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USING THE JACOBI LAST MULTIPLIER APPROACH TO LINEARIZE THE MATHEW-LAKSHMANAN OSCILLATOR EQUATION

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ABSTRACT

The Mathews–Lakshmanan (ML) oscillator is a remarkable nonlinear dynamical system that preserves several features of the linear harmonic oscillator while exhibiting inherent nonlinearity. Owing to its exact solvability, linearizability, and relevance in classical and quantum mechanics, the ML oscillator has attracted significant research interest across physics, engineering, and applied mathematics. Parallel to this, the Jacobi Last Multiplier (JLM) method originally developed by Carl Gustav Jacobi has re-emerged as a powerful analytical tool for deriving Lagrangians, identifying first integrals, and revealing variational structures of nonlinear differential equations. In this study, we apply the JLM framework to the ML oscillator in order to construct its corresponding Lagrangian and perform an explicit linearization.

Keywords: Linearization, Mathew-Lakshmanan Oscillator Equation, Differential Equation, Jacobi Last Multiplier

INTRODUCTION

The nonlinear oscillator model known as the Mathews–Lakshmanan (ML) oscillator equation was first presented by P. M. Mathews and M. Lakshmanan in 1974. It is noteworthy because it is linearizable under appropriate transformations and admits accurate harmonic-type solutions, in contrast to the majority of nonlinear oscillators. To put it briefly, it bridges the gap between complicated nonlinear systems and linear harmonic oscillators, which makes it useful in both theoretical and applied sciences.

The Jacobi Last Multiplier (JLM) approach on the other hand, is a classical method introduced by Carl Gustav Jacobi in the 19th century, for solving ordinary differential equations (ODEs). By providing a methodical approach to determining a Lagrangian, the Jacobi Last Multiplier Method converts a second-order nonlinear ODE into a variational problem. Because of this, it is an effective tool in nonlinear dynamics, mechanics, and mathematical physics.

A Mathews-Lakshmanan-type oscillator with $m(x) = 1/[1 + (\lambda x)^2]$ is used in a work that investigates classical and quantum position-dependent mass (PDM) systems using Hamiltonian factorization and canonical transformations (Tainá & González-borrero, 2023). According to the authors, when the phase space was analyzed classically, the trajectories show progressively more noticeable abnormalities as the energy and λ values rise. An examination of wave functions and probability densities was presented along with the solution to the ambiguous ordering problem for the PDM oscillator in the quantum domain.

The authors of a different work suggested the novel approach in a wide category of nonpolynomial oscillators and velocity-dependent potential systems, which are commonly encountered in mechanical and physical contexts (Kabilan & Venkatesan, 2023). The findings are helpful for examining the impact of damping on the nonlinear behavior as well as for studying energy transfer events for this class of nonlinear systems. The design and fault detection of mechanical systems and structures that this nonlinear model may represent depend on these findings. As a result, academics from a variety of disciplines, including the cognitive sciences and engineering, have been interested in studying the dynamics of nonlinear systems.

An overview of some recent developments in the identification and generation of finite dimensional integrable nonlinear dynamical systems that display intriguing oscillatory and other solution features, such as quantum aspects, was provided in another article. The authors specifically focus on nonlinear oscillators of the Lienard type, as well as their coupled and generalized forms. Mathews-Lakshmanan oscillators, modified Emden equations, isochronous oscillators, and generalizations are examples of specific systems (Lakshmanan & Chandrasekar, 2014). They also briefly discuss nonstandard Hamiltonian and Lagrangian formulations of certain of these systems.

A work uses the case study of a two-dimensionally linked Mathews-Lakshmanan oscillator (abbreviated as M-L oscillator) to demonstrate the theory and techniques of analytical mechanics that may be successfully used to the analysis of various nonlinear nonconservative systems (Guangbao & Guangtao, 2020). They added that, the Lagrangian and Hamiltonian function in the form of rectangular coordinates of the two-dimensional M-L oscillator was directly created from an integral of the two-dimensional M-L oscillators, in accordance with the inverse problem approach of Lagrangian mechanics. The authors continued that, the Lagrange function, the initial integral, and the two-dimensional M-L oscillator motion differential equation are expressed by introducing the vector form variables. Consequently, it was demonstrated that the three-dimensional M-L oscillator may be reduced to the two-dimensional case, and the two-dimensional M-L oscillator was immediately extended to the three-dimensional case.

To get the Lagrangians of any second-order differential equation, Nucci & Tamizhmani (2013) employed the Jacobi's technique, which entails computing the Jacobi Last Multiplier. Utilizing the relationships between a mechanical system's Lie symmetries, Jacobi Last Multiplier, and Lagrangian to generate alternate Lagrangians and first integrals is possible when symmetry is abundant (Nucci & Leach, 2008). A Liénard-type nonlinear oscillator serves as the example. They also illustrate the possible incompatibilities between the general solution and the first integrals of a dynamical system.

Madhav Rao created a method almost 70 years ago that connects the Jacobi Last Multiplier and its Lagrangian of a second-order ordinary differential equation, which we use to get the Lagrangians of the Painlevé equations (Choudhury et al., 2009). In fact, the Lagrangians of a large number of the Painlevé–Gambier classification equations are obtained using this method. The authors determine the corresponding Hamiltonian functions by applying the usual Legendre transformation. Despite their often-non-standard shape, these Hamiltonians were shown to be constants of motion. To get the pertinent Lagrangians for second-order Liénard class equations, they employed a novel transformation. Examples of some particular situations were given, together with the conserved quantity (first integral) that results from the associated Noetherian symmetry.

The linearization of the Mathews–Lakshmanan (ML) oscillator equation with the Jacobi Last Multiplier (JLM) method is the main topic of this study.

MATERIALS AND METHODS

Method of Jacobi Last Multiplier

Second-order nonlinear differential equations can be analyzed classically using the Jacobi Last Multiplier (JLM), which offers a methodical approach to obtaining a Lagrangian and, in some situations, aids in linearizing or simplifying the equation. Regarding an equation of this type:

$$y'' = f(x, y, y'), \quad (1)$$

the following is satisfied by the Jacobi Last Multiplier $M(x, y, y')$:

$$M(y'' - f(x, y, y')) = \frac{dL}{dx}, \quad (2)$$

where the Lagrangian function is given as $L(x, y, y')$.

From equation (1), the JLM fulfills

$$\frac{d}{dx}(\ln M) + \frac{\partial f}{\partial y'} = 0, \quad (3)$$

or similar to this,

$$\frac{dM}{dx} = -M\frac{\partial f}{\partial y'}. \quad (4)$$

This is a first-order linear ordinary differential equation in M.

In summary, given a second-order ordinary differential equation in equation (1), compute

$$\frac{\partial f}{\partial y'}.$$

Solve the JLM from equation (3)

$$\frac{d}{dx}(\ln M) = -\frac{\partial f}{\partial y'}.$$

Integrating the above equation with respect to x:

$$\ln M = -\int \frac{\partial f}{\partial y'} dx + C,$$

so that

$$M(x, y, y') = C\, exp\left(-\int \frac{\partial f}{\partial y'} dx\right).$$

Jacobi showed that M can also be represented as a determinant constructed from solutions of the system of first-order ODEs equivalent to $y'' = f(x, y, y')$. That system is:

$$\dot{x} = 1,$$

$$\dot{y} = y',$$

$$\dot{y}' = f(x, y, y').$$

If we have two independent first integrals $\phi_1(x, y, y')$ and $\phi_2(x, y, y')$, then the Jacobi multiplier can be written as

$$M = \begin{pmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial y'} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial y'} \\ 1 & y' & f(x, y, y') \end{pmatrix}.$$

By definition, the Jacobi last multiplier $M(x, y, y')$ satisfies

$$\frac{\partial^2 L}{\partial y'^2} = M.$$

That means the Lagrangian $L(x, y, y')$ can be reconstructed by integrating twice with respect to y'.

Integrating once with respect to y', one has that:

$$\frac{\partial L}{\partial y'} = \int M dy' + \phi_1(x, y),$$

where $\phi_1(x, y)$ is an arbitrary function of x and y (since it vanishes under $\partial/\partial y'$).

Integrating again with respect to y', we have:

$$L(x, y, y') = \int\left(\int M dy' + \phi_1(x, y)\right)dy' + \phi_2(x, y),$$

where $\phi_2(x, y)$ is another arbitrary function of x and y.

Thus,

$$L(x, y, y') = \int\int M(x, y, y')d(y'^2) + \phi_1(x, y)y' + \phi_2(x, y).$$

Therefore, the Lagrangian is determined up to gauge terms that do not affect the equations of motion.

Once we have L, we verify the ordinary differential equation by applying the Euler–Lagrange equation:

$$\frac{d}{dx}\left(\frac{dL}{dy'}\right) - \frac{dL}{dy} = 0.$$

The resulting equation should reproduce the original second–order ordinary differential equation $y'' = f(x, y, y')$.

The condition $\frac{\partial^2 L}{\partial y'^2} = M$ ensures that the Euler–Lagrange equation is consistent with the Jacobi last multiplier equation

$$\frac{dM}{dx} + M\frac{\partial f}{\partial y'} = 0.$$

Thus, M directly connects the variational (Lagrangian) structure to the dynamics of the nonlinear ODE.

RESULTS AND DISCUSSION

The Mathews–Lakshmanan oscillator equation is given as

$$y'' - \frac{\lambda y}{1+\lambda y^2}y'^2 + \omega^2 y(1 + \lambda y^2) = 0, \quad (5)$$

where λ and ω are real parameters.

First, rewrite the equation in the form

$$y'' = f(y, y').$$

That is

$$y'' = \frac{\lambda y}{1+\lambda y^2}y'^2 - \omega^2 y(1 + \lambda y^2). \quad (6)$$

The Jacobi Last Multiplier $M(x)$ satisfies equation (3). Computing $\frac{\partial f}{\partial y'}$, one has that:

$$f(y, y') = \frac{\lambda y}{1+\lambda y^2}y'^2 - \omega^2 y(1 + \lambda y^2) \Rightarrow \frac{\partial f}{\partial y'} = 2\left(\frac{\lambda y}{1+\lambda y^2}y'\right).$$

Therefore,

$$\frac{d}{dx}(\ln M) + 2\left(\frac{\lambda y}{1+\lambda y^2}y'\right) = 0. \quad (7)$$

Consider $M = M(x)$ (no y' dependence) then;

$$\frac{d}{dx}(\ln M(y)) = \frac{d}{dy}(\ln M(y))y'.$$

Thus,

$$\frac{d}{dx}(\ln M)y' + 2\left(\frac{\lambda y}{1+\lambda y^2}y'\right) = 0 \Rightarrow \left(\frac{d}{dx}(\ln M)y' + 2\frac{\lambda y}{1+\lambda y^2}\right)y' = 0.$$

Since $y' \neq 0$ in general:

$$\frac{d}{dx}(\ln M) = -2\left(\frac{\lambda y}{1+\lambda y^2}\right).$$

On integration of both sides, one has that:

$$\ln M = -\int 2\left(\frac{\lambda y}{1+\lambda y^2}\right)dy. \quad (8)$$

But

$$\int \frac{2\lambda y}{1+\lambda y^2}dy = \int \frac{d(1+\lambda y^2)}{1+\lambda y^2} = \ln(1 + \lambda y^2).$$

So,

$$\ln M = -\ln(1 + \lambda y^2) \Rightarrow M(y) = \frac{1}{1+\lambda y^2}. \quad (9)$$

To construct the Lagrangian using the JLM is the next thing to do. For second order ODEs, the JLM $M(y)$ is related to the Lagragian $L(y, y')$ by:

$$M(y) = \frac{\partial^2 L}{\partial y'^2}. \quad (10)$$

So, one integrates $M(y)$ twice to obtain the Lagragian. The first integration gives

$$\frac{\partial L}{\partial y'} = \int M(y)dy' = \int \frac{1}{1+\lambda y^2}dy' = \frac{1}{(1+\lambda y^2)}y' + c_1(y).$$

Second integration produces

$L = \int \left(\frac{1}{(1+\lambda y^2)} y' + c_1(y) \right) dy' = \frac{1}{2(1+\lambda y^2)} y'^2 + c_1(y)y' + c_2(y).$

One can discard the total derivative terms (like $c_1(y)y'$) because they do not contribute to the Euler-Lagrange equations. So, the Lagrangian is

$$L = \frac{1}{2(1+\lambda y^2)} y'^2 - \frac{1}{2}\omega^2 y^2. \qquad (11)$$

Now, we find a coordinate transformation that transforms the nonlinear equation into a linear one. Define a new coordinate:

$$X = \frac{y}{\sqrt{1+\lambda y^2}}. \qquad (12)$$

Differentiating equation (12) with respect to x using the quotient rule, one has that:

$$X' = \frac{y'(1+\lambda y^2) - \lambda y^2 y'}{(1+\lambda y^2)^{3/2}} = \frac{y'}{(1+\lambda y^2)^{1/2}}.$$

One can use the product and chain rules to differentiate the equation above to have:

$$X'' = \frac{y''}{(1+\lambda y^2)^{1/2}} - \frac{\lambda y y'^2}{(1+\lambda y^2)^{3/2}} = \frac{1}{(1+\lambda y^2)^{3/2}}\left[y''(1+\lambda y^2) - \lambda y y'^2\right]. \qquad (13)$$

Recall that $y'' = \frac{\lambda y}{1+\lambda y^2} y'^2 - \omega^2 y(1+\lambda y^2)$. Therefore, equation (13) can be simplified to become

$$X'' = \frac{-\omega^2 y (1+\lambda y^2)^2}{(1+\lambda y^2)^{3/2}} = -\omega^2 y (1+\lambda y^2)^{1/2}.$$

Now, recall from equation (12) that

$$X = \frac{y}{\sqrt{1+\lambda y^2}} \Longrightarrow y = \frac{X}{\sqrt{1-\lambda X^2}},$$

invertible only when $\lambda X^2 < 1$. Therefore,

$$y(1+\lambda y^2)^{1/2} = X \Longrightarrow X'' = -\omega^2 X.$$

Thus, the transformed equation is:

$$X'' + \omega^2 X = 0. \qquad (14)$$

Equation (14) is the classical linear harmonic oscillator. The general solution of equation (14) is

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

where A and B are constants determined by the initial conditions.

The Mathews–Lakshmanan (ML) oscillator is a remarkable nonlinear oscillator whose dynamics resemble those of the simple harmonic oscillator despite its nonlinearity. Starting from the nonlinear ODE

$$y'' - \frac{\lambda y}{1+\lambda y^2} y'^2 + \omega^2 y(1+\lambda y^2) = 0,$$

the Jacobi Last Multiplier (JLM) technique provides a systematic way to determine whether the equation admits a Lagrangian formulation. Computing the multiplier yields

$$M(y) = \frac{1}{1+\lambda y^2},$$

which immediately leads to the Lagrangian

$$L = \frac{1}{2}\frac{(y')^2}{1+\lambda y^2} - \frac{1}{2}\omega^2 y^2.$$

This Lagrangian shows that the ML oscillator behaves like a particle with a position-dependent effective mass

$$m(y) = 1 + \lambda y^2.$$

Thus, the nonlinearity arises entirely from a variable mass term, while the potential energy remains quadratic.

A key result is that the nonlinear equation can be exactly linearized by the coordinate transformation

$$X = \frac{y}{1+\lambda y^2},$$

which converts the equation into the linear harmonic oscillator

$$X'' + \omega^2 X = 0.$$

This means the ML oscillator is point-transformable to a linear system and therefore exactly solvable. Its solutions can be written explicitly in terms of trigonometric functions, and the general motion retains a constant period

$$T = \frac{2\pi}{\omega},$$

independent of the amplitude. This property isochrony makes the ML oscillator a rare example of a nonlinear system whose oscillations do not change period with amplitude.

The transformation is valid only when $\lambda X^2 < 1$, which places a bound on the amplitude when $\lambda > 0$. For $\lambda \leq 0$, no such restriction occurs.

In summary, the results show that:

i. A Lagrangian exists and corresponds to a system with position-dependent mass.
ii. The nonlinear ML oscillator is exactly linearizable, revealing hidden simplicity behind its nonlinear form.
iii. The oscillator is isochronous, sharing fundamental behavior with the simple harmonic oscillator despite its nonlinearity.

These properties explain why the ML oscillator is widely studied and why it fits naturally into geometric linearization theory.

CONCLUSION

The linearization of the Mathews–Lakshmanan (ML) oscillator equation using the Jacobi Last Multiplier (JLM) method is considered in this study. The Jacobi Last Multiplier is used to construct the appropriate Lagrangian. The Mathews–Lakshmanan oscillator equation is reduced to the second-order classical linear harmonic oscillator equation with an appropriate transformation.

REFERENCES

Choudhury, A. G., Guha, P., & Khanra, B. (2009). On the Jacobi Last Multiplier, integrating factors and the Lagrangian formulation of differential equations of the Painlevé – Gambier classification. *Journal of Mathematical Analysis and Applications*, *360*(2), 651–664. https://doi.org/10.1016/j.jmaa.2009.06.052

Guangbao, W., & Guangtao, D. (2020). The Lagrangian and Hamiltonian for the Two-Dimensional Mathews-Lakshmanan Oscillator. *Advances in Mathematical Physics*, *2020*, 1–6. https://doi.org/10.1155/2020/2378989

Kabilan, R., & Venkatesan, A. (2023). Vibrational Resonance in a Damped Bi-harmonic Driven Mathews – Vibrational Resonance in a Damped Bi - harmonic Driven Mathews – Lakshmanan Oscillator. *Journal of Vibration Engineering & Technologies*, *September*, 1–10. https://doi.org/10.1007/s42417-023-00897-6

Lakshmanan, M., & Chandrasekar, V. K. (2014). Generating Finite Dimensional Integrable Nonlinear Dynamical Systems. *The European Physical Journal Special Topics*, *July*, 1–28. https://doi.org/10.1140/epjst/e2013-01871-6

Nucci, M. C., & Leach, P. G. L. (2008). The Jacobi Last Multiplier and its applications in mechanics. *PHYSICA SCRIPTA*, *065011*(78), 1–6. https://doi.org/10.1088/0031-8949/78/06/065011

Tainá, R., & González-borrero, P. P. (2023). Classical and quantum systems with position-dependent mass: An application to a Mathews-Lakshmanan-type oscillator. *Licença Creative Commons*, *45*(e20230172), 1–12. https://doi.org/10.1590/1806-9126-RBEF-2023-0172

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