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AN INTRODUCTION TO THE CONCEPT OF FUZZY MULTISET BITOPOLOGY

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ABSTRACT
In this paper, we introduce the concept of fuzzy multiset bitopological spaces. Moreover, the notions of fuzzy submultiset of bitopological space, union, intersection and Hausdorf fuzzy multiset bitopology are studied and presented. The importance of this approach is that, the class of fuzzy multiset bitopological spaces is wider and more general than the class of classical bitopological spaces.

Keywords: Fuzzy Set, Multiset, Fuzzy Multiset, Topology, Fuzzy Topology, Multiset Topology, Fuzzy Multiset Topology, Bitopology, Multiset Bitopology, Fuzzy Multiset Bitopology

INTRODUCTION
Primarily in the area of set theory, fuzzy mathematics differs from conventional mathematics. Fuzzy mathematics had been introduced many years ago, it is full of topics, and it is used widely in many sectors such as vehicles and traffic system. Zadeh introduced the concept of fuzzy set. The notion associates each element with its grade of membership. In 1986 Chang introduced the concept of fuzzy set topology which provides a natural framework for generating many of the concepts of general topology to fuzzy topological spaces and its development. Topology has enormous applications on fuzzy set. In classical set, repetitions of any elements are not allowed. Nevertheless, if we tend to relax this restriction and permit recurrent occurrence of any element, then we find a mathematical tool which we call bags or multiset. In the real world, it is essential, as there are identical things like in a statistical survey repeated data, in water molecule repetitions of hydrogen atoms, repetitions of roots in polynomial and many more. De Bruijin suggested the word multiset (or bag). Grish and John gave the topological structure on multiset and studied some basic properties. Rajish and Sunil Introduced the new concept of subspace multiset topology on a submultiset and compared this subspace multiset topology with already had concepts and analyzed the situation in which they coincide. Again, Rajish, Sunil and Sovan discussed the open cover of a submultiset in to different subspace of multiset topology. Furthermore, Huque, Bhattacharaya and Tripathy presented the notion of closed subspace mixed multiset topologies. After the introduction of multisets, Yager studied the notion of fuzzy multisets (or fuzzy bags). A fuzzy multiset is a collection which simultaneously deals with quantities and degrees of membership of the elements it contains. Bhattacharya [9] initiated the topological structure on fuzzy multisets and studied some of its basic properties and established concepts of general topology on crisp sets to fuzzy multiset topology. The concept of bitopological space was initiated and introduced by American Mathematician J. C. Kelly (John Clive Kelly) in1963. Kelly [10] defined bitopological space as a non-empty set X equipped with two (potentially different) topologies τ_1 and τ_2, denoted as (X, τ_1, τ_2). The concept of multiset bitopological spaces was introduced by Omar and others in 2015. Omar [11] defined the concept of multiset bitopological space as a triple (M, τ_1, τ_2), where M is a non-empty multiset and τ_1, τ_2 are arbitrary multiset bitopologies on M. Tella and Usman [12] studied and extended the concept of multiset topological space to multiset subspace and introduced operations on the various classes of multiset bitopological space to include union and intersection of multiset bitopological space. Tella and Usman [13] again, introduced the concept of Hausdorff bitopological space, root (support) set of a multiset topological space and established that a multiset topological space and it submultiset space are genaralizations of a classical topological space and its subspace via root set respectively.Following the existing literature, we discovered that the ideas of fuzzy submultiset of bitopological space, algebraic operations of fuzzy multiset bitopology, among others have not been adequately examined. Hence in this paper, some deviations between fuzzy multiset bitopology and ordinary bitopology are given. Moreover, the concept of fuzzy multiset bitopological space is introduced. Furthermore, the notions of fuzzy submultiset of bitopological space, union, intersection and Hausedorf fuzzy multiset bitopological space are studied.

MATERIALS AND METHODS
In this section, the preliminary concepts relevant to the main results are collated.

Definition 1: If X is a universe of discourse and x is a particular element of X, then a fuzzy set A defined on X can be written as a collection of ordered pairs $A = \{(x, \mu_{A(x)}), x \in X\}$, where $\mu_A: X \to [0, 1]$ is called the membership function.

Definition 2: The support of fuzzy set $\tilde{A}$ is a subset of universe X, in which each has a membership degree to $\tilde{A}$ greater than zero i.e., Supp $(\tilde{A}) = \{x/x \in X, \mu_{\tilde{A}}(x) > 0$

Definition 3: A multiset M drawn from X is represented by a count function C_M defined as
$C_M: X \to N = \{0,1,2,3, \dots\}$. For $x \in X$, $C_M(x)$ denotes the number of times the element x in the multiset M occurs. The representation of the multiset M drawn from $X = \{x_1, x_2, x_3, \dots, x_n\}$ is $\{x_1, x_2, \dots, x_n\}_{a_1, a_2, \dots, a_n}$.

Definition 4: A domain X is defined as a set of elements from which multisets are constructed. The multiset space $[X]^W$ is the set of all multisets whose elements are in X such that no element in the multisets occurs more than w times.

Definition 5: Let M and N be two multisets. M is called sub multiset (submset) of N written as $M \subseteq N$ or $N \supseteq M$, if $C_M(x) \le C_N(x)$ for all $x \in X$

Definition 6: Let $M \in [X]^W$ be a multiset. The power multiset $P\ (M)$ of M is the multiset of all submultisets of M.The power set of a multiset is the support set of the power multiset and is denoted by $P^*\ (M)$.

Definition 7: Let X be a non-empty set, a fuzzy multiset A drawn from X is characterized by a function 'count membership' of A denoted by CM_A such that $CM_A: X \to Q$ where Q is the set of all crisp multiset drawn from the unit interval $[0, 1]$. Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from $[0, 1]$.
Note that: Throughout this work, we use $\mathcal{M}$ to represents fuzzy multiset

Definition 8: Let F be a fuzzy subset of X. A collection τ of fuzzy subset of F i.e,. $\tau \subset U_F$ is called a fuzzy topology on F if it satisfying the following conditions:
(i) $\emptyset, F \in \tau$;
(ii) $U, V \in \tau \Longrightarrow U \cap V \in \tau$;
(iii) $\{U_i : i \in J\} \subset \tau \Longrightarrow U_{i\in J}\, U_i \in \tau$.
it is called a fuzzy topology on F. The pair (F,τ) is called a fuzzy topological space, members of τ are called fuzzy open sets and their compliments referred to F are called a fuzzy closed set of (F,τ). The family of all fuzzy closed sets in (F,τ) is denoted by τ'_F.
Note: Unless otherwise mentioned by fuzzy topological spaces we shall mean it in the sense of the above definition and (F,τ) will denote a fuzzy topological space

Definition 9: Let $M \in [X]^W$ and $\tau \subseteq P^*\ (M)$. Then, τ is called a multiset topology (for short M-topology) of M if τ satisfies the following properties
i. The multiset M and empty multiset $\emptyset$ are in τ;
ii. The multiset union of the elements of any sub collection of τ is in τ;
iii. The multiset intersection of the elements of any finite sub collection of τ is in τ.
Mathematically, a multiset topological space is an ordered pair (M, τ) consisting of a multiset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M.
Note: τ is an ordinary set whose elements are multisets. Multiset topology is abbreviated as an M- topology.

Definition 10: Let τ be the collection of subsets of fuzzy multiset drawn from the set X.
Then τ is called fuzzy multiset topology if the following conditions are satisfied
i. $\mathcal{M}, \emptyset \in \tau$;
ii. $A \cap B \in \tau$ for any $A, B \in \tau$;
iii. $\cup A \in \tau$ for any arbitrary family $\{A_i: i \in I\} \in \tau$.
The ordered pair $(\mathcal{M}, \tau)$ is called fuzzy multiset topological space (FMT).
Example: Let $X = \{a, b, c, d\}$ be a non-empty set and $\mathcal{M} = \{3/(0.1, a),\ 2/(0.2, b),\ 2/(0.3, c),\ 1/(0.5, d)\}$ be fuzzy multiset. Then,
$\tau = \{\mathcal{M}, \emptyset, \{3/(0.1, a)\}, \{2/(0.2, b)\}, \{2/(0.2, b), 2/(0.3, c\}, \{3/(0.1, a), 2/(0.2, b), 2/(0.3, c)\}$.

Definition 11: A bitopological space is defined as non-empty set X equipped with two (potentially different) topologies τ_1 and τ_2, denoted as (X, τ_1, τ_2). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$.
Example: Let $X = \{a, b, c, d\}$ be a non-empty set. Then,
$\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}\}$
$\tau_2 = \{X, \emptyset, \{a\}, \{c\}, \{a, d\}, \{a, c, d\}\}$ is bitopological space drawn from X

Definition 12: A multiset bitopological space is a triple (M, τ_1, τ_2), where $M \in M(X)$ and τ_1 and τ_2 are arbitrary multiset topologies on M.
Example: Let $X = \{a, b, c, d\}$ be a non-empty set and $M = \{3/a, 2/b, 2/c, 1/d\}$ be multiset
Then, $\tau_1 = \{M, \emptyset, \{3/a\}, \{2/b\}, \{3/a, 2/c\}, \{3/a, 2/b, 2/c\}\}$
$\tau_2 = \{\{M, \emptyset, \{3/a\}, \{2/c\}\{3/a, 2/b\}, \{3/a, 2/b, 2/c\}\}$ is multiset bitopological space on X

RESULTS AND DISCUSSION

In this section, we present the main results of our research

Definition 13 Fuzzy sub multiset space
Let $(\mathcal{M}, \tau)$ be a fuzzy multiset topological space, such that $\mathcal{M} \in \mathcal{M}(X)$, $\mathcal{N} \subseteq \mathcal{M}$. Then, $(\mathcal{N}, \tau_{\mathcal{N}})$, where, $\tau_{\mathcal{N}} = \{U \in \mathcal{M}(X): U = \mathcal{N} \cap V, , V \in \tau\}$
is called a fuzzy submultiset space of the fuzzy multiset topological space.
***Example 1*:** $X = \{a, b, c, d\}$ be a set, let and $\mathcal{M} = \{3/(0.1, a),\ 2/(0.2, b),\ 2/(0.3, c),\ 1/(0.5, d)\}$ be fuzzy multiset. Then,
$\tau = \{\mathcal{M}, \emptyset, \{3/(0.1, a)\}, \{2/(0.2, b)\}, \{2/(0.2, b), 2/(0.3, c\}$ $\{3/(0.1, a), 2/(0.2, b), 2/(0.3, c)\}$. And so $\mathcal{N} \subseteq \mathcal{M}$
$\mathcal{N} = \{2/(0.2, b), 2/(0.3, c)\ 1/(0.5, d)\}$. Then,
$\mathcal{N} \cap \mathcal{M} = \mathcal{N}$, $\mathcal{N} \cap \emptyset = \emptyset$, $\mathcal{N} \cap \{3/(0.1, a)\} = \emptyset$, $\mathcal{N} \cap \{2/(0.2, b)\} = \{2/(0.2, b)\}$,
$\mathcal{N} \cap \{2/(0.2, b), 2/(0.3, c\} = \{2/(0.2, b), 2/(0.3, c\}$,
$\mathcal{N} \cap \{3/(0.1, a), 2/(0.2, b), 2/(0.3, c)\} = \{2/(0.2, b), 2/(0.3, c)\}$. Hence $\tau_{\mathcal{N}}$ to $\mathcal{N}$ is
$\tau_{\mathcal{N}} = \{\mathcal{M}, \emptyset, \{2/(0.2, b)\}, \{2/(0.2, b), 2/(0.3, c\}\}$.
Proposition 1: Let $(\mathcal{M}, \tau)$ be a fuzzy multiset topological space and $\mathcal{N}, \mathcal{M} \in \mathcal{M}(X)$ and $\mathcal{N} \subseteq \mathcal{M}$. Then, the fuzzy sub multiset space $(\mathcal{N}, \tau_{\mathcal{N}})$ is fuzzy multiset topological space.
Proof
Let $(\mathcal{M}, \tau) \in FMT$ and $\mathcal{N}, \mathcal{M} \in \mathcal{M}(X)$ such that $\mathcal{N} \subseteq \mathcal{M}$. Then we show that
$\mathcal{N} \subseteq \mathcal{M} \in FMT$. Since by definition, $\tau_{\mathcal{N}} = \{U \in \mathcal{M}(X): U = \mathcal{N} \cap V, , V \in \tau\}$. But
$\emptyset = \mathcal{N} \cap \emptyset$ (since $\emptyset$ is an open set in $(\mathcal{M}, \tau)$ then $\emptyset$ is also open in $(\mathcal{N}, \tau_{\mathcal{N}})$. Again
$\mathcal{N} = \mathcal{N} \cap \mathcal{M}$ and $\mathcal{M}$ is open in $(\mathcal{M}, \tau)$, then $\mathcal{N}$ is also open in $(\mathcal{N}, \tau_{\mathcal{N}})$. Thus, $\emptyset, \mathcal{N} \in \tau_{\mathcal{N}}$.
Now let (A_α) be a collection of open sets in $\tau_{\mathcal{N}}$. Then, for each $\alpha \in I$ there exist and open set
$B_\alpha \in \tau$ such that $A_\alpha = \mathcal{N} \cap B_\alpha$. This implies that
$\cup_{\alpha\in I} A_\alpha = \cup_{\alpha\in I} \mathcal{N} \cap B_\alpha) = \mathcal{N} \cap (\cup_{\alpha\in I} (B_\alpha)) \in \tau_{\mathcal{N}}$ since $\cup_{\alpha\in I} (B_\alpha) \in \tau$. Therefore the arbitrary union $\cup_{\alpha\in I} A_\alpha$ is open in $\tau_{\mathcal{N}}$.
Once again, suppose A_1….., A_n are open fuzzy multisets in $\tau_{\mathcal{N}}$. Then, There exist
B_1……,$B_n \in \tau$ such that $A_i = \mathcal{N} \cap B_i$. This implies that
$\cap_{i\neq1}^n A_i = \cap_{i\neq1}^n (\mathcal{N} \cap B_i) = \mathcal{N} \cap (\cap_{i\neq1}^n B_i) \in \tau_{\mathcal{N}}$ (since $\cap_{i\neq1}^n B_i \in \tau_{\mathcal{N}}$).
Therefore, $\cap_{i\neq1}^n A_i$ is open in $\tau_{\mathcal{N}}$. Hence $(\mathcal{N}, \tau_{\mathcal{N}})$ is a fuzzy multiset topological space.
In particular, $(\mathcal{N}, \tau_{\mathcal{N}}) \in FMT$

Definition 14 Union of fuzzy multiset topology
Let $(\mathcal{M}, \tau_1), (\mathcal{N}, \tau_2) \in FMT$,
where, $\mathcal{N} \in \mathcal{M}(X)$. The union of these fuzzy multisets topological spaces denoted as

$(\mathcal{M}, \tau_1) \cup (\mathcal{N}, \tau_2)$ is given by $(\mathcal{M}, \tau_1) \cup (\mathcal{N}, \tau_2) = (\mathcal{M} \cup \mathcal{N}, \tau_1 \cup \tau_2)$ where
$\tau_1 \cup \tau_2 = \{U \cup V / U \in \tau_1, V \in \tau_2\}$.

Proposition 2: Let $\mathcal{M}$ be a fuzzy multiset topology and $\mathcal{N} \in \mathcal{M}(X)$. Then,
$(\mathcal{M}, \tau_1) \wedge (\mathcal{N}, \tau_2) \in FMT \Rightarrow (\mathcal{M}, \tau_1) \cup (\mathcal{M}, \tau_2) \in FMT$.

Proof
Now, $(\mathcal{M}, \tau_1) \cup (\mathcal{N}, \tau_2) = (\mathcal{M} \cup \mathcal{N}, \tau_1 \cup \tau_2)$, where $\tau_1 \cup \tau_2 = \{U \cup V / U \in \tau_1, V \in \tau_2\}$.
Note that $U \subseteq \mathcal{M}, V \subseteq \mathcal{N} \Rightarrow U \cup V \subseteq \mathcal{M} \cup \mathcal{N}$. Since $\emptyset, \mathcal{M} \in \tau_1$ and $\emptyset, \mathcal{N} \in \tau_2$. Then, we now have $\emptyset \cup \emptyset = \emptyset, \mathcal{N} \in \tau_1 \cup \tau_2$.
Let $\{U_\alpha \cup V_\beta\}$ be a collection of open sets in $\tau_1 \cup \tau_2$ for each $\alpha\beta \in I$.
Now, $\cup \{U_\alpha \cup V\} = (\cup U_\alpha) \cup (\cup V_\beta)$ since $\cup U_\alpha \in \tau_1$ and $\cup V_\beta \in \tau_2$ (by hypothesis).
Then, we have $(\cup U_\alpha) \cup (\cup V_\beta) \in \tau_1 \cup \tau_2$ (by definition). In particular, $\cup \{ U_\alpha \cup V_\beta\} \in \tau_1 \cup \tau_2$
Let $\{ U_1 \cup V_1, U_2 \cup V_2, \ldots.., U_n \cup V_n\}$ be any finite collection in $\tau_1 \cup \tau_2$. Then,
we have $\cap_{i=1}^{n} (U_i \cup V_i) = (\cap_{i=1}^{n} U_i) \cup (\cap_{i=1}^{n} V_i)$.
Since $\cap_{i=1}^{n} U_i \in \tau_1$ and $\cap_{i=1}^{n} V_i \in \tau_2$ (by hypothesis) then, we have
$\cap_{i=1}^{n} (U_i \cup V_i) = \{(\cap_{i=1}^{n} U_i) \cup (\cap_{i=1}^{n} V_i)\} \in \tau_1 \cup \tau_2$.
This shows that $(\mathcal{M}, \tau_1) \cup (\mathcal{N}, \tau_2) \in FMT$.

Definition 15 Intersection of fuzzy multiset topology
Let $(\mathcal{M}, \tau_1)$ and $(\mathcal{N}, \tau_2) \in FMT$,
where $\mathcal{N} \in \mathcal{M}(X)$. The intersection of these fuzzy multiset topological spaces, denoted as
$(\mathcal{M}, \tau_1) \cap (\mathcal{N}, \tau_2)$ is given by $(\mathcal{M}, \tau_1) \cap (\mathcal{N}, \tau_2) = (\mathcal{M} \cap \mathcal{N}, \tau_1 \cap \tau_2)$, where
$\tau_1 \cap \tau_2 = \{U \cap V / U \in \tau_1, V \in \tau_2\}$.

Proposition 3: Let $\mathcal{M}$ be a fuzzy multiset topology and $\mathcal{N} \in \mathcal{M}(X)$. Then,
$(\mathcal{M}, \tau_1) \wedge (\mathcal{N}, \tau_2) \in FMT \Rightarrow (\mathcal{M}, \tau_1) \cap (\mathcal{N}, \tau_2) \in FMT$

Proof
Now, $(\mathcal{M}, \tau_1) \cap (\mathcal{N}, \tau_2) = (\mathcal{M} \cap \mathcal{N}, \tau_1 \cup \tau_2)$, where $\tau_1 \cap \tau_2 = \{U \cap V / U \in \tau_1, V \in \tau_2\}$.
Note that $U \subseteq \mathcal{M}, V \subseteq \mathcal{N} \Rightarrow U \cap V \subseteq \mathcal{M} \cup \mathcal{N}$, since $\emptyset, \mathcal{M} \in \tau_1$ and $\emptyset, \mathcal{N} \in \tau_2$.
We now have $\emptyset \cap \emptyset = \emptyset, \mathcal{N} \in \tau_1 \cap \tau_2$.
Let $\{U_\alpha \cap V_\beta\}$ be a collection of open sets in $\tau_1 \cap \tau_2$ for each $\alpha\beta \in I$.
Now, $\cup \{U_\alpha \cap V\} = (\cap U_\alpha) \cap (\cap V_\beta)$, since $\cap U_\alpha \in \tau_1$ and $\cap V_\beta \in \tau_2$ (by hypothesis).
Then, we have $(\cap U_\alpha) \cap (\cap V_\beta) \in \tau_1 \cap \tau_2$ (by definition).
In particular, $\cap \{ U_\alpha \cap V_\beta\} \in \tau_1 \cap \tau_2$
Let $\{ U_1 \cap V_1, U_2 \cap V_2, \ldots.., U_n \cap V_n\}$ be any finite collection in $\tau_1 \cap \tau_2$. Then,
we have $\cap_{i=1}^{n} (U_i \cap V_i) = (\cap_{i=1}^{n} U_i) \cap (\cap_{i=1}^{n} V_i)$, since $\cap_{i=1}^{n} U_i \in \tau_1$ and $\cap_{i=1}^{n} V_i \in \tau_2$
(by hypothesis). Then, we have $\cap_{i=1}^{n} (U_i \cap V_i) = \{(\cap_{i=1}^{n} U_i) \cap (\cap_{i=1}^{n} V_i)\} \in \tau_1 \cap \tau_2$.
This shows that $(\mathcal{M}, \tau_1) \cap (\mathcal{N}, \tau_2) \in FMT$
We now introduce the concept of fuzzy multiset bitopological space as follows:

Definition 16 Fuzzy multiset bitopological space
A Fuzzy multiset bitopological space is a
triple $(\mathcal{M}, \tau_1, \tau_2)$, where $\mathcal{M} \in \mathcal{M}(X)$ and τ_1, τ_2 are fuzzy multiset topologies on $\mathcal{M}$. We denote
the class of fuzzy multiset bitopological space by $FMBT$.

Example 2: Let $X = \{a, b, c, d\}$ be a non-empty set and
$\mathcal{M} = \{3/(0.1, a), 1/(0.2, b), 2/(0.3, c), 2/(0.5, d)\}$ be fuzzy multiset. Then,
$\tau_1 = \{\mathcal{M}, \emptyset, \{3/(0.1, a)\}, \{1/(0.2, b)\}, \{3/(0.1, a), 2/(0.3, c)\},$
$\{3/(0.1, a), 1/(0.2, b), 2/(0.3, c)\}\}$ and
$\tau_2 = \{\mathcal{M}, \emptyset, \{3/(0.1, a)\}, \{2/(0.3, c)\}, \{3/(0.1, a), 2/(0.5, d)\},$
$\{3/(0.1, a), 2/(0.3, c), 2/(0.5, d)\}\}$. Here, τ_1 and τ_2 are fuzzy
multiset topologies. This shows that $(\mathcal{M}, \tau_1, \tau_2)$ is fuzzy multiset bitopological space.

Definition 17 Fuzzy submultiset space of bitopological space
Let $(\mathcal{M}, \tau_1, \tau_2)$ be a fuzzy
multiset bitopological space and $\mathcal{M}, \mathcal{N} \in \mathcal{M}(X)$, such that $\mathcal{N} \subseteq \mathcal{M}$. Then, the sub fuzzy
multiset space $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}})$, where $\tau_{1\mathcal{N}} = \{A = U \cap \mathcal{N}, U \in \tau_1$ and
$\tau_{2\mathcal{N}} = \{B = V \cap \mathcal{N}, V \in \tau_2\}$ is called a fuzzy submultiset space of bitopological space.

Theorem 4: Let $(\mathcal{M}, \tau_1, \tau_2) \in FMBT$ and $\mathcal{M}, \mathcal{N} \in \mathcal{M}(X)$, $\mathcal{N} \subseteq \mathcal{M}$.
Then, $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}}) \in FMBT$.

Proof
Since $\tau_{1\mathcal{N}} = \{A = U \cap \mathcal{N}, U \in \tau_1$, and $\tau_{2\mathcal{N}} = \{B = V \cap \mathcal{N}, V \in \tau_2$ (by definition), and also
$(\mathcal{N}, \tau_{1\mathcal{N},} \tau_{2\mathcal{N}}) \in FMT$ (by Proposition 1). In particular $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}}) \in FMBT$ (by definition).

Definition 18 Union and intersection of fuzzy multiset bitopology
Let $(\mathcal{M}, \tau_1, \tau_2)$,
$(\mathcal{M}_1, \tau_{11}, \tau_{12})$ and $(\mathcal{M}_2, \tau_{21}, \tau_{22})$ be fuzzy multiset bitopological space, where $\mathcal{M}_1, \mathcal{M}_2$
$\mathcal{M}, \mathcal{N} \in \mathcal{M}(X)$. We defined union ($\cup$) and intersection ($\cap$) as follows:
(i) ***Union***
$(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cup (\mathcal{M}_2, \tau_{21}, \tau_{22}) = ((\cup_{i=1}^{2} \mathcal{M}_i, \tau_{1i}, \tau_{2i}), i = 1, 2, \ldots..,$ where
$\tau_{1i} \cup \tau_{2i} = \{U_{1i} \cup V_{2i} / U_{11} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$.
(ii) ***Intersection***
$(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cap (\mathcal{M}_2, \tau_{21}, \tau_{22}) = ((\cap_{i=1}^{2} \mathcal{M}_i, \tau_{1i}, \tau_{2i}), i = 1, 2, \ldots\ldots,$ where
$\tau_{1i} \cap \tau_{2i} = \{U_{1i} \cap V_{2i} / U_{11} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$.

Proposition 5: Let $(\mathcal{M}, \tau_1, \tau_2)$, $(\mathcal{M}_1, \tau_{11}, \tau_{12})$, $(\mathcal{M}_2, \tau_{21}, \tau_{22}) \in FMBT$, where
$\mathcal{M}_{1,} \mathcal{M}_2, \mathcal{M}, N \in \mathcal{M}(X)$. Then,
(i) $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cup (\mathcal{M}_2, \tau_{21}, \tau_{22}) \in FMBT$
(ii) $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cap (\mathcal{M}_2, \tau_{21}, \tau_{22}) \in FMBT$

Proof
(i) $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cup (\mathcal{M}_2, \tau_{21}, \tau_{22}) = (\cup_{i=1}^{2} \mathcal{M}_i, \tau_{1i} \cup \tau_{2i})$, $i = 1, 2, \ldots..,$ where
$\tau_{1i} \cup \tau_{2i} = \{U_{1i} \cup V_{2i} / U_{11} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$ (by definition).
Now, $(\mathcal{M}_{i,} \tau_{1i} \cup \tau_{2i}) \in FMBT$, since $\mathcal{M}, \tau_1 \cup \tau_2) \in FMT$. In particular, $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cup (\mathcal{M}_2, \tau_{21}, \tau_{22}) \in FMBT$ (by definition).
(ii) $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cap (\mathcal{M}_2, \tau_{21}, \tau_{22}) = (\cap_{i=1}^{2} \mathcal{M}_i), \tau_{1i}, \tau_{2i})$, $i = 1, 2, \ldots.,$ where
$\tau_{11} \cap \tau_{21} = \{U_{1i} \cap V_{2i} / U_{1i} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$ (by definition).
Now, this shows that $(\cap_{i=1}^{2} \mathcal{M}_i, \tau_{1i}, \tau_{2i}) \in FMBT$, since $(\mathcal{M}, \tau_1 \cap \tau_2) \in FMT$.
Therefore in particular, $(\mathcal{M}_1, \tau_{11}, \tau_{12}) \cap (\mathcal{M}_2, \tau_{21}, \tau_{22}) \in FMBT$

Definition 19: Hausdorff fuzzy multiset bitopological space

Let $(\mathcal{M}, \tau_1, \tau_2) \in FMBT$.

Then, $(\mathcal{M}, \tau_1, \tau_2)$ is a Hausdorff fuzzy multiset bitopological space if and only if for any simple fuzzy multisets $\{\{k_1/(x,\ \mu_{(x)})\}, \{k_2/(y, \mu_{(y)})\}\} \in \mathcal{M}$, where $x \neq y$ and $\mu_{(x)} \neq \mu_{(y)}$, there exist $U \in \tau_1$ and $V \in \tau_2$ such that

$\{k_1/(x,\ \mu_{(x)})\} \subseteq U$, $\{k_2/(y, \mu_{(y)})\} \in V$ and $U \cap V = \emptyset$

We denote the class of Hausdorff fuzzy multiset bitopological space by $HFMBT$

Theorem 6: Let $(\mathcal{M}, \tau_1,\ \tau_2) \in HFMBT$ and $\mathcal{N} \in \mathcal{M}(X)$ such that $\mathcal{N} \subseteq \mathcal{M}$. Then $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}}) \in HFMBT$

Proof

Let $(\mathcal{M}, \tau_1,\ \tau_2) \in HFMBT$ and $(\mathcal{N}, \tau_{1\mathcal{N}}\ \tau_{2\mathcal{N}})$ be it's sub fuzzy multiset space. Then, we show that $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}})$ is Hausdorff fuzzy multiset topological space clearly $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}}) \in FMBT$ (by definition), $\{\{k_1/(x,\ \mu_{(x)})\}, \{k_2/(y, \mu_{(y)})\}\} \subseteq \mathcal{N}$ such that $x \neq y$ since $\mathcal{N} \subseteq \mathcal{M}$, then $\{\{k_1/(x,\ \mu_{(x)})\}, \{k_2/(y, \mu_{(y)})\}\} \subseteq \mathcal{M}$ since$(\mathcal{M}, \tau_1,\ \tau_2) \in HFMBT$ (by hypothesis). And also $A \in \tau_1$ and $B \in \tau_2$ such that $\{k_1/(\mu_{(x)}, x)\} \subseteq A$ and $\{k_2/(\mu_{(y)}, y)\} \subseteq B$ and $A \cap B = \emptyset$.

But $A \in \tau \Longrightarrow \mathcal{N} \cap A \in \tau_{1\mathcal{N}}$ and $B \in \tau_2 \Longrightarrow \mathcal{N} \cap B \in \tau_{2\mathcal{N}}$.

In particular $\{k_1/(x,\ \mu_{(x)})\} \subseteq \mathcal{N} \cap A$ and $\{k_2/(y, \mu_{(y)})\} \subseteq \mathcal{N} \cap B$.

But $(\mathcal{N} \cap A) \cap (\mathcal{N} \cap B) = \mathcal{N} \cap (A \cap B) = \mathcal{N} \cap \emptyset = \emptyset$.

Hence, $(\mathcal{N}, \tau_{1\mathcal{N}}, \tau_{2\mathcal{N}}) \in HFMBT$.

CONCLUSION

In this article, some deviations between fuzzy multiset bitopology and ordinary bitopology were given. Moreover, the concept of fuzzy multiset bitopological space was introduced. Furthermore, the notions of fuzzy submultiset of bitopological space, algebraic operations of fuzzy multiset bitopology and Hausdorf fuzzy multiset bitopological space were studied and presented.

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