=\frac{f(y)}{S(y)}$, (Klein & Moeschberger, 2003; Lawless, 2003) where $f(y)$ and $S(y)$ is the PDF and survival function respectively of the given distribution. Thus,

$$h(y)=\frac{f(y)}{S(y)}=\frac{\frac{\pi}{2}\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]}{1-\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]}$$

$$h(y)=\frac{\frac{\pi}{2}\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]}{(1+\lambda y)^{(\theta+1)}\left\{1-\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]\right\}} \quad (17)$$

This function can be used to model the instantaneous failure rate in reliability engineering or survival analysis. The behavior of $h(y)$ (e.g., increasing, decreasing, or bathtub-shaped) depends on the parameters α, θ, λ.

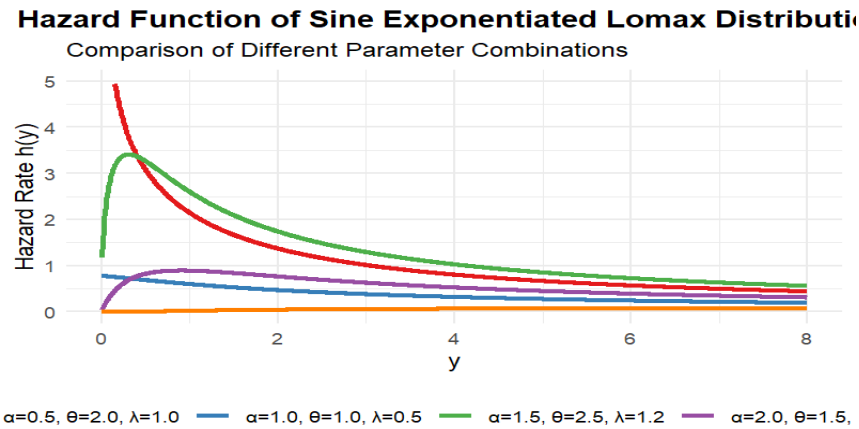

Figure 4: Hazard Function of S-EL Distribution with Different Parameter Combinations

Based on the hazard function plot comparing different parameter combinations for the S-EL distribution (figure 4), the curves reveal how the instantaneous failure rate is influenced by varying the power parameter (α), shape parameter (θ), and scale parameter (λ). The combination α=2.0, θ=1.5, λ=0.8 produces a hazard curve that increases initially and stabilize or decline, reflecting a distribution with increasing failure rate early on that moderate over time. In contrast, combinations with lower α and higher θ, such as α=0.5, θ=2.0, λ=1.0, might show a more constant or slowly increasing hazard, indicating a different failure pattern. The curves collectively demonstrate how higher α values can amplify the hazard rate initially, θ influences the tail behavior affecting long-term risk, and λ scales the spread of failures, resulting in hazard functions that range from rapidly increasing to more steady patterns, highlighting the model's flexibility in modeling diverse failure processes.

Cumulative Hazard Function of S-EL Distribution

Given the survival function of distribution, the cumulative hazard function H(y) of a distribution is obtain using the relation $H(y)=-\ln[S(y)]$ (Collet, 2015, Kleinbaum & Klein, 2012). Thus, the cumulative hazard function of s-EL distribution was obtained using the relation:

$$H(y)=-\ln\left(1-\sin\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]\right) \quad (18)$$

This is the explicit form of the cumulative hazard function for the S-EL distribution.

Quantile Function

The quantile function $Q(p)$, is obtain by inverting the CDF (Nair et al., 2022). Specifically, for a given probability $p\in(0,1)$, $Q(p)$ is obtained by solving

$$F_s(Q(p);\xi)=p \quad (19)$$

where $\xi=\alpha,\theta,\lambda$ denotes the parameter vector.

Substitute the CDF:

$$\sin\left[\frac{\pi}{2}\{[1-(1+\lambda Q(p))^{-\theta}]^{\alpha}\}\right]=p$$

$$[1-(1+\lambda Q(p))^{-\theta}]^{\alpha}=\frac{2}{\pi}arcsin(p)$$

$$1-(1+\lambda Q(p))^{-\theta}=(\frac{2}{\pi}arcsin(p))^{\frac{1}{\alpha}}$$

$$Q(p)=\frac{1}{\lambda}\left(\sqrt[\frac{-1}{\theta}]{1-\left(\frac{2}{\pi}\arcsin(p)\right)^{\frac{1}{\alpha}}}-1\right) \quad (20)$$

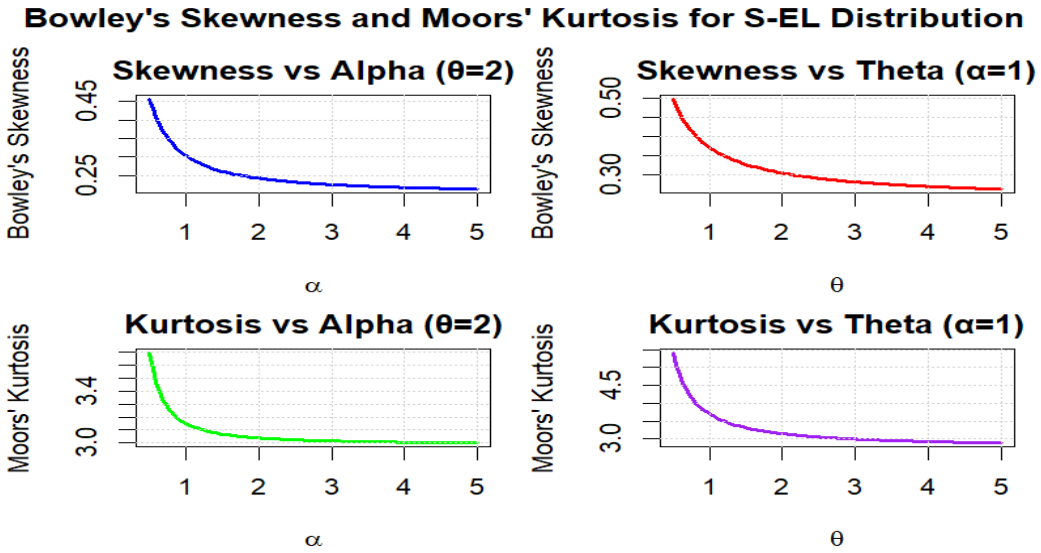

Figure 5: Bowle's and Moors' Kurtosis for S-EL Distribution

The plots for Bowley's skewness and Moors' kurtosis in figure 5 illustrate how the shape of the S-EL distribution changes with its parameters. As the power parameter alpha increases while theta is held constant at 2, both skewness and kurtosis decrease, indicating that the distribution becomes more symmetric and less peaked with lighter tails. Conversely, when the shape parameter theta increases while alpha is fixed at 1, skewness and kurtosis values rise, reflecting a shift towards greater asymmetry, heavier tails, and a more pronounced peak. These trends show that alpha acts as a flattening parameter that reduces extreme characteristics, while theta enhances the distribution's skewness and tail weight, making it more leptokurtic.

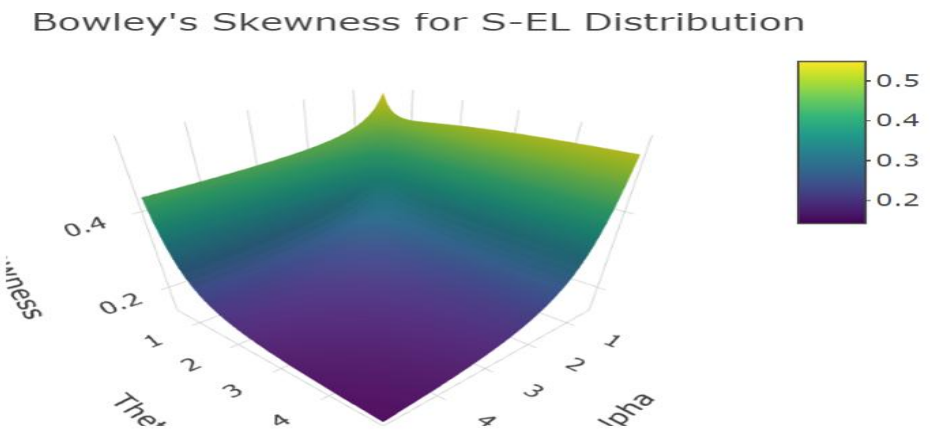

Figure 6: Bowle's Skewness for S-EL Distribution

Based on the Bowley's skewness plot for the S-EL distribution in figure 6, the values indicate the direction and degree of asymmetry in the model. Skewness values around 0.2 to 0.4 suggest a moderate right-skew, where the tail on the right side of the distribution is longer, meaning outcomes are more spread out towards higher values. The presence of values near 1.5 to 1.7 indicates a much stronger right-skew, where the distribution is highly asymmetric with a heavy tail extending to the right. This range demonstrates that the S-EL distribution can model varying levels of skewness, from moderately asymmetric to highly skewed, depending on the parameter combinations, highlighting its flexibility in capturing diverse data behaviors.

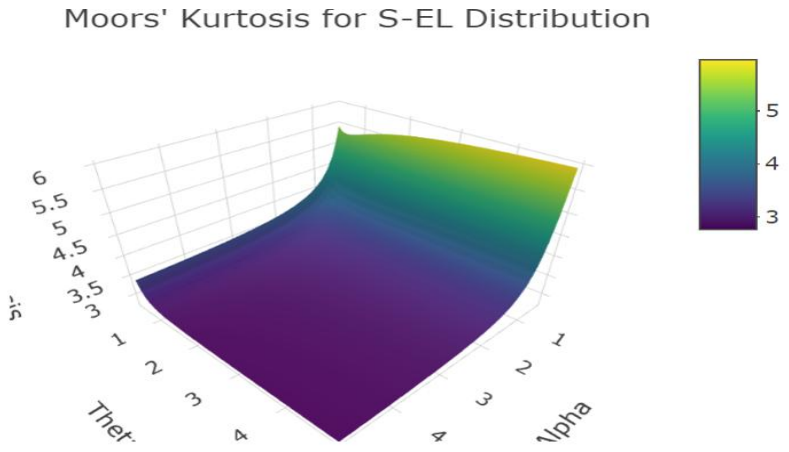

Figure 7: Moors' Kurtosis for S-EL Distribution

This plot of Moors' kurtosis for the S-EL distribution in figure 7 shows values ranging from approximately 3 to 5.5, indicating a shift from moderate (mesokurtic) to extreme (leptokurtic) tail behavior and peak sharpness. Kurtosis increases as the shape parameters theta or alpha approach their maximum values, demonstrating the distribution's flexibility in modeling heavier tails and more concentrated data peaks under extreme parameter conditions.

Estimation of Parameters of S-EL Distribution

Let $y=y_1,y_2...,y_n$ be a random sample of size n from the sine-exponentiated Lomax Distribution with pdf given in 4.12 where $\xi=(\alpha,\theta,\lambda)$ the distributional parameters, the likelihood function is given by:

$$L(\xi,y_1,y_2...,y_n)=\prod_{i=1}^{n}[f_s(y;\xi)] \quad (21)$$

$$L(y_1,y_2...,y_n;\xi)=\prod_{i=1}^{n}\left[\frac{\pi}{2}\left\{\frac{\alpha\theta\lambda[1-(1+\lambda y)^{-\theta}]^{\alpha-1}}{(1+\lambda y)^{(\theta+1)}}\right\}\cos\left[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}\right]\right]$$

The log-likelihood is given by:

$$L(y;\xi)=\sum_{i=1}^{n}\left[\ln\left(\frac{\pi}{2}\right)+\ln(\alpha)+\ln(\theta)+\ln(\lambda)+(\alpha-1)\ln[1-(1+\lambda y)^{-\theta}]-(\theta+1)\ln(1+\lambda y)+\ln(\cos[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}])\right]$$

$$L(y;\xi)=n\ln\left(\frac{\pi}{2}\right)+n\ln(\alpha)+n\ln(\theta)+n\ln(\lambda)+(\alpha-1)\sum_{i=1}^{n}\ln[1-(1+\lambda y)^{-\theta}]-(\theta+1)\sum_{i=1}^{n}\ln(1+\lambda y)+\sum_{i=1}^{n}\ln(\cos[\frac{\pi}{2}\{[1-(1+\lambda y)^{-\theta}]^{\alpha}\}])$$

$$\frac{\partial L(y;\xi)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln[1-(1+\lambda y)^{-\theta}]-\frac{\pi}{2}\sum_{i=1}^{n}\tan\{\frac{\pi}{2}[1-(1+\lambda y)^{-\theta}]^{\alpha}\}[1-(1+\lambda y)^{-\theta}]^{\alpha}\ln[1-(1+\lambda y)^{-\theta}]$$

$$\frac{\partial L(y;\xi)}{\partial\theta}=\frac{n}{\theta}+(\alpha-1)\sum_{i=1}^{n}\frac{(1+\lambda y)^{-\theta}\ln(1+\lambda y)}{1-(1+\lambda y)^{-\theta}}-\sum_{i=1}^{n}\ln(1+\lambda y)-\frac{\pi}{2}\sum_{i=1}^{n}\tan\left\{\frac{\pi}{2}[1-(1+\lambda y)^{-\theta}]^{\alpha}\right\}\alpha[1-(1+\lambda y)^{-\theta}]^{\alpha-1}(1+\lambda y)^{-\theta}\ln(1+\lambda y)$$

$$\frac{\partial L(y;\xi)}{\partial\lambda}=\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{\theta y(1+\lambda y)^{-\theta-1}}{1-(1+\lambda y)^{-\theta}}-(\theta+1)\sum_{i=1}^{n}\frac{y}{1+\lambda y}-\frac{\pi}{2}\sum_{i=1}^{n}\tan\left\{\frac{\pi}{2}[1-(1+\lambda y)^{-\theta}]^{\alpha}\right\}\alpha[1-(1+\lambda y)^{-\theta}]^{\alpha-1}\theta y(1+\lambda y)^{-\theta-1}$$

To solve for the parameters $\xi=(\alpha,\theta,\lambda)$ by maximizing the log-likelihood function, we set the partial derivatives of the log-likelihood with respect to each parameter to zero and solve the resulting system of equations. However, due to the complexity of the log-likelihood function and its derivatives, an analytical solution is not feasible. Instead, we use

numerical optimization methods such as gradient descent, Newton-Raphson, or other iterative algorithms. Below, I outline the steps to solve for the parameters numerically.

Simulation Studies

Simulation Design

This study employed two simulation designs to evaluate the performance of maximum likelihood estimation for the S-EL distribution. In both designs, data were generated from the S-EL model using the true parameter values $\alpha = 2.0$, $\theta = 1.5$, and $\lambda = 0.8$, with each scenario replicated N = 1,000 times. For every generated dataset, the S-EL distribution was fitted using MLE, and the resulting parameter estimates were assessed using the empirical bias, mean square error (MSE), average length (AL) of the 95% confidence intervals, and their corresponding coverage probabilities (CP). Case Study I: Examined small-to-moderate sample sizes with n = 50, 100, 250, and 500. All 50 simulation runs successfully converged for each sample size, and the repeated estimation allowed for evaluation of estimator behavior under limited data conditions. Case Study II: Considered larger samples with n = 250, 500, 1,000, and 2,000 to assess improvements in estimator stability and interval performance as sample size increases. Across both designs, the metrics (bias, MSE, AL, and CP) were used to examine the finite-sample properties of the estimators and to determine how sample size influences estimation accuracy and inferential reliability for the S-EL distribution.

Table 1: Simulation Results of S-EL Distribution with n = 50, 100, 250, 500

Parameter	Sample Size	True Value	Mean Estimate	Bias	MSE	AL	CP
α	50	2.0	2.997	0.997	5.961	9.293	0.96
θ	50	1.5	2.124	0.624	3.809	11.723	0.88
λ	50	0.8	1.830	1.030	6.522	10.398	0.92
α	100	2.0	2.341	0.341	1.811	4.262	0.92
θ	100	1.5	1.832	0.332	1.603	5.011	0.96
λ	100	0.8	1.134	0.334	1.449	4.701	0.90
α	250	2.0	1.999	-0.001	0.122	1.421	0.98
θ	250	1.5	1.638	0.137	0.176	1.610	0.98
λ	250	0.8	0.813	0.013	0.191	1.571	0.90
α	500	2.0	2.005	0.004	0.054	0.980	0.94
θ	500	1.5	1.563	0.063	0.046	0.989	1.00
λ	500	0.8	0.797	-0.003	0.054	1.047	0.94

Source: Authors Compilation

The simulation results in Table 1 demonstrate that as sample size increases from n = 50 to n = 500, the parameter estimates show substantial improvement in accuracy: biases decrease dramatically (e.g., from approximately 1.0 to near zero for α and λ), MSEs reduce significantly (from around 6.0 to under 0.06), confidence interval lengths contract substantially (from over 9.0 to about 1.0), while coverage probabilities remain close to the nominal 95% level across all sample sizes.

Table 2: Simulation Results of S-EL Distribution with n = 250, 500, 1000 and 2000

Parameter	Sample Size	True Value	Mean Estimate	Bias	MSE	AL	CP
α	250	2.0	2.105	0.105	0.174	1.538	0.945
θ	250	1.5	1.604	0.104	0.217	1.550	0.950
λ	250	0.8	0.909	0.109	0.221	1.708	0.910
α	500	2.0	2.037	0.038	0.075	1.016	0.955
θ	500	1.5	1.539	0.0392	0.054	0.954	0.930
λ	500	0.8	0.842	0.042	0.094	1.106	0.935
α	1000	2.0	2.013	0.013	0.029	0.694	0.945
θ	1000	1.5	1.526	0.026	0.042	0.662	0.935
λ	1000	0.8	0.820	0.020	0.040	0.751	0.920
α	2000	2.0	2.018	0.018	0.016	0.488	0.950
θ	2000	1.5	1.500	0.0001	0.013	0.444	0.940
λ	2000	0.8	0.823	0.023	0.019	0.526	0.965

Source: Authors Compilation

The simulation results in Table 2 demonstrate excellent asymptotic properties of the maximum likelihood estimators: as sample size increases from n = 250 to n = 2000, all three parameters show consistent improvement in estimation accuracy. The biases decrease substantially (e.g., for α from 0.105 to 0.018, for θ from 0.104 to near zero, and for λ from 0.109 to 0.023), MSEs show a clear decreasing trend (reducing by approximately 90% across all parameters), confidence interval lengths contract significantly (by about 68-74% across parameters), while coverage probabilities remain consistently close to the nominal 95% level, indicating well-calibrated uncertainty estimation. The parameter λ shows slightly lower coverage probabilities at smaller sample sizes but improves to excellent coverage (96.5%) at n = 2000, demonstrating the consistency and efficiency of the estimation procedure.

Applications

Application to Daily Percentage Returns of the S & P 500 Stock Market Index

The daily percentage returns of the S&P 500 stock market index represent the day-to-day price fluctuations of the index, expressed as a percentage change from the previous trading day's closing value. The data is available in *Ecdat* R package.

Table 3: Descriptive Statistics of Daily Percentage Returns of the S & P 500 Stock Market Index Data Set

Category	Statistic	Value
Central Tendency	Mean	0.0004
	Median	0.0005
Dispersion	Standard Dev.	0.0109
	Range	-0.2280 to 0.0871
Quartiles	1st Quartile	-0.0048
	3rd Quartile	0.0058
Extreme Values	Minimum	-0.2280
	Maximum	0.0871
Tail Percentiles	95th Percentile	0.0163
	99th Percentile	0.0252
Distribution Shape	Skewness	-3.4976
	Kurtosis	77.4558
	Excess Kurtosis	74.4558
Normality	Shapiro-wilk (p-value)	0.84139(p<0.01)
	Anderson-Darlin(p-value)	36.4589(p<0.01)

Source: Authors Compilation

The descriptive statistics results of S&P 500 daily returns presented in Table 3 exhibited extreme characteristics, with strong negative skewness (-3.50) and exceptionally high kurtosis (77.46), indicating severe left asymmetry and heavy tails far exceeding normality. The mean (0.0004) and median (0.0005) are near zero but overshadowed by high volatility (SD = 0.0109) and extreme values (range: -0.228 to 0.087), while quartiles (-0.0048 to 0.0058) and tail percentiles (95th: 0.0163, 99th: 0.0252) confirm pronounced tail risks. Normality tests (Shapiro-Wilk, Anderson-Darling) emphatically reject normality ($p < 0.01$) (Table 4.3). These features extreme skewness, leptokurtosis, and heavy tails make the S-EL distribution highly suitable, as its power (α) and shape (θ) parameters can capture asymmetric heavy-tailed behavior, while its scale parameter (λ) adapts to volatility clustering, providing a superior fit for financial return modeling beyond conventional distributions.

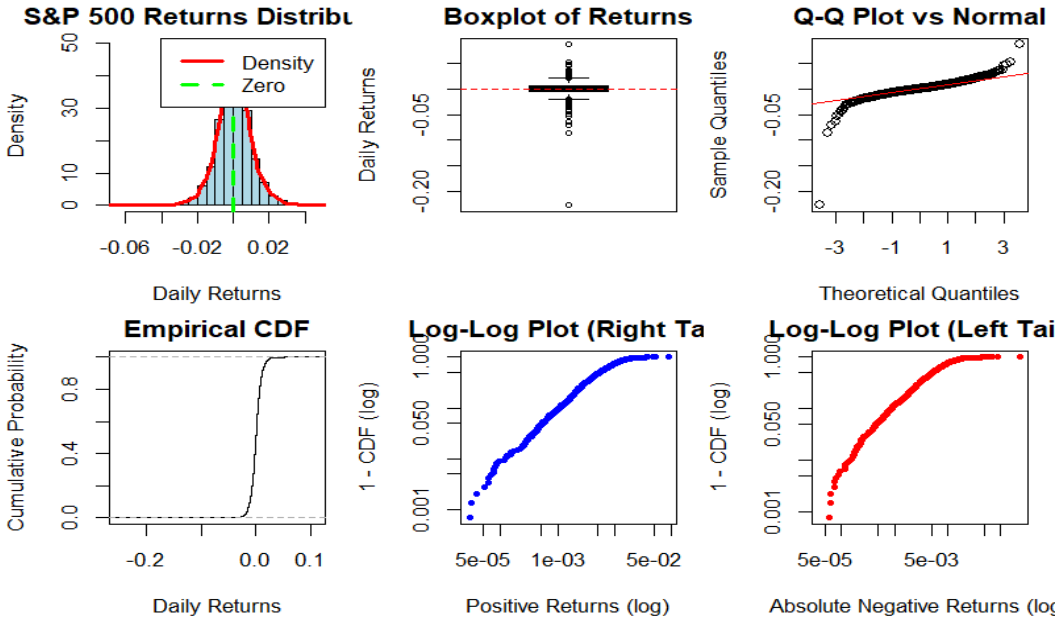

Figure 8: Distribution, Box Plot, Q-q Plot, Empirical CDF, Log-log Plot Right and Left Tail

The figure 8 analyzes the distribution of S&P 500 daily percentage returns, revealing pronounced non-normal characteristics. The density plot shows strong peakedness (leptokurtosis) and heavy tails, with excess concentration around zero and extreme values exceeding ±0.05. The boxplot confirms asymmetry with negative skewness and numerous outliers. The Q-Q plot deviates sharply from the normal distribution diagonal, especially in the tails, rejecting normality. The empirical CDF exhibits steep central rises and gradual tail extensions, indicating high probability mass near zero with persistent tail risks. Both log-log plots demonstrate linear patterns for absolute positive and negative returns, confirming power-law decay in the tails a hallmark of financial returns showing extreme movements are more frequent than predicted by normal distributions. These visualizations validate the heavy-tailed, asymmetric nature of market returns, critical for risk management modeling.

Table 4: MLE and Goodness of Fit Measure for Daily Percentage Returns of the S & P 500 Stock Market Index Data Set

Distribution	Parameters	Estimate	Goodness of Fit Measures
S-EL	α	1.094	AIC (21885.91)
	θ	6.342	CAIC (21885.91)
	λ	14.539	BIC (21897.77)
			HQIC (21890.19)
			MLE (-10940.96)
Exponentiated Lomax	α	17.665	AIC (21915.33)
	θ	1.157	CAIC (21915.34)
	λ	9.137	BIC (21933.12)
			HQIC (21921.76)
			MLE (-10954.67)
Lomax	α	33.339	AIC (21886.89)
	θ	-	CAIC (21886.89)
	λ	4.276	BIC (21898.75)
			HQIC (21891.17)
			MLE (-10941.45)
Weibull	α	-	AIC (21935.51)
	θ	1.046	CAIC (21935.52)
	λ	0.008	BIC (21953.31)
			HQIC (21941.94)
			MLE (-10964.76)
Burr	α	8.000	AIC (140401.77)
	θ	0.300	CAIC (140401.78)
	λ	2.000	BIC (140419.56)
			HQIC (140408.20)
			MLE (-70197.89)
Exponentiated Exponential	α	1.099	AIC (21891.79)
	θ	-	CAIC (21891.80)
	λ	146.928	BIC (21903.66)
			HQIC (21896.08)
			MLE (-10943.90)

Source: Authors Compilation

Table results in Table 4 indicates that, the S-EL distribution demonstrates the best fit for the S&P 500 returns data among the compared models, achieving the lowest AIC (21885.91), CAIC (21885.91), and competitive BIC (21897.77) and HQIC (21890.19) values. Its superior performance is further supported by the highest log-likelihood (-10940.96), indicating a closer alignment with the data's heavy-tailed and skewed characteristics. While the Lomax distribution shows similar AIC performance (21886.89), the S-EL's additional parameters provide greater flexibility in capturing the complexities of financial returns. In contrast, the Exponentiated Lomax, Weibull, Burr, and Exponentiated Exponential models exhibit notably higher information criteria values, confirming their weaker fit. This establishes the S-EL as the most appropriate model for this dataset, balancing parsimony and accuracy.

Application to Amounts of Damage Caused by Earthquakes

The dataset contains recorded amounts of financial damage (typically in millions of dollars) caused by historical earthquakes. It is used for modeling heavy-tailed loss distributions.

Table 5: Descriptive Statistics and Magnitude Distribution of Earthquake Data

Category	Statistic	Value
Central Tendency	Mean magnitude	4.62
	Median magnitude	4.60
Dispersion	Standard Dev.	0.40
	Range	4.0 – 6.4
Quartiles	1st Quartile	4.30
	3rd Quartile	4.90
Extreme Values	Minimum magnitude	4.00
	Maximum magnitude	6.40
Distribution Shape	Skewness	0.77
	Kurtosis	3.51
Magnitude categories	Major ($\geq$5.0)	198(19.8%)
	Moderate (4.0 – 4.9)	802(80.2%)
	Minor ($<$5.0)	0(0.0%)

Source: Authors Compilation

The descriptive statistics in Table 5 revealed that the earthquake magnitude data exhibits moderate right-skewness (skewness = 0.77) and near-normal tail behavior (kurtosis = 3.51), with nearly identical mean (4.62) and median (4.60) values indicating central symmetry, and moderate dispersion (SD = 0.40, IQR = 4.30–4.90). The concentration of events in moderate magnitudes (80.2% in 4.0–4.9) alongside meaningful but non-extreme tail events (19.8% ≥ 5.0, max = 6.4) suggests a distribution with manageable yet non-negligible tail risk. The S-EL distribution is well-suited to model this data, as its shape parameter (θ) can capture the observed skewness, while the power parameter (α) provides flexibility to adapt to the moderate tail behavior, offering a superior fit over symmetric models for this asymmetric but not overly heavy-tailed dataset.

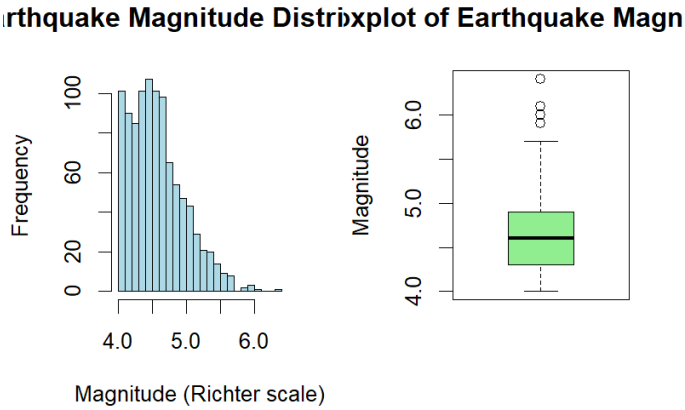

Figure 9: Distribution and Box Plot of Earthquake Magnitude Data

The distribution of financial damage from historical earthquakes in figure 9 exhibits a highly right-skewed and heavy-tailed pattern, making it well-suited for modeling with the S-EL distribution. This heavy-tailed asymmetry underscores why conventional symmetric distributions fail to quantify seismic risk accurately, while the S-EL's flexibility provides a robust framework for estimating extreme-value probabilities, improving disaster preparedness and resource allocation for future seismic events.

Table 6: MLE and Goodness of Fit Measure for Earthquake Data Set

Distribution	Parameters	Estimate	Goodness of Fit Measures
S-EL	α	15.0000	AIC (-5422.07)
	θ	7.1341	CAIC (-5422.06)
	λ	0.1000	BIC (-5412.25)
			HQIC (-5418.34)
			MLE (2713.03)
Exponentiated Lomax	α	8.1385	AIC (-3264.67)
	θ	15000	CAIC (-3264.65)
	λ	0.1000	BIC (-3249.95)
			HQIC (-3259.08)
			MLE (1635.34)
Lomax	α	2.6353	AIC (-2842.83)
	θ	-	CAIC (-2842.81)
	λ	0.1000	BIC (-2828.11)
			HQIC (-2837.24)
			MLE (1424.42)
Weibull	α		AIC (1270.48)
	θ	10.6724	CAIC (1270.49)
	λ	4.8126	BIC (1280.29)
			HQIC (1274.21)
			MLE (-633.24)
Burr	α	14.1523	AIC (-902.47)
	θ	35.9915	CAIC (-902.45)
	λ	3.4376	BIC (-887.75)
			HQIC (-896.88)
			MLE (454.24)
Exponentiated Exponential	α	6.3800	AIC (-893.93)
	θ	-	CAIC (-893.92)
	λ	3.0325	BIC (-884.12)
			HQIC (-980.20)
			MLE (448.97)

Distribution	Parameters	Estimate	Goodness of Fit Measures
Pareto (Type I0	α	-	AIC (-1124.75)
	θ	-	CAIC (-1124.74)
	λ	4.000	BIC (-1114.94)
			HQIC (-1121.02)
			MLE (564.38)
Generalized Pareto	α	-0.2445	AIC (-33.75)
	θ	0.3700	CAIC (-33.73)
	λ	-	BIC (-23.93)
			HQIC (-30.02)
			MLE (18.8700)

Source: Authors Compilation

The result in Table 6 indicates that, the S-EL distribution provides the best fit for the earthquake magnitude data, achieving the lowest AIC (-5422.07) and CAIC (-5422.06) values, along with competitive BIC (-5412.25) and HQIC (-5418.34) scores. Its superior performance is further confirmed by the highest log-likelihood (2713.03), indicating a strong alignment with the data's moderate skewness and near-normal tail behavior. While the Exponentiated Lomax and Lomax models show reasonable fits, they are substantially outperformed by the S-EL across all information criteria. The Weibull, Burr, Exponentiated Exponential, Pareto, and Generalized Pareto distributions exhibit notably poorer performance, with significantly higher AIC values and lower log-likelihoods. This establishes the S-EL as the most suitable model for capturing the earthquake magnitude distribution, effectively balancing flexibility and parsimony.

Application to Remission Time of Bladder Cancer Data Set

The third set of dataset represents to remission times (in months) of a random sample of 128 bladder cancer patients and the data was extracted from Adekunle et al. (2023).

Table 7: Descriptive Statistics and Survival Characteristics of Remission Time Data Set

Category	Statistic	Value
Central Tendency	Mean	8.52
	Median	6.05
Dispersion	Standard Dev.	7.86
	Range	0.08 – 43.01
Quartiles	1st Quartile	3.32
	3rd Quartile	11.54
Extreme Values	Minimum	0.08
	Maximum	43.01
Distribution Shape	Skewness	1.88
	Kurtosis	7.06
Survival Probabilities	>12 months	0.230(23.0%)
	>24 months	0.056(5.6%)
	> 36 months	0.016(1.6%)

Source: Authors Compilation

The descriptive statistics results for remission time data presented in Table 7 exhibited strong right-skewness (skewness = 1.88) and high kurtosis (7.06), indicating a heavy-tailed distribution where most patients experience shorter remission periods (median = 6.05 months) but a subset survives significantly longer, as evidenced by the substantial gap between the mean (8.52) and median. The large standard deviation (7.86) and wide range (0.08–43.01 months) further highlight this variability, while the survival probabilities confirm extended tail behavior with 23% exceeding 12 months, 5.6% exceeding 24 months, and 1.6% exceeding 36 months (Table 7). These characteristics pronounced skewness, heavy tails, and heterogeneous survival times make the S-EL distribution ideally suited for modeling, as its parameters can flexibly capture the observed asymmetry (via shape parameter θ), tail weight (via power parameter α), and scale (via λ), providing a nuanced fit for biomedical survival analysis beyond conventional distributions.

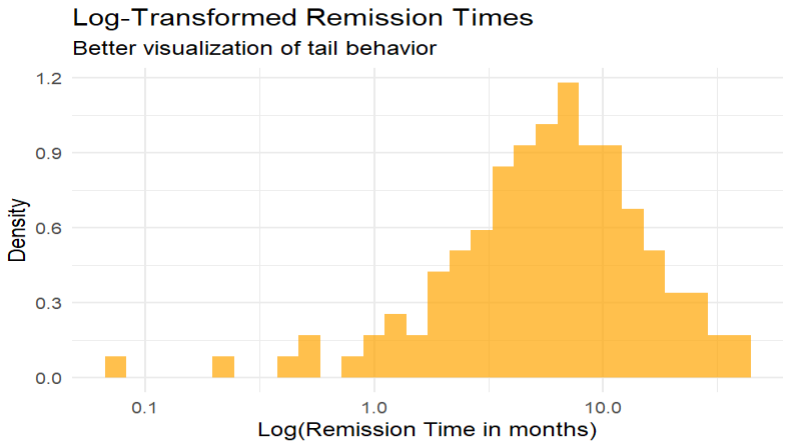

Figure 10: Visualization of Tail Behavior of Remission Times

The log-transformed remission times for bladder cancer patients plot in figure 10 revealed a right-skewed distribution with a long tail, indicating that while most patients experience remission within a shorter timeframe, a subset exhibits significantly prolonged remission periods. This heavy-tailed behavior suggests underlying heterogeneity in disease progression or treatment response, emphasizing the need for models that accommodate extreme values. The S-EL distribution, with its flexibility in capturing skewness and tail weight, would be well-suited to characterize such data, providing improved risk stratification and prognostic insights for clinical planning.

Table 8: MLE and Goodness of Fit Measure for Remission Time Data Set

Distribution	Parameters	Estimate	Goodness of Fit Measures
S-EL	α	1.4179	AIC (-813.93)
	θ	3.4291	CAIC (-813.83)
	λ	0.0315	BIC (-808.26)
			HQIC (-811.63)
			MLE (408.97)
Exponentiated Lomax	α	1.5170	AIC (-779.73)
	θ	9.4671	CAIC (-779.53)
	λ	0.0180	BIC (-771.22)
			HQIC (-776.27)
			MLE (392.86)
Lomax	α	1.8288	AIC (-779.65)
	θ	-	CAIC (-779.46)
	λ	0.1000	BIC (-771.14)
			HQIC (-776.20)
			MLE (392.83)
Weibull	α	-	AIC (791.77)
	θ	1.1537	CAIC (791.86)
	λ	8.9834	BIC (797.44)
			HQIC (794.07)
			MLE (-393.88)
Burr	α	2.4481	AIC (-780.79)
	θ	0.4978	CAIC (-780.59)
	λ	9.8760	BIC (-772.28)
			HQIC (-777.33)
			MLE (393.40)
Exponentiated Exponential	α	1.3533	AIC (-782.23)
	θ	-	CAIC (-782.13)
	λ	0.1417	BIC (-776.56)
			HQIC (-779.93)
			MLE (393.12)

Source: Authors Compilation

From Table 8 it can be seen that the S-EL distribution provides the best fit for the remission time data, achieving the lowest AIC (-813.93) and CAIC (-813.83) values, along with superior BIC (-808.26) and HQIC (-811.63) scores compared to all other models. Its performance is further validated by the highest log-likelihood (408.97), indicating the strongest alignment with the data's right-skewed and heavy-tailed characteristics. While the Exponentiated Lomax, Lomax,

Burr, and Exponentiated Exponential distributions show relatively similar fits, they are consistently outperformed by the S-EL across all information criteria. The Weibull distribution exhibits the weakest fit with a positive AIC (791.77), confirming its inadequacy for this dataset. These results demonstrate the S-EL's flexibility in capturing the complex survival patterns of remission times, making it the most suitable model for this biomedical application.

Applications to Waiting Time to the Start of the Next Eruption (in Minutes)

The variable 'waiting time to the start of the next eruption' refers to the time interval in minutes between the conclusion of one geyser eruption and the beginning of the subsequent one. This continuous measurement is a fundamental component of the Old Faithful geyser dataset, a classic time series in statistical analysis. It is renowned for its bimodal distribution, which reveals two predominant clustering patterns of intervals—shorter and longer waits—that reflect the geyser's distinct eruptive states. This characteristic makes the waiting time a frequently used example for demonstrating statistical concepts such as probability distributions, density estimation, and predictive modeling.

Table 9: Descriptive Statistics and Distributive Shape Analysis of Geyser Waiting Time Data Set

Category	Statistic	Value
Central Tendency	Mean waiting Time	70.90
	Median waiting Time	76.00
Dispersion	Standard Dev.	13.59
	Range	43 – 96
	Minimum	43
	Maximum	96
	Coefficient of Variation	0.192
Distribution Shape	Skewness Coeff.	-0.416
	Kurtosis Coeff.	1.857
	Excess Kurtosis	-1.143
Normality Tests	Shapiro-Wilk (p-value)	0.922(p<0.001)
	Anderson Darlin (p-value)	8.614(p<0.001)
Tail Characteristics	Hill estimator (Tail index)	26.572
Mean Excess function	At 50th Percentile	6.61
	At 75th Percentile	4.37
	At 90th Percentile	3.91
	At 95th Percentile	2.58

Source: Authors Compilation

The descriptive statistics results for geyser waiting time data presented in Table 9 exhibited moderate left-skewness (-0.416) and light-tailed behavior (excess kurtosis = -1.143), with a mean (70.90) lower than the median (76.00) indicating a concentration of longer waiting times. The low coefficient of variation (0.192) and narrow range (43–96) suggest limited dispersion, while normality tests (Shapiro-Wilk, Anderson-Darling) strongly reject Gaussianity ($p < 0.001$). The Hill estimator (26.572) and declining mean excess values (from 6.61 at the 50th percentile to 2.58 at the 95th) confirm light tails with no extreme outliers (table 4.9). For this dataset, the S-EL distribution remains applicable but may require parameter constraints (e.g., lower α to reduce tail weight) to adapt to its light-tailed, left-skewed nature, offering flexibility for capturing its non-normal yet bounded characteristics where conventional models fail.

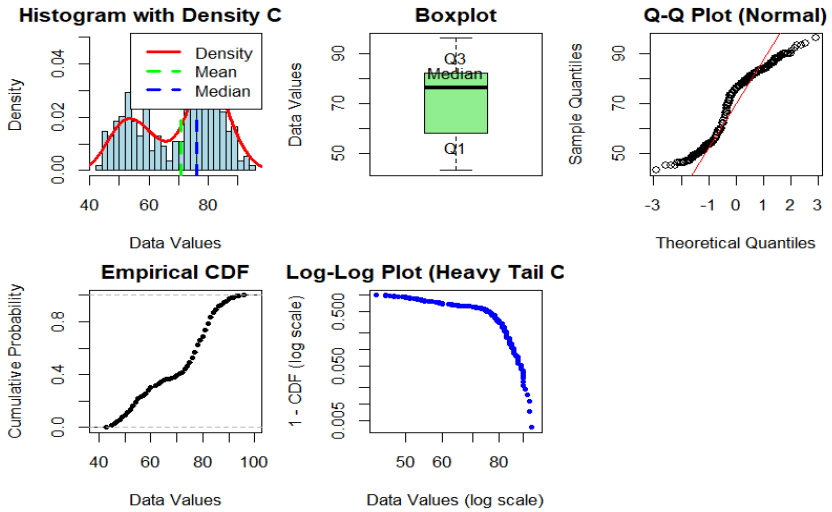

Figure 11: Histogram, Boxplot, Q-Q plot, Empirical CDF and Log-log Plot Heavy Tail

The visualization of the Geyser waiting time data in figure 11 reveals a bimodal distribution with heavy-tailed characteristics. The histogram and density plot show two distinct peaks, indicating two prevalent waiting time patterns, while the boxplot confirms positive skewness and outliers. The Q-Q plot deviates significantly from normality, especially

in the tails, rejecting a Gaussian fit. The empirical CDF displays a steep initial rise followed by a gradual tail, suggesting concentrated probability mass at lower times with prolonged waiting events. The log-log plot of the survival function exhibits a linear decay pattern, validating power-law behavior in the upper tail. This heavy-tailed, multimodal structure underscores the need for flexible distributions like the S-EL to capture both the common short waits and rare extreme delays accurately.

Table 10: MLE and Goodness of Fit Measure for Geyser Waiting time Data Set

Distribution	Parameters	Estimate	Goodness of Fit Measures
S-EL	α	2.000	AIC (-2.8392e+03)
	θ	1.5001	CAIC (-2.8391e+03)
	λ	0.0119	BIC (-2.8284e+03)
			HQIC (-2.8348e+03)
			MLE (1.4226e+03)
Exponentiated Lomax	α	2.0001	AIC (-2.7675e+03)
	θ	1.5001	CAIC (-2.7674e+03)
	λ	0.0119	BIC (-2.7566e+03)
			HQIC (-2.7631e+03)
			MLE (1.3867e+03)
Pareto (Type I)	α	5.7859e+13	AIC (8.9682e+17)
	θ	-	CAIC (8.9682e+17)
	λ	1.6454e+14	BIC (8.9682e+17)
			HQIC (8.9682e+17)
			MLE (-4.4841e+17)
Weibull	α	-	AIC (2.1740e+03)
	θ	6.4463	CAIC (2.1741e+03)
	λ	76.4036	BIC (2.1812e+03)
			HQIC (2.1769e+03)
			MLE (-1.0850e+03)
Burr	α	36.8873	AIC (-2.1414e+03)
	θ	0.1184	CAIC (-2.1413e+03)
	λ	86.9482	BIC (-2.1306e+03)
			HQIC (-2.1370e+03)
			MLE (1.0737e+03)

Source: Authors Compilation

Table 10 indicates that, the S-EL distribution demonstrates the best fit for the geyser waiting time data, achieving the lowest AIC (-2839.2) and CAIC (-2839.1) values, along with superior BIC (-2828.4) and HQIC (-2834.8) scores compared to all other models. Its performance is further validated by the highest log-likelihood (1422.6), indicating the strongest alignment with the data's light-tailed and left-skewed characteristics. The Exponentiated Lomax shows a moderately good fit but is consistently outperformed by the S-EL across all metrics. The Burr distribution performs respectably but remains inferior to the S-EL, while the Weibull and Pareto models exhibit poor fits, with the Pareto producing implausible parameter estimates and extremely high information criteria values. These results highlight the S-EL's adaptability even for datasets with less extreme tails, effectively capturing the geyser waiting time's unique distributional properties.

CONCLUSION

This study introduces the Sine-Exponentiated Lomax (S-EL) distribution, a novel statistical model specifically developed to address the challenges of modeling heavy-tailed and right-skewed data across various disciplines. The distribution integrates trigonometric transformation with an exponentiated baseline to enhance flexibility in capturing extreme values and complex data patterns. Through comprehensive derivation of its mathematical properties and rigorous parameter estimation via maximum likelihood methods, the S-EL demonstrates robust theoretical foundations. Simulation studies confirm the consistency and efficiency of its estimators, while applications to real-world datasets validate its superior performance over existing models. The S-EL distribution effectively bridges the gap between methodological sophistication and practical utility, offering researchers a powerful tool for reliable statistical modeling in fields requiring accurate representation of extreme-value phenomena.

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