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COMPARATIVE BAYESIAN AND CLASSICAL ESTIMATION OF THE SCALE PARAMETER IN THE WEIBULL POWER FUNCTION DISTRIBUTION

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ABSTRACT

We investigate the scale parameter of the Weibull power function distribution (WPFD) via both Bayesian and traditional statistical methodologies. Diverse estimations for the scale parameter were derived from the Bayesian framework, employing three distinct loss functions. The findings revealed that quadratic loss functions, utilising Jeffrey's and Gamma priors, consistently yielded better results than precautionary and squared error loss functions, irrespective of sample size. As the sample size increased, the estimation errors diminished, and the calculated values converged towards the true scale parameter. In conclusion, the Bayesian estimates for the scale parameter, particularly those utilising Jeffrey's and Gamma priors with a quadratic loss function, demonstrated superior performance compared to other estimation techniques.

Keywords: Weibull exponential distribution, Maximum likelihood estimation, Bayesian estimation, Gamma prior, Jeffrey prior

INTRODUCTION

Within the realm of probability and statistics, the Weibull power function distribution (WPFD) represents an advanced form of the exponential distribution. The necessity for extending the exponential distribution arises primarily from its inherent characteristic of a constant failure rate, which limits its applicability in modelling certain real-world phenomena. The WPFD offers enhanced flexibility and is thus more suitable for describing a wider array of practical situations. Tahir et al. (2016) conducted an in-depth examination of the WPFD, exploring its properties and potential applications. From a Bayesian perspective, the estimation process necessitates the choice of a prior distribution for the parameters under consideration. Conversely, classical statistical methods do not require any prior information about these parameters. There is no universally accepted method within Bayesian statistics to definitively determine which prior is superior. The choice of prior often reflects the analyst's preference. However, if existing in sequence concerning the parameter(s) is available, utilising an informative prior is generally recommended; or else, a non-informative prior may be considered. This research aims to investigate the application of Gamma and Jeffrey's priors for parameter estimation in this circumstance. The ongoing efforts to develop novel distributions capable of modelling dynamic data evolution are critically important. Consequently, numerous distributions have emerged, including the Weibull power function distribution (Tahir *et al*., 2016), Type I half Logistic Topp-Leone exponential distribution (Adepoju *et al*., 2023), and others such as the Type I half Logistic Topp-Leone Inverse Lomax model (Adepoju *et al*., 2024a) and Cosine Marshall Olkin Weibull distribution (Adepoju *et al*., 2024b). The classical approach to parameter estimation, unlike its Bayesian counterpart, does not rely on pre-existing information about the parameter. This is evident in the works of ZeinEldin *et al*. (2019), Yilmaz *et al*. (2021), and others. In contrast, the Bayesian methodology mandates a suitable selection of prior information for the parameters. Parameter estimation for the exponential distribution and its variations has been extensively explored. Oguntunde *et al.* (2015) estimated Weibull-exponential model parameters using Maximum Likelihood. Aliyu and Yahaya (2016) investigated the scale parameter of the Generalised Rayleigh distribution with non-informative priors under various loss functions. Ieren and Oguntunde (2018) estimated the scale parameter of the Weibull exponential distribution using Jeffrey and uniform priors. Danrimi, along with Abubakar (2023), proposed a Bayesian method for the two-parameter Weibull distribution, finding Bayesian estimates superior to Maximum Likelihood. Liu *et al*. (2021) compared classical and Bayesian methods for power function distribution, concluding that Bayesian estimates were more efficient. Adepoju *et al*. (2021a, 2021b) also favoured Bayesian methods, particularly with quadratic loss functions. Monte Carlo simulations by Eraikhuemen *et al*. (2020a, 2020b), Ieren *et al*. (2020), Preda *et al*. (2010), and Dey (2010) similarly supported Bayesian approaches over Maximum Likelihood Estimates (MLEs). Some other distributions were developed and found to be powerful, making them a more useful candidate in various fields such as medical, engineering, survival analysis, insurance, hydrology, economics, and so on. Such a model can be found in Sadiq *et al* (2022), Sadiq *et al* (2024), Sadiq *et al* (2023a), Kajuru *et al* (2023), Sadiq *et al* (2023b), Mohammed *et al*. (2025), Sadiq *et al* (2023c), Obafemi *et al* (2024), Habu *et al* (2024), Semary *et al* (2025), Sadiq *et al*. (2025a) and Abd Elgawad *et al*. (2025), Sadiq *et al*. (2025b), Mohammed *et al*. (2025) to mention but few.

Although the Weibull Power Function Distribution (WPFD) has been recognised for its flexibility in modelling reliability and lifetime data, existing research has largely concentrated on its structural properties, general estimation strategies, and simultaneous estimation of multiple parameters. However, there is a noticeable lack of studies that specifically examine and compare classical and Bayesian estimation techniques for the scale parameter of the WPFD. While Bayesian methods have been successfully applied to related Weibull-type distributions, the role of different prior distributions and loss functions in improving the estimation accuracy of the WPFD scale parameter has not been systematically investigated. In particular, empirical evidence comparing the performance of Jeffreys and Gamma priors with classical maximum likelihood estimation, especially under small and moderate sample sizes, is currently absent from the literature. This gap calls attention to the need for a focused study on scale-parameter estimation within the WPFD framework.

The primary aim of this study is to conduct a comparative evaluation of classical and Bayesian estimation methods for the scale parameter of the Weibull Power Function Distribution. Specifically, the study investigates how Jeffreys and Gamma priors perform under three different loss functions within the Bayesian framework. Additionally, the study seeks to assess and compare the efficiency, bias, and overall accuracy of these estimators through an extensive simulation analysis across varying sample sizes.

MATERIALS AND METHODS

The Weibull-Power function distribution's probability density function (pdf) and cumulative distribution function (cdf) are given by formulas (1) and (2), as established by Tahir *et al.* (2016). The parameters α and β be identified as scale parameters of the WPFD.

$$f_{WPFD}(k) = \frac{ABC^D Dk^{DB-1}}{(C^D-k^D)^{B+1}} e^{-A\left[\frac{k^D}{C^D-k^D}\right]^B} \quad (1)$$

$$F_{WPFD}(k) = 1 - e^{-A\left[\frac{k^D}{C^D-k^D}\right]^B} \quad (2)$$

Estimation Method

We detail the estimation of the Weibull-power function distribution's scale parameter via Maximum Likelihood and Bayesian approaches.

Maximum Likelihood Estimation

For a sample from population K with a specified PDF, the chance function H(K|C, D, B, A) quantifies the combined density of its random variables. Equation (3) provides this function for the WPFD.

$$H(K|C,D,B,A) \propto$$

$$(ABC^D D)^n \prod_{i=1}^{n}\left(\frac{k^{DB-1}}{(C^D-k^D)^{B+1}}\right) e^{-A\left[\frac{k^D}{C^D-k^D}\right]^B} \quad (3)$$

The chance function for A is given by;

$$H(K|A) = \eta A^n e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}$$

Where $\eta = (\theta\alpha^\beta\beta)k_i^{\beta\theta-1}(\alpha^\beta - k_i^\beta)^{-\theta-1}$ is a constant independent of the scale parameter$A$.

The log-likelihood function,$p = log\, H\,(K|A)$, is therefore given as:

$$p = n\, log\, A - A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B$$

Differentiating p partially with respect to$A$, and solving for $\hat{A}$, yields the MLE for the scale parameter, as shown in equation (4)

$$\hat{A} = n\left(\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{-1} \quad (4)$$

In deriving the likelihood function and the closed-form maximum likelihood estimator (MLE) for the scale parameter A, we explicitly assume that the shape parameters α, β and B are known. This assumption is necessary because the closed-form expression for the MLE of A (Equation 4) is valid only under known nuisance parameters. When α is unknown, the likelihood no longer separates cleanly with respect to A, and a closed-form estimator cannot be obtained; numerical optimisation would be required instead. Therefore, the classical estimator presented in this study is derived under the condition that α is known.

"For the purpose of deriving a closed-form MLE, we assume that the parameter α is known. Under this assumption, the log-likelihood simplifies accordingly and leads to the analytic estimator presented in Equation (4)."

Bayesian Estimation Method

Posterior Distributions

In statistical inference, the likelihood function is defined as the joint probability mass otherwise density function of the observed data, where the parameters are treated as variables and the data as fixed. For independent observations$\underset{\sim}{k} = (k_1, k_2, \ldots, k_3)$, the likelihood function can be expressed as:

$$H(\underset{\sim}{k}|C,D,B,A) = P(k_1,k_2,\ldots,k_n|C,D,B,A) = \prod_{i=1}^{n} P(\underset{\sim}{k}|C,D,B,A)$$

Bayes' Theorem is employed to derive the posterior model$P(A|\underset{\sim}{k})$, which quantifies the updated probability model of the parameter λ given the observed data.

$$P(A|\underset{\sim}{k}) = \frac{p(A)L(\underset{\sim}{k}|A)}{g(\underset{\sim}{k})} \quad (5)$$

Here, g(k) is defined as the marginal distribution of K.

$$g(k) = \int_{-\infty}^{\infty} p(A)H(\underset{\sim}{k}|A)$$

Here, $p(A)$denotes the prior model, and $H(\underset{\sim}{k}|B)$represents the likelihood function.

Posterior Distribution of the Scale Parameter under the Assumption of Jeffrey's Prior

The non-informative Jeffrey's prior designed for the scale parameter A of the WPFD is given by:

$$p(A) \propto \frac{1}{A}; 0 < A < \infty \quad (6)$$

Utilising Jeffrey's prior, the posterior distribution of the scale parameter $A$for a given dataset is defined as:

$$p(A|\underset{\sim}{k}) = \frac{p(A)H(A|\underset{\sim}{k})}{\int_0^\infty p(A)H(A|\underset{\sim}{k})dA} \quad (7)$$

Now, let

$$K = \int H(\underset{\sim}{k}|A)P(A)dA \quad (8)$$

Substituting for $P(A)$ and $H(\underset{\sim}{k}|A)$; we have:

$$K = \eta\int_0^\infty A^{n-1} e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B} d\alpha \quad (9)$$

Through integration by substitution in equation (9), we obtain:

Let $u = A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B \Rightarrow A = \frac{u}{\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}$$

Upon substituting AanddA into Equation (9) and performing subsequent simplification, we obtain:

$$K = \eta\int_0^\infty \left(\frac{u}{\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}\right)^{n-1} e^{-u} \frac{du}{\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}$$

$$K = \eta\frac{1}{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n}\int_0^\infty u^{n-1}e^{-u}du \quad (10)$$

Also recall that $\int_0^\infty w^{a-1}e^{-w}dt = \Gamma(a)$and that $\int_0^\infty w^a e^{-w}dt = \int_0^\infty w^{a+1-1}e^{-w}dt = \Gamma(a+1)$

Hence;

$$M = \frac{\eta\Gamma(n)}{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n} \quad (11)$$

By substituting M,$P(A)$, and $H(\underline{k}|A)$into equation (7) and performing the necessary simplifications, the posterior distribution under Jeffrey's prior is derived as:

$$P(\mathrm{A}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n}\mathrm{A}^{n-1}\mathrm{e}^{-\mathrm{A}\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}}}{\Gamma(n)} \quad (12)$$

Posterior Distribution of the Scale Parameter under the Assumption of Gamma Prior

As a conjugate prior for the scale parameter $A$of the Weibull-Power function distribution (WPFD), the gamma prior is distinct as:

$$p(A)=\frac{a^b}{\Gamma(b)}A^{b-1}e^{-a\alpha};a,b,A>0 \quad (13)$$

Under a gamma prior, the posterior distribution of the scale parameter $A$for a given dataset is defined as:

$$p(A|\underline{k})=\frac{p(A)L(A|\underline{k})}{\int_0^\infty p(A)L(A|\underline{k})dA} \quad (14)$$

Now, let

$$M_3=\int_0^\infty p(A)H(A|\underline{k})dA \quad (15)$$

Substituting for $p(A)$ and $H(A|\underline{k})$; we have:

$$M_3=\int_0^\infty \frac{a^b}{\Gamma(b)}A^{b-1}e^{-aA}\left(\eta A^n e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}\right)dA$$

$$M_3=\frac{\eta a^b}{\Gamma(b)}\int_0^\infty A^{n+b-1}\,e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}dA \quad (16)$$

Through integration by substitution in equation (16), we obtain:

Let $u=A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)\Rightarrow A=\frac{u}{\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}$

$$\frac{du}{dA}=a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B$$

$$dA=\frac{du}{a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}$$

Upon substituting $A$and dAinto Equation (16) and performing subsequent simplification, we obtain:

$$M_3=\frac{\eta a^b}{\Gamma(b)}\int_0^\infty\left(\frac{u}{\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}\right)^{n+b-1}e^{-u}\frac{du}{\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}$$

$$M_3=\frac{\eta a^b}{\Gamma(b)}\frac{1}{\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}}\int_0^\infty u^{n+b-1}\,e^{-u}du$$

$$M_3=\frac{\eta a^b\Gamma(a+b)}{\Gamma(b)\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}} \quad (17)$$

By substitutingM_3,$p(A)$, and $H(A|\underline{k})$ into Equation (14) and performing the necessary simplifications, the posterior distribution under a gamma prior is consequent as:

$$p(A|\underline{k})=\frac{\frac{\eta a^b}{\Gamma(b)}A^{n+b-1}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\frac{\eta a^b\Gamma(a+b)}{\Gamma(b)\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}}}$$

$$p(A|\underline{k})=\frac{A^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\Gamma(a+b)} \quad (18)$$

Bayesian Estimation under Jeffrey's Prior Using Three Loss Functions

Estimation of the WPFD's scale parameter is performed under three specified loss functions, employing the posterior distribution derived via Jeffrey's prior.

Using Squared Error Loss Function (SELF$_u$)

Employing the Squared Error Loss Function (SELF$_u$) and Jeffrey's prior, the derivation of the Bayes estimator yields:

$$A_{SELFu}=E(A|\underline{k})$$

$$E(A|\underline{k})=\int_0^\infty Ap(A|\underline{k})\,dA \quad (19)$$

For Jeffrey's prior, it is noted that:

$$P(\mathrm{A}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n}\mathrm{A}^{n-1}\mathrm{e}^{-\mathrm{A}\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}}}{\Gamma(n)}$$

Upon substituting $P(A|\underline{k})$into Equation (19), we obtain:

$$E(\mathrm{A}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n}}{\Gamma(n)}\int_0^\infty \mathrm{A}^{n}\mathrm{e}^{-\mathrm{A}\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}}d\mathrm{A} \quad (20)$$

Applying the method of integration by substitution to Equation (20) and performing subsequent algebraic simplification yields:

$$E(\mathrm{A}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{-1}}{\Gamma(n)}\int_0^\infty u^{n+1-1}\mathrm{e}^{-u}du$$

$$A_{SELF}=E(A|\underline{k})=n\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-1} \quad (21)$$

Under Quadratic Loss Function (QLF$_u$)

For Bayesian estimation of the scale parameter, we adopt the normalised quadratic loss $(L(\theta,\delta)=(\delta-\theta)^2/\theta^2)$. This loss penalises estimation error in relative rather than absolute terms, which is appropriate for positive scale parameters where multiplicative precision is more meaningful. Normalised or relative quadratic loss has been used in Bayesian decision-theoretic analyses (Zellner, 1986; Varian, 1975; Alduais, 2021; Ishaq *et al.*, 2021) and provides inference that is invariant under rescaling of the parameter.

The Bayes estimator is derived using the Quadratic Loss Function (QLF$_u$) and Jeffrey's prior, resulting in:

$$A_{QLFu}=\frac{E(A^{-1}|\underline{k})}{E(A^{-2}|\underline{k})}=\frac{\int_0^\infty A^{-1}P(A|\underline{k})dA}{\int_0^\infty A^{-2}P(A|\underline{k})dA}$$

$$E(A^{-1}|\underline{k})=\int_0^\infty A^{-1}P(A|\underline{k})dA \quad (22)$$

For Jeffrey's prior, it is noted that:

$$P(\mathrm{A}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n}\mathrm{A}^{n-1}\mathrm{e}^{-\mathrm{A}\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}}}{\Gamma(n)}$$

Upon substituting $P(A|\underline{k})$into Equation (22), we obtain:

$$E(\mathrm{A}^{-1}\mid\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n}}{\Gamma(n)}\int_0^\infty \mathrm{A}^{n-2}\mathrm{e}^{-\mathrm{A}\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}}d\mathrm{A} \quad (23)$$

Applying integration by substitution to Equation (23) and subsequently simplifying the expression yields:

$$E(A^{-1}|\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]}{(n-1)} \quad (24)$$

Similarly;

$$E(A^{-2}|\underline{k})=\int_0^\infty A^{-2}P(A|\underline{k})dA \quad (25)$$

For Jeffrey's prior, it is noted that:

$$P(A\mid \underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n A^{n-1}e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}}{\Gamma(n)}$$

Upon substituting $P(A|\underline{k})$ into Equation (25), we obtain:

$$E\left(A^{-2}\mid \underline{k}\right)=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n}{\Gamma(n)}\int_0^\infty A^{n-3}e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}dA \quad (26)$$

By applying the method of integration by substitution to Equation (26) and simplifying the result, we obtain:

$$E(A^{-2}|\underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^2}{(n-1)(n-2)} \quad (27)$$

Note that

$$A_{QLFu}=\frac{E(A^{-1}|\underline{k})}{E(A^{-2}|\underline{k})}$$

This indicates that

$$A_{QLFu}=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]}{(n-1)}\div\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^2}{(n-1)(n-2)}$$

$$A_{QLFu}=\frac{(n-2)}{\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B} \quad (28)$$

Using Precautionary Loss Function (PLF$_u$)

The Bayes estimator, derived via the Precautionary Loss Function (PLFu) and Jeffrey's prior, is given by:

$$A_{PLFu}=\{E(A^2)\}^{\frac{1}{2}}=\{E(A^2|\underline{k})\}^{\frac{1}{2}}=\sqrt{E(A^2|\underline{k})}$$

$$E(A^2|\underline{x})=\int_0^\infty A^2p(A|\underline{k})dA \quad (29)$$

For Jeffrey's prior, it is noted that:

$$P(A\mid \underline{k})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n A^{n-1}e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}}{\Gamma(n)}$$

Upon substituting $P(A|\underline{k})$ into Equation (29), we obtain:

$$E\left(A^{2}\mid \underline{k}\right)=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^n}{\Gamma(n)}\int_0^\infty A^{n+1}e^{-A\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B}dA \quad (30)$$

By applying integration by substitution to Equation (30) and simplifying the result, we arrive at:

$$E\left(A^{2}\mid \underline{k}\right)=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-2}}{\Gamma(n)}\int_0^\infty u^{n+2-1}e^{-u}du$$

$$E(A^2|\underline{k})=n(n+1)\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-2} \quad (31)$$

$$A_{PLFu}=[n(n+1)]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-1} \quad (32)$$

Bayesian Estimation under Gamma Prior Using Three Loss Functions

The scale parameter of the WPFDs is estimated via three distinct loss functions, leveraging the posterior distribution derived from a Gamma prior.

Using Squared Error Loss Function (SELF$_u$)

Employing the Squared Error Loss Function (SELF$_u$) and a Gamma prior, the derivation of the Bayes estimator yields:

$$A_{SELFu}=E(A|\underline{k})$$

$$E(A|\underline{k})=\int_0^\infty Ap(A|\underline{k})\,dA \quad (33)$$

For the gamma prior, it is noted that:

$$p(A|\underline{k})=\frac{A^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\Gamma(n+b)}$$

Upon substituting $P(A|\underline{k})$ into Equation (33), we obtain:

$$E\left(A\mid \underline{k}\right)=\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{n+b}}{\Gamma(n+b)}\int_0^\infty A^{n+b}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}dA \quad (34)$$

Integrating Equation (34) by substitution and simplifying gives

$$E\left(A\mid \underline{k}\right)=\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-1}}{\Gamma(n+b)}\int_0^\infty u^{n+b+1-1}e^{-u}du$$

$$A_{SELF}=E(A|\underline{k})=(n+b)\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-1} \quad (35)$$

Under Quadratic Loss Function (QLF$_u$)

The Bayes estimator, derived using a Gamma prior and a Quadratic Loss Function (QLF$_u$), is given by:

$$A_{QLFu}=\frac{E(A^{-1}|\underline{x})}{E(A^{-2}|\underline{x})}=\frac{\int_0^\infty A^{-1}P(A|\underline{k})dA}{\int_0^\infty A^{-2}P(A|\underline{k})dA}$$

$$E(A^{-1}|\underline{k})=\int_0^\infty A^{-1}P(A|\underline{k})dA \quad (36)$$

As a reminder, when considering a Gamma prior

$$p(A|\underline{k})=\frac{A^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\Gamma(n+b)}$$

By substituting $P(A|\underline{k})$ into Equation (36) and subsequently simplifying, we obtain:

$$E(A^{-1}|\underline{x})=\frac{\left[\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]}{(n+b-1)} \quad (37)$$

Similarly;

$$E(A^{-2}|\underline{k})=\int_0^\infty A^{-2}P(A|\underline{k})dA \quad (38)$$

As a reminder, when considering a Gamma prior

$$p(A|\underline{k})=\frac{A^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\Gamma(n+b)}$$

By substituting $P(A|\underline{k})$ into Equation (38) and subsequently simplifying, we obtain:

$$E\left(A^{-2}\mid \underline{x}\right)=\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{n+b}}{\Gamma(n+b)}\int_0^\infty A^{n+b-3}e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}dA \quad (39)$$

Through integration by substitution of Equation (39) and subsequent simplification, we obtain:

$$E(A^{-2}|\underline{k})=\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^2}{(n+b-1)(n+b-2)} \quad (40)$$

As a reminder

$$A_{QLFu}=\frac{E(A^{-1}|\underline{k})}{E(A^{-2}|\underline{k})}$$

This implies that

$$A_{QLFu}=\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]}{(n+b-1)}\div\frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^2}{(n+b-1)(n+b-2)}$$

$$A_{QLFu} = \frac{(n+b-2)}{a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B} \quad (41)$$

Using Precautionary Loss Function (PLF_u)

We obtain the Bayes estimator by employing a Precautionary Loss Function (PLF_u) and assuming a Gamma prior, as follows:

$$A_{PLFu} = \{E(A^2)\}^{\frac{1}{2}} = \sqrt{E(A^2|\underline{k})}$$

$$E(A^2|\underline{k}) = \int_0^{\infty} A^2 p(A|\underline{k})dA \quad (42)$$

As a reminder, when considering a Gamma prior

$$p(A|\underline{k}) = \frac{A^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)^{n+b} e^{-A\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right)}}{\Gamma(n+b)}$$

Upon substituting $P(A|\underline{k})$ into Equation (42) and simplifying, we find:

$$E\left(\mathrm{A}^2 \mid \underline{k}\right) = \frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{n+b}}{\Gamma(n+b)} \int_0^{\infty} \mathrm{A}^{n+b+1} \mathrm{e}^{-\mathrm{A}\left(a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right)} d\mathrm{A} \quad (43)$$

By performing integration by substitution in Equation (43) and simplifying the expression, we obtain:

$$E\left(\mathrm{A}^2 \mid \underline{k}\right) = \frac{\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^{\mathrm{B}}\right]^{-2}}{\Gamma(n+b)} \int_0^{\infty} u^{n+b+2-1} \mathrm{e}^{-u} du$$

$$E(A^2|\underline{k}) = \frac{\Gamma(n+b+2)\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-2}}{\Gamma(n+b)}$$

$$A_{PLFu} = \{E(A|\underline{k})\}^{\frac{1}{2}} = \left\{(n+b)(n+b+1)\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-2}\right\}^{\frac{1}{2}}$$

$$A_{PLF} = [(n+b)(n+b+1)]^{\frac{1}{2}}\left[a+\sum_{i=1}^{n}\left[\frac{k^D}{C^D-k^D}\right]^B\right]^{-1}$$

RESULTS AND DISCUSSION

Simulation Results

We conducted a Monte Carlo simulation study to evaluate the performance of classical and Bayesian estimators for the scale parameter of the Weibull Power Function Distribution (WPFD) under various priors and loss functions. Each simulation scenario was replicated 5,000 times to ensure stable and reliable estimates of the average value and Mean Squared Error (MSE) for each estimator. The results were computed by averaging the outcomes across all replications, allowing for an accurate comparison of estimator bias, variance, and overall efficiency. The sample sizes considered were (n = 25, 100, 300, 500), reflecting small to moderately large datasets commonly encountered in practice.

"All reported estimates and MSEs in Tables 1, 2 and 3 are based on 5,000 Monte Carlo replications to ensure numerical stability and reliable assessment of estimator performance."

Table 1: Simulation Results for Estimation of the Scale Parameter using Various Priors and Loss Functions with $A = 0.5, B = 2.5, C = 0.5, D = 0.5, a = 1.0$ and $b = 1$

n	Measures	MLE	Jeffrey's Prior			Gamma Prior		
			SELFu	QLFu	PLFu	SELFu	QLFu	PLFu
25	Estimate	0.4433	0.4434	0.4771	0.4699	0.439	0.4789	0.4471
	MSE	0.0747	0.0714	0.0697	0.076	0.0705	0.069	0.0727
100	Estimate	0.4452	0.4453	0.489	0.479	0.4454	0.495	0.4499
	MSE	0.0718	0.0708	0.0614	0.0719	0.0704	0.0687	0.0712
300	Estimate	0.4707	0.4763	0.4957	0.482	0.4445	0.4979	0.4548
	MSE	0.0715	0.0707	0.0609	0.071	0.07	0.0679	0.0707
500	Estimate	0.486	0.4833	0.4977	0.4894	0.4443	0.4982	0.481
	MSE	0.0711	0.0705	0.0544	0.0709	0.0684	0.0666	0.0707

Table 1 presents the average estimates and Mean Squared Errors (MSEs) for the scale parameter of the WPFD under classical MLE and Bayesian estimation using Jeffreys and Gamma priors across different loss functions and sample sizes. Overall, the results show that all estimators improve as the sample size increases, with decreasing MSE values indicating enhanced estimation accuracy and stability.

For small samples (e.g., $n = 25$), all methods tend to slightly underestimate the true scale parameter (A = 0.5), although Bayesian estimators under the Quadratic Loss Function (QLF) and Precautionary Loss Function (PLF) generally perform better than MLE in terms of lower MSE. The QLF in particular yields consistently lower MSEs for both Jeffreys and Gamma priors, demonstrating its robustness even at small sample sizes.

As the sample size increases to $n = 100$ and beyond, Bayesian estimators, especially those associated with Gamma and Jeffreys priors under the QLF, continue to show improved performance. They produce estimates that converge more closely to the true parameter value, while also achieving the smallest MSEs compared to MLE and other Bayesian loss functions. Across all sample sizes, the Bayesian QLF estimator with the Gamma prior achieves the lowest or near-lowest MSE, confirming its superior accuracy and efficiency. For larger samples $n = 300$ and $n = 500$, the estimates from all methods become more stable, with values approaching the true parameter A. However, the QLF-based Bayesian estimators maintain their advantage, consistently producing the minimum MSE, confirming their asymptotic superiority. The PLF and SELF estimators also exhibit improved precision with increasing n, but their performance remains slightly less efficient than QLF-based estimators.

The results in Table 1 demonstrate that Bayesian estimation, particularly with Gamma and Jeffreys priors under the Quadratic Loss Function, provides the most accurate and reliable estimates for the scale parameter of the WPFD across different sample sizes. The MLE method remains competitive but is consistently outperformed by Bayesian QLF estimators in terms of MSE reduction and convergence to the true parameter.

Table 2: Simulation Results for Estimation of the Scale Parameter using Various Priors and Loss Functions with $A = 0.5, B = 0.5, C = 2.5, D = 0.5, a = 1.0$ and$b = 1.0$

n	Measures	MLE	Jeffrey's Prior			Gamma Prior		
			SELFu	QLFu	PLFu	SELFu	QLFu	PLFu
25	Estimate	0.4431	0.4436	0.4774	0.4699	0.4392	0.479	0.4472
	MSE	0.0746	0.0713	0.0691	0.076	0.0704	0.069	0.0726
100	Estimate	0.4455	0.4456	0.4892	0.4796	0.4456	0.4954	0.4499
	MSE	0.0717	0.0707	0.0613	0.0719	0.0703	0.0687	0.0712
300	Estimate	0.4706	0.4766	0.4959	0.4825	0.4445	0.4979	0.4549
	MSE	0.0715	0.0706	0.0608	0.071	0.0701	0.0678	0.0706
500	Estimate	0.4862	0.4836	0.4979	0.4896	0.4445	0.4984	0.4811
	MSE	0.071	0.0704	0.0542	0.0708	0.0684	0.0665	0.0705

Table 2 presents the average estimates and Mean Squared Errors (MSEs) for the scale parameter of the WPFD under classical Maximum Likelihood Estimation (MLE) and Bayesian estimation with Jeffreys and Gamma priors across three different loss functions, for varying sample sizes.
The results indicate a pattern similar to Table 1. For small sample sizes $n = 25$, all estimators tend to slightly underestimate the true scale parameter $A = 0.5$. Among Bayesian approaches, the Quadratic Loss Function (QLF) consistently provides estimates closer to the true value and yields lower MSEs compared to the Self (SELFu) and Precautionary Loss Function (PLFu) estimators. This demonstrates that QLF is robust even when sample information is limited.
As the sample size increases to $n = 100$ and $n = 300$, Bayesian QLF estimators under both Jeffreys and Gamma priors continue to outperform other estimators in terms of accuracy and precision. Estimates converge steadily toward the true scale parameter, while the MSEs decrease, indicating improved efficiency. The SELF and PLF estimators also improve with sample size, but their MSEs remain slightly higher than those of the QLF-based estimators.
For large samples $n = 500$, all methods produce stable estimates approaching the true parameter. However, the QLF-based Bayesian estimators retain a consistent advantage, showing the lowest MSEs across all sample sizes. MLE estimates, while improving with larger samples, are slightly less precise than the QLF-based Bayesian estimates, confirming the asymptotic efficiency of Bayesian QLF estimators under the considered priors.
The results in Table 2 reinforce the superiority of Bayesian estimation with QLF, particularly using Jeffreys and Gamma priors, for estimating the scale parameter of the WPFD. The consistent reduction in MSEs with increasing sample size highlights both the efficiency and reliability of these estimators.

Table 3: Simulation Results for Estimation of the Scale Parameter using Various Priors and Loss Functions with $A = 2.5, B = 0.5, C = 0.5, D = 1.0, a = 1.0$ and $b = 1.0$

n	Measures	MLE	Jeffrey's Prior			Gamma Prior		
			SELFu	QLFu	PLFu	SELFu	QLFu	PLFu
25	Estimate	2.4378	2.4128	2.4583	2.447	2.4422	1.8815	2.4061
	MSE	0.5705	0.5695	0.5591	0.5613	0.5695	0.5599	0.5613
100	Estimate	2.441	2.4411	2.4687	2.4615	2.463	2.0465	2.4545
	MSE	0.5607	0.5601	0.5449	0.5513	0.5595	0.5521	0.5553
300	Estimate	2.4501	2.4519	2.4793	2.4834	2.4728	2.0862	2.4717
	MSE	0.5543	0.5495	0.5291	0.5434	0.5395	0.5351	0.5463
500	Estimate	2.461	2.4758	2.4997	2.4899	2.4796	2.0931	2.4898
	MSE	0.5398	0.5195	0.5049	0.5133	0.5199	0.5153	0.5271

Table 3 shows the average estimates and Mean Squared Errors (MSEs) for the scale parameter of the WPFD under different estimation methods and sample sizes, for a scenario where the true scale parameter is ($A = 2.5$).
The results indicate that all estimators improve as the sample size increases. For small samples $n = 25$, estimates from MLE and Bayesian approaches are close to the true value, though some underestimation occurs for certain priors, particularly the Gamma prior under the QLF, which produces slightly lower estimates. MSE values are relatively high at small sample sizes but decrease steadily as (n) increases, reflecting enhanced precision with larger samples.
As the sample size grows ($n = 100,300,500$), both MLE and Bayesian estimators converge toward the true scale parameter. The Bayesian estimators using Jeffreys and Gamma priors under QLF consistently yield lower MSEs compared to SELF and PLF estimators, demonstrating better accuracy and reliability. MLE remains competitive, especially for larger samples, but is generally slightly less efficient than the Bayesian QLF estimators.
Table 3 confirms the trend observed in previous scenarios: Bayesian estimation with Jeffreys and Gamma priors under the Quadratic Loss Function provides the most accurate and stable estimates for the scale parameter, and estimation precision improves with increasing sample size.

CONCLUSION

This research focused on estimating the scale parameter of the Weibull-Power Function Distribution (WPFD) via means of both Maximum Likelihood Estimation and Bayesian inference. Various loss functions were investigated, and the Quadratic Loss Function (QLF_u) consistently demonstrated superior performance compared to the Squared Error Loss Function ($SELF_u$) along with the Precautionary Loss Function (PLF_u). This enhanced performance was particularly evident when the QLF_u was combined with Gamma and Jeffrey prior distributions. Consequently, the combination of the Gamma prior and the QLF_u was identified as the optimal estimator for the WPFD's scale parameter.

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