

STATISTICAL PROPERTIES OF LOMAX-INVERSE EXPONENTIAL DISTRIBUTION AND APPLICATIONS TO REAL LIFE DATA

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ABSTRACT

This paper proposes a Lomax-inverse exponential distribution as an improvement on the inverse exponential distribution in the form of Lomax-inverse Exponential using the Lomax generator (Lomax-G family) with two extra parameters to generalize any continuous distribution (CDF). The probability density function (PDF) and cumulative distribution function (CDF) of the Lomax-inverse exponential distribution are defined. Some basic properties of the new distribution are derived and extensively studied. The unknown parameters estimation of the distribution is done by method of maximum likelihood estimation. Three real-life datasets are used to assess the performance of the proposed probability distribution in comparison with some other generalizations of Lomax distribution. It is observed that Lomax-inverse exponential distribution is more robust than the competing distributions, inverse exponential and Lomax distributions. This is an evident that the Lomax generator is a good probability model.

Keywords: Lomax-G family, Lomax-Inverse Exponential, Generalization, Maximum Likelihood Estimation, Properties.

INTRODUCTION

There are several ways of improving a distribution function one of which is by adding one or more parameters to the distribution to make the resulting distribution richer and more flexible for modeling data. In literature, there are many areas where distribution functions are generalized, for instance in the area of generalization of exponential distribution, Oguntunde and Adejumo(2015) worked on transmuted inverse exponential, Lomax-exponential distribution derived and studied by Ieren and Kuhe (2019), the odd generalization exponential-exponential distribution proposed by Maiti and Pramanik(2015), Abdullahi *et al.*(2018) studied the transmuted odd generalized exponential-exponential distribution, the transmuted exponential distribution by Owoloko *et al.* (2015), Sandya and Prasanth[30] proposed Marshall-Olkin discrete uniform distribution, an extended Lomax distribution by Lemonte and Cordeiro(2013), Exponential Lomax distribution by El-Bassiouny *et al.*(2015), Lomax exponential distribution by Ijaz *et al.*(2019), a Lomax-inverse Lindley distribution by Ieren *et al.*(2019), a new Generalization of Lomax distribution by Mundher and Ahmed(2017), Odd Lindley-Rayleigh distribution by Ieren *et al.*(2020), Oguntunde *et al.*(2015) derived and studied Weibull-exponential distribution, etc.

Cordeiro *et al.* (2014) proposed a Lomax generator with two extra positive parameters to generalize any continuous baseline distribution. Some special models such as the Lomax-normal, Lomax-Weibull, Lomax-log-logistic and Lomax-Pareto distributions were discussed. They presented some properties of the Lomax generator as well as some entropy measures and discussed the estimation of unknown parameters of the model by maximum likelihood method. They also proposed a modification process based on the marginal Lomax exponential distribution and defined a log-lomax-weibull regression model for censored data. The importance of the new generator was illustrated by means of three real data sets, for more details interested reader(s) can refer to their papers.

Keller and Kamath (1982) developed the Inverse Exponential distribution which is a modified version of the Exponential distribution that can model data sets with inverted bathtub failure rate, but its inability to properly model data sets that are highly skewed or that have fat tails has been noticed in the work of Abouammoh and Alshingiti (2009) where the Generalised Inverse Exponential distribution was introduced.

According to Cordeiro *et al.*(2014), the Lomax-G family (Lomax-based generator) cumulative density function (CDF) and the probability density function (PDF) for any continuous probability distribution are given respectively as:

$$F(x) = 1 - \left\{ \frac{\beta}{\beta - \log[1 - G(x)]} \right\}^{\alpha} \quad (1)$$

$$\text{And } f(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1 - G(x)]\{\beta - \log[1 - G(x)]\}^{\alpha+1}}, \quad (2)$$

where $g(x)$ and $G(x)$ are the PDF and CDF of any continuous distribution to be generalized respectively and $\alpha > 0$ and $\beta > 0$ are the two additional shape parameters of the Lomax-G family of distribution respectively.

Researches conducted on Lomax distribution by other authors have been documented in the literature. Balakrishnan & Ahsanullah (1994) discussed some important properties and moments of Lomax distribution. Al-Awadhi and Ghitany (2001) provided the discrete Poisson-Lomax distribution. Abd-Elfattah et al;(2007) studied the Bayesian and non-Bayesian estimation procedure of the reliability of Lomax distribution. Marshall-Olkin extended Lomax distribution that was introduced by Ghitany et al; (2007). The optimal times of changing stress level for simple stress plans under a cumulative exposure model for the Lomax distribution was determined by Hassan and Al-Ghandi (2009)studied the optimal times of changing stress level for k-level step stress accelerated life tests based on adaptive type-II progressive hybrid censoring with product's lifetime following Lomax distribution.

of the probability density function (PDF) and the cumulative distribution function (CDF) of the newly proposed Lomax-inverse exponential distribution and its plots. Then some basic properties of the Lomax-inverse exponential distribution are studied and parameter estimation is done using maximum likelihood method. We provided an illustration of the potentiality of the proposed model and two other competing models using three real life datasets. Finally we gave concluding remark.

The structure of this article is as follows, introduction of definition

MATERIAL AND METHOD

Mathematical definition of the Lomax-inverse Exponential Distribution

We introduce the CDF and PDF of the Lomax-inverse exponential distribution using the steps proposed by Cordeiro et al, (2014). According to them, the CDF and PDF of the Lomax-G family are defined for any continuous distribution as follows:

$$F(x) = \int_0^{-\log[1-G(x)]} \alpha\beta^\alpha \frac{dt}{(\beta+t)^{\alpha+1}} \tag{3}$$

Solving the integral above, equation (5) becomes

$$F(x) = 1 - \left\{ \frac{\beta}{\beta - \log[1 - G(x)]} \right\}^\alpha \tag{4}$$

The corresponding probability density function (PDF) of the Lomax is given as

$$f(x) = \alpha\beta^\alpha \frac{g(x)}{[1 - G(x)]\{\beta - \log[1 - G(x)]\}^{\alpha+1}}, \tag{5}$$

where g(x) and G(x) are the PDF and CDF of any continuous distribution to be generalized respectively and $\alpha > 0$ and $\beta > 0$ are the two additional shape parameters of the Lomax-G family of distribution respectively.

The inverse exponential distribution with parameter $\theta > 0$ has the cumulative distribution function (CDF) and probability density function (PDF) given by:

$$G(x) = e^{-\frac{\theta}{x}} \tag{6}$$

$$\text{and } g(x) = \frac{\theta}{x^2} e^{-\frac{\theta}{x}} \tag{7}$$

respectively. For $x > 0, \theta > 0$ where θ is the scale parameter of the inverse exponential distribution.

To obtain the cumulative density function and probability density function of the Lomax inverse exponential distribution(LOMINEXD), then substitute equation (6) and (7) into equation (4) and (5) and simplify as follows:

$$F(x) = 1 - \beta^\alpha \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} \tag{8}$$

$$\text{Again using equation (7) we obtain the PDF of LOMINEXD as } f(x) = \alpha\beta^\alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}} \right]^{-1} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-(\alpha+1)} \tag{9}$$

Therefore equation (8) and (9) are the CDF and PDF of the newly proposed distribution (LOMINEXD), respectively, where $\alpha > 0$ and $\beta > 0$ are the shape parameters and θ is a scale parameter.

Model validity check

Recall that the total area under a pdf curve is always equal to 1 ($\int_0^\infty f(X)dx = 1$).

Proof:

$$\int_0^\infty \alpha \beta^\alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}}\right]^{-1} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+1)} dx$$

$$= \alpha \beta^\alpha \theta \int_0^\infty x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}}\right]^{-1} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+1)} dx$$

Using integration by substitution

$$\text{Let } y = \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+1)} = u^{-(\alpha+1)}$$

$$u = \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] = \beta - \log t$$

$$t = 1 - e^{-\frac{\theta}{x}}$$

This implies that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx} = (\alpha + 1) \cdot -\frac{1}{t} \cdot \frac{\theta}{x^2} e^{-\frac{\theta}{x}}$$

$$\frac{dy}{dx} = \frac{-(\alpha + 1)\theta e^{-\frac{\theta}{x}} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+1)}}{x^2 \left[1 - e^{-\frac{\theta}{x}}\right]}$$

$$dx = -\frac{x^2 \left[1 - e^{-\frac{\theta}{x}}\right]}{(\alpha + 1)\theta e^{-\frac{\theta}{x}} \left\{ \beta - \log \left(1 - e^{-\frac{\theta}{x}}\right) \right\}^{-(\alpha+2)}} dy$$

$$\int_0^\infty f(x) dx = \alpha \beta^\alpha \theta \int_0^\infty \frac{e^{-\frac{\theta}{x}}}{x^2 \left[1 - e^{-\frac{\theta}{x}}\right]} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+1)} * -\frac{x^2 \left(1 - e^{-\frac{\theta}{x}}\right)}{(\alpha + 1)\theta e^{-\frac{\theta}{x}} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\}^{-(\alpha+2)}} dx$$

$$= -\frac{\alpha \beta^\alpha}{(\alpha + 1)} \int_0^\infty \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}}\right] \right\} dx$$

$$= -\frac{\alpha \beta^\alpha}{\alpha + 1} \int_0^\infty u dy$$

Recall that;

$$y = u^{-(\alpha+1)}, \text{ which implies that}$$

$$y^{\frac{1}{\alpha+1}} = u.$$

$$\int_0^\infty f(x)dx = -\frac{\alpha\beta^\alpha}{\alpha+1} \int_0^\infty y^{-\frac{1}{\alpha+1}} dy = -\frac{\alpha\beta^\alpha}{\alpha+1} \left[\frac{y^{\frac{\alpha}{\alpha+1}}}{\frac{\alpha}{\alpha+1}} \right]_0^\infty = -\beta^\alpha \left[y^{\frac{\alpha}{\alpha+1}} \right]_0^\infty$$

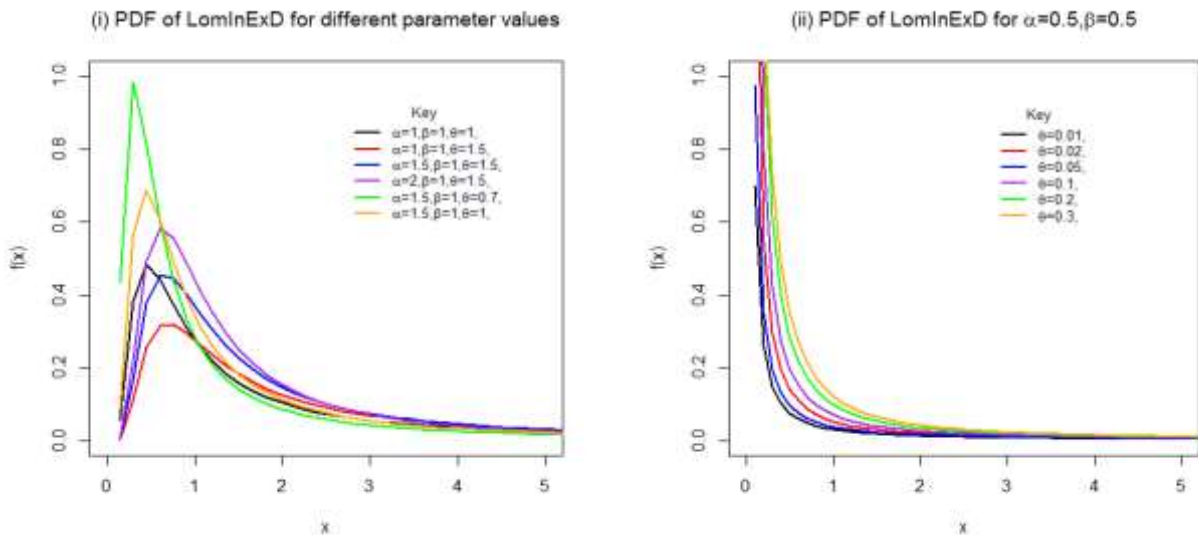
We then substitute the limit into the function

$$f(x)dx = -\beta^\alpha \left\{ \lim_{x \rightarrow \infty} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} - \lim_{x \rightarrow x} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} \right\} = -\beta^\alpha (0 - \beta^{-\alpha}) = 1$$

Hence the proof.

Graphs of probability density function (PDF) and cumulative density function (CDF) of LOMINEXD

Given some values for the parameters $\theta, \alpha, \beta > 0$, we provide some possible curves for the probability density function and the cumulative distribution function of the LOMINEXD as shown in Figure 1 and 2 below:



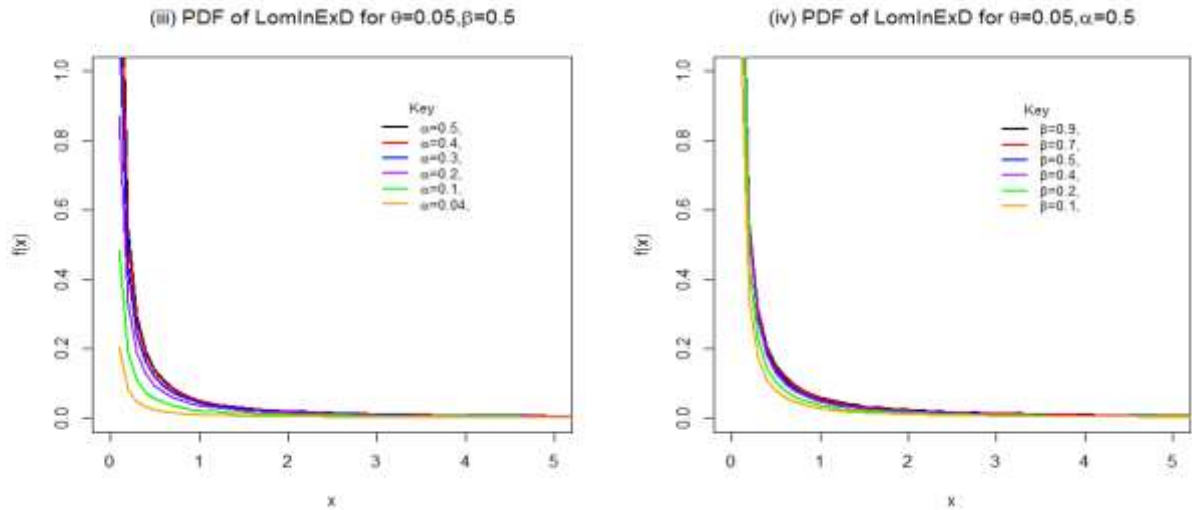


Fig1: PDF of the LOMINEXD for different values of the parameters.

From Figure 1, we observe that the *LOMINEXD* distribution takes various shapes and is always right-skewed depending on values of the parameters as seen in plots (i), (ii), (iii) and (iv). This means that distribution can be very useful for datasets with different shapes.

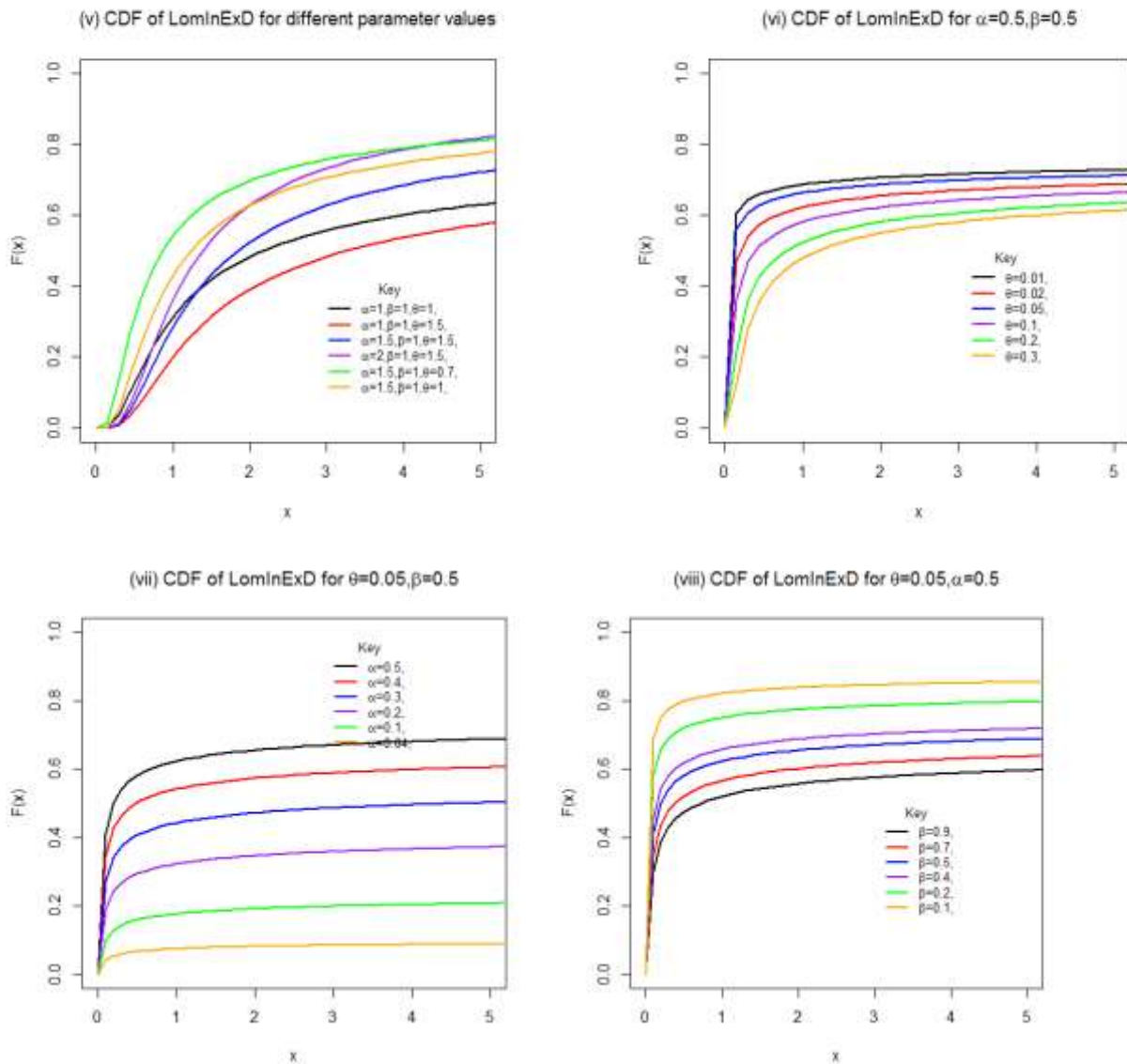


Fig. 2: CDF of the LOMINEXD for different values of the parameters.

From figure 2 in plots (v), (vi), (vii) and (viii) above, we see that the cdf increases when X increases, and approaches 1 when X turns to infinity, as expected.

Some statistical Properties of the Lomax-Inverse Exponential Distribution

Quantile Function

Suppose $F(x)$ is the cumulative distribution function (CDF) of the Lomax-inverse exponential distribution and if is inverted it gives the quantile function as follows:

$$F(X) = u$$

This implies that the quantile function follows the form:

$$Q(u) = F^{-1}(x)$$

$$1 - \beta^\alpha \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} = u$$

Simplify above equation and solve for X gives the quantile function of the LOMINEXD as

$$Q(u) = \theta \left[\beta - \beta(1-u)^{\frac{1}{\alpha}} \right]^{-1} \tag{10}$$

From equation (10), when $u=0.5$

$$Median = \theta \left\{ \beta \left[1 - (0.5)^{\frac{1}{\alpha}} \right] \right\}^{-1}$$

The corresponding first quartile and third quartile can also be obtained by making the substitution of $u = 0.25$ and $u = 0.75$, respectively, into Equation (10).

Reliability Analysis

Survival Function

Survival function is chance that a component or an individual will survive a given time. Therefore the survival function is given by:

$$F(x) = P(X \leq x) = \int_0^{\infty} f(x)dx$$

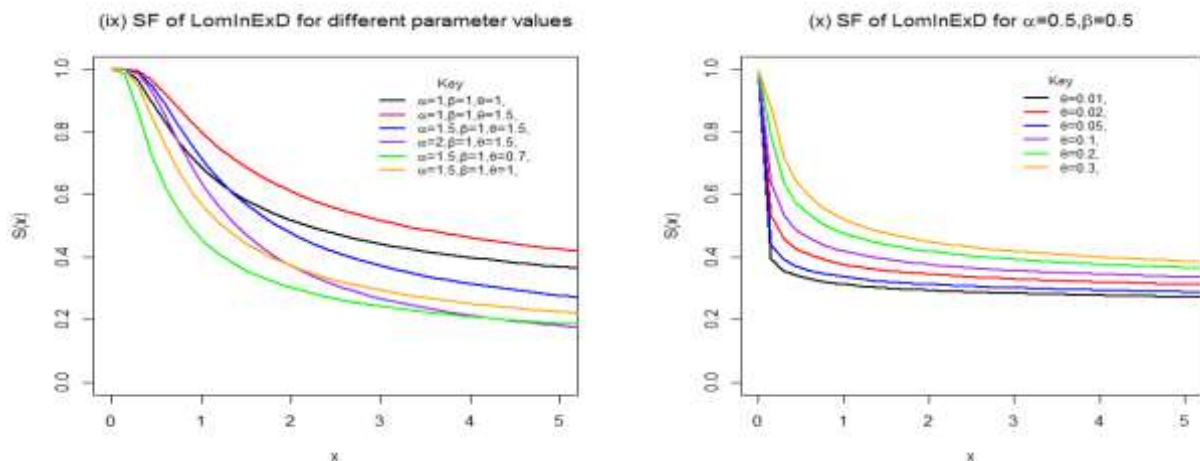
$$S(x) = P(X \geq x) = 1 - F(x) \tag{11}$$

For the Lomax inverse exponential distribution it is given as

$$S(x) = \beta^\alpha \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} \tag{12}$$

For $x > 0, \alpha, \beta, \theta > 0$.

Below are the plots of the survival function at chosen parameter values in Figure 3.



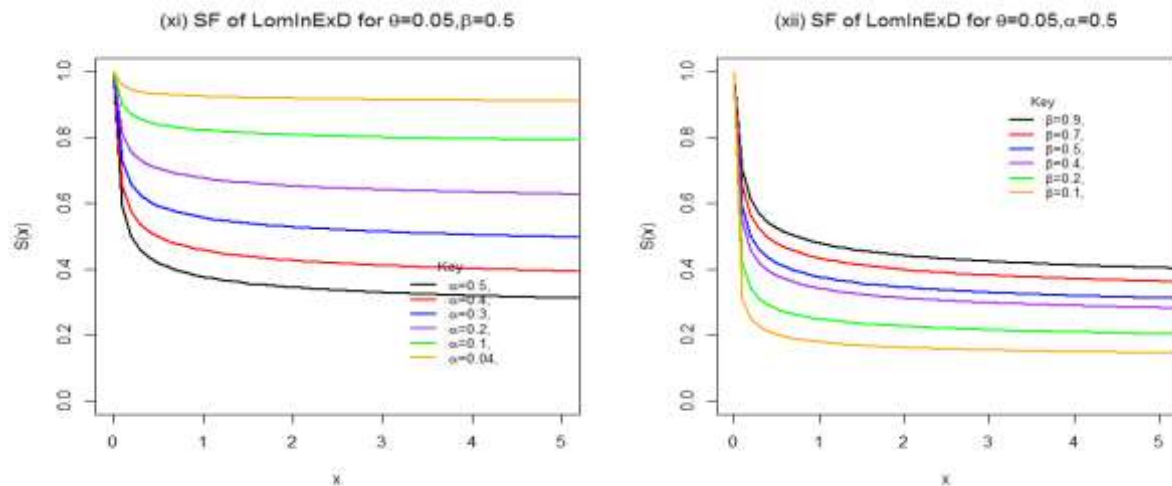


Figure 3: The survival function of the LOMINEXD for different values of the parameters.

The Figure 3 above shows the probability of survival for a random variable that follows a LOMINEXD and as seen in plots (ix), (x), (xi) and (xii), the survival plots decreases the values of the random variable increase. It implies that the LOMINEXD can be used to model random variables whose survival rate decreases as their ages increase.

Hazard Function

Hazard function is the probability that a component will experience an event, say failure or death for a given interval of time. The hazard function is defined as follows;

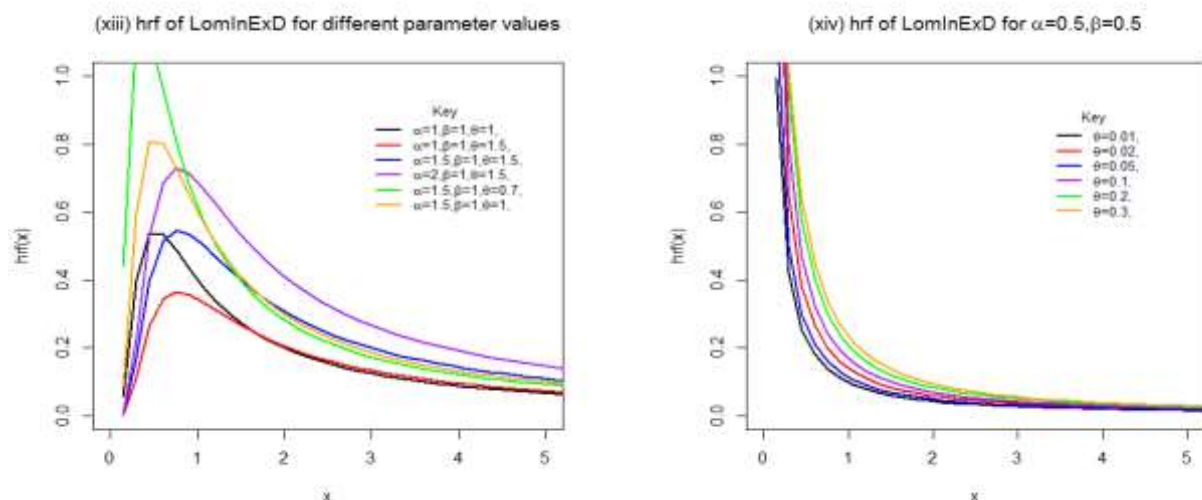
$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \tag{13}$$

Therefore, the corresponding failure rate for the Lomax inverse exponential distribution is express as:

$$h(x) = \alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}} \right]^{-1} \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-1} \tag{14}$$

For $x > 0, \alpha, \beta, \theta > 0$.

The following is a plot of the hazard function at chosen parameter values are in Figure 4



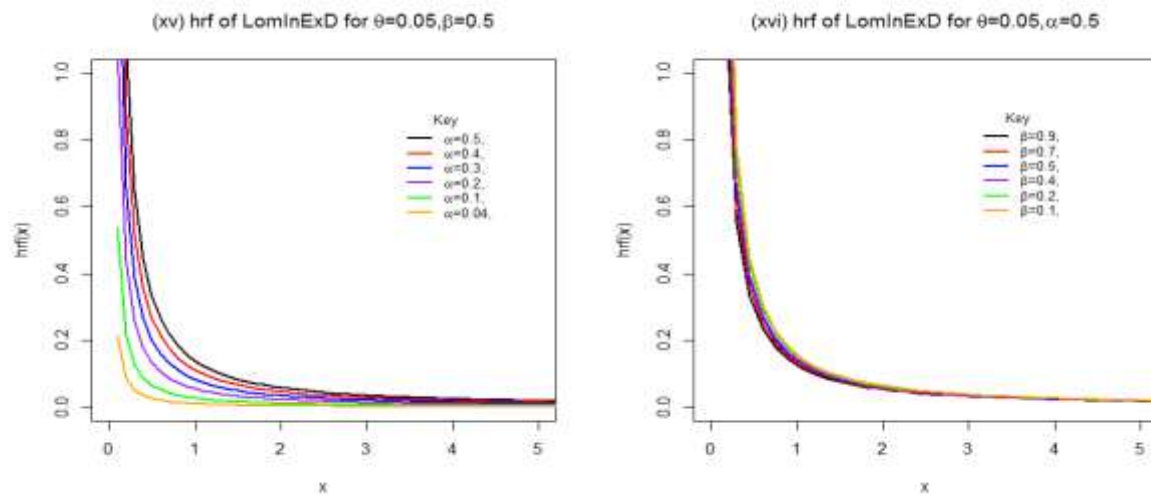


Figure 4: The hazard function of the LOMINEXD for different values of the parameters.

Figure 4 shows that the failure rate for any random variable following a LOMINEXD decreases as time increases as shown in plots (xiii), (xiv), (xv) and (xvi), that is, as time goes on, probability of death decreases. This implies that the LOMINEXD can be used to model random variables whose failure rate decreases as the age increases.

Order Statistics

The PDF of the *i*th order statistic for a random sample X_1, \dots, X_n from a distribution function $F(x)$ and an associated PDF $f(x)$ is

$$f_{i:n}(X) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+i-1} \tag{15}$$

Now, take $f(x)$ and $F(x)$ in equation (15) to be the probability density function (PDF) and cumulative distribution function (CDF) of the Lomax inverse exponential distribution with parameters $\alpha, \beta,$ and θ as defined in equation(9). Therefore, the probability density function of the *i*th order statistic for a random sample X_1, \dots, X_n from the Lomax inverse exponential distribution is given as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \frac{\alpha\beta^\alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}}\right]^{-1}}{\left\{\beta - \log\left[1 - e^{-\frac{\theta}{x}}\right]\right\}^{\alpha+1}} \left[1 - \beta^\alpha \left\{\beta - \log\left[1 - e^{-\frac{\theta}{x}}\right]\right\}^{-\alpha}\right]^{k+i-1}$$

Hence, the PDF of the minimum order statistic $X_{(1)}$ and the maximum order statistic $X_{(n)}$ of the Lomax inverse exponential distribution are respectively given by

$$f_{x_{(1)}}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\alpha\beta^\alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}}\right]^{-1}}{\left\{\beta - \log\left[1 - e^{-\frac{\theta}{x}}\right]\right\}^{\alpha+1}} \left[1 - \beta^\alpha \left\{\beta - \log\left[1 - e^{-\frac{\theta}{x}}\right]\right\}^{-\alpha}\right]^k \tag{16}$$

and

$$f_{x(1)}(x) = n \frac{\left[\alpha \beta^\alpha \theta x^{-2} e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}} \right]^{-1} \right]}{\left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{\alpha+1}} \left[1 - \beta^\alpha \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-\alpha} \right]^{n-1} \tag{17}$$

Estimation of Parameters

Let X_1, X_2, \dots, X_n denote random samples drawn from the Lomax inverse exponential distribution with parameters α, β and θ as defined in equation (9). The parameter estimation of the Lomax inverse exponential distribution was done by maximum likelihood method as presented below.

$$L(x_1, \dots, x_n : \alpha, \beta, \theta) = \prod_{i=1}^n \frac{\left[\alpha \beta^\alpha \theta x_{i=1}^{-2} e^{-\frac{\theta}{x_i}} \left[1 - e^{-\frac{\theta}{x_i}} \right]^{-1} \right]}{\left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right\}^{\alpha+1}}$$

$$L(X | \alpha, \beta, \theta) = \frac{\alpha^n \beta^{n\alpha} \theta^n \prod_{i=1}^n \left\{ x_i^{-2} e^{-\frac{\theta}{x_i}} \left[1 - e^{-\frac{\theta}{x_i}} \right]^{-1} \right\}}{\prod_{i=1}^n \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right\}^{\alpha+1}}$$

$$\ln L(X | \alpha, \beta, \theta) = n \log \alpha + n \alpha \log \beta + n \log \theta - 2 \sum_{i=1}^n \log x_i - \theta \sum_{i=1}^n x_i^{-1} - \sum_{i=1}^n \log \left[1 - e^{-\frac{\theta}{x_i}} \right]$$

$$- (\alpha + 1) \sum_{i=1}^n \log \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right\}$$

$$\frac{\delta \ln L(X | \alpha, \beta, \theta)}{\delta \alpha} = \frac{n}{\alpha} + n \log \beta - \sum_{i=1}^n \log \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right\} \tag{18}$$

$$\frac{\delta \ln L(X | \alpha, \beta, \theta)}{\delta \beta} = \frac{n \alpha}{\beta} - (\alpha + 1) \sum_{i=1}^n \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right\}^{-1} \tag{19}$$

$$\frac{\delta \ln L(X | \alpha, \beta, \theta)}{\delta \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i^{-1} - \sum_{i=1}^n \left\{ \frac{x_i^{-1} e^{-\frac{\theta}{x_i}}}{1 - e^{-\frac{\theta}{x_i}}} \right\} - (\alpha + 1) \sum_{i=1}^n \left\{ \frac{\left(1 - e^{-\frac{\theta}{x_i}} \right)^{-1} x_i^{-1} e^{-\frac{\theta}{x_i}}}{\left(\beta - \log \left[1 - e^{-\frac{\theta}{x_i}} \right] \right)} \right\} \quad (20)$$

Setting $\frac{\partial \ln(\alpha\beta\theta)}{\partial \alpha} = 0$, $\frac{\delta \ln L(\alpha, \beta, \theta)}{\delta \beta} = 0$ and $\frac{\delta \ln L(\alpha, \beta, \theta)}{\delta \theta} = 0$, and solving for the parameters will give the maximum

likelihood estimators of α , β and θ respectively. However, the solution of the above equations is not in closed form and hence can only be gotten with the help of a software such as R.

RESULTS AND DISCUSSION

Illustration of the proposed distribution and other two models are presented in this section by using three data sets. The comparison of the performance of the proposed distribution Lomax-inverse Exponential distribution, the Lomax distribution and the inverse exponential distribution is done. The probability density function of the LOMINEXD distribution is given as

$$f(x) = \alpha \beta^\alpha \theta x^2 e^{-\frac{\theta}{x}} \left[1 - e^{-\frac{\theta}{x}} \right] \left\{ \beta - \log \left[1 - e^{-\frac{\theta}{x}} \right] \right\}^{-(\alpha+1)} \quad (21)$$

The pdf of the LomD distribution is given as:
$$f(x) = \frac{\alpha}{\beta} \left[1 + \left(\frac{x}{\beta} \right) \right]^{-(\alpha+1)} \quad (22)$$

The PDF of the InExD distribution is given as:

$$f(x) = \frac{\theta}{x^2} e^{-\frac{\theta}{x}} \quad (23)$$

Dataset I: This data represents the remission times (in months) of a random sample of 128 bladder cancer patients adopted from the work of Rady et al;(2016). It has previously been used by Lee and Wang (2003), Ieren and Chukwu (2018) and Abdullahi et al;(2018). It is given as follows:

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Dataset II: This data represents the survival times of a group of patients suffering from head and neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT) (Efron (1988), Shanker et al. (2015), Oguntunde et al. (2015)).

12.20 23.56 23.74 25.87 31.98 37 41.35 47.38 55.46 58.36 63.47 68.46 78.26 74.47 81.43 84 92 94 110 112 119 127 130 133 140 146 155 159 173 179 194 195 209 249 281 319 339 432 469 519 633 725 817 1776
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Dataset III: This data (on the strength of 1.5cm glass fibers) comprising 63 data items was originally obtained by workers at the UK national physical laboratory and it has been used by Bourguignon et al; (2014), Oguntunde et al;(2015), Smith and Naylor (1987), Barreto-Souza et al;(2011) as well as Affiy and Aryal (2016). The data is as follows:

0.55, 1.28, 1.51, 1.61, 1.70, 2.00, 0.74, 1.29, 1.52, 1.62, 1.70, 2.01, 0.77, 1.30, 1.53, 1.62, 1.73, 2.24, 0.81, 1.36, 1.54, 1.63, 1.76, 0.84, 1.39, 1.55, 1.64, 1.76, 0.93, 1.42, 1.55, 1.66, 1.77, 1.04, 1.48, 1.58, 1.66, 1.78, 1.11, 1.48, 1.59, 1.66, 1.81, 1.13, 1.49, 1.60, 1.67, 1.82, 1.24, 1.49, 1.61, 1.68, 1.84, 1.25, 1.50, 1.61, 1.68, 1.84, 1.27, 1.50, 1.61, 1.69, 1.89
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Table 1: Descriptive summary of dataset I, II and III

Dataset	N	Minimum	Q ₁	Midian	Q ₃	Mean	Maximum	Variance	Skewness	Kurtosis
I	128	0.800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537
II	44	12.20	67.21	128.5	219.0	223.48	1776.00	93286.4	3.38382	13.5596
III	63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

From Table 2 above, it is clear that the first dataset (dataset I) is skewed to the right or positively skewed and the second dataset (dataset II) is negatively skewed, that is, skewed to the left and would be good for flexible models like LOMINEXD. The performance of the models was done using some criteria: the AIC (Akaike Information Criterion) and CAIC (Consistent Akaike Information Criterion). The formulas for these statistics are respectively given as:

$$AIC = -2\hat{\lambda} + 2k \quad \text{and} \quad CAIC = -2\hat{\lambda} + \frac{2kn}{(n - k - 1)},$$

where $\hat{\lambda}$ is the value of the log-likelihood function evaluated at the maximum likelihood estimates (MLEs), k is the number of model parameters and n is the sample size.

A goodness-of-fit test in order to confirm which distribution fits the data better, we apply the Kolmogorov-Smirnov ($K-S$) statistics was also used to confirm which distribution fits the data better. Further information about these statistics can be obtained from Chen and Balakrishnan (2018). This statistics can be computed as:

$$K - S = D = \sup |F_n(x) - F_0(x)|$$

where $F_n(x)$ and $F_0(x)$ are the empirical and observed distribution functions respectively and n is the sample size.

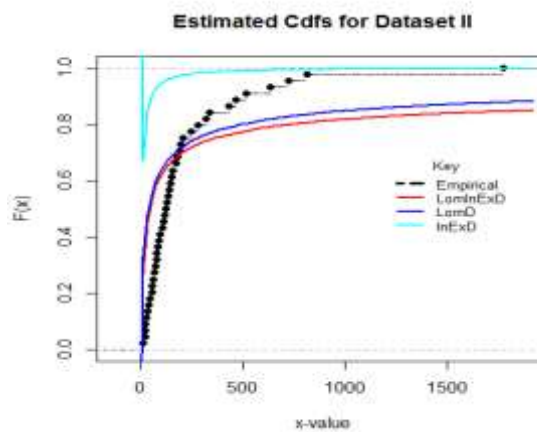
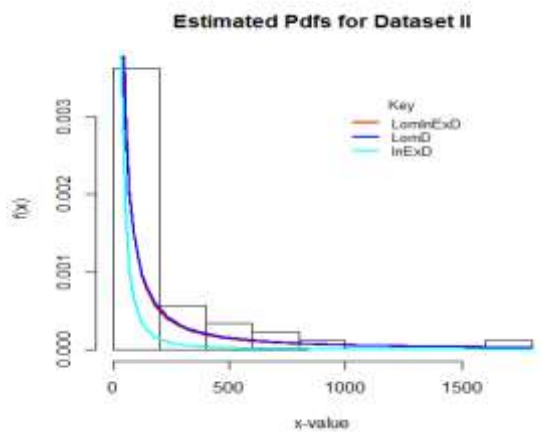
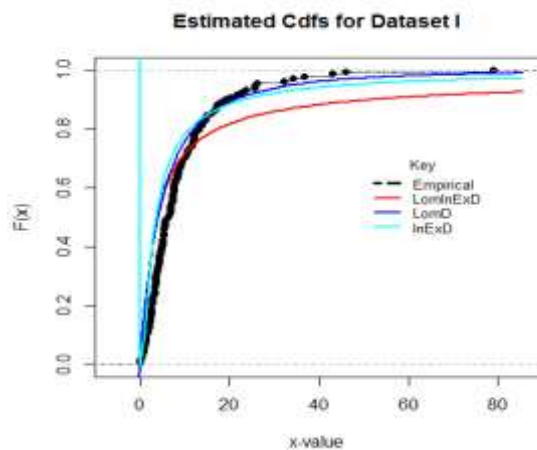
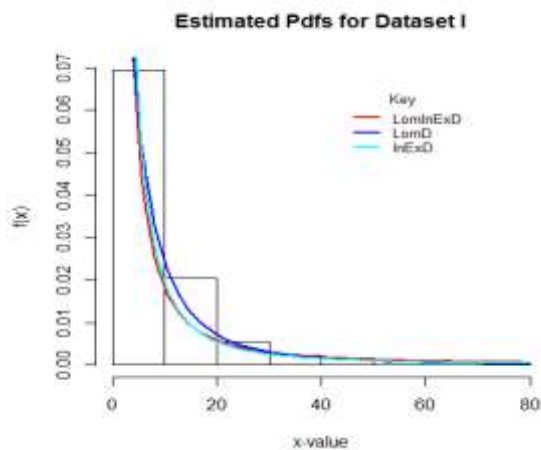
The required computations are carried out using the R package “Adequacy Model” which is freely available from <http://cran.r-project.org/web/package>. In decision making, the model with the lowest values for these statistics would be chosen as the best fitted model.

Table 2: Maximum Likelihood Parameter Estimates for datasets I, II & III

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
DATASET I			
LomInExD	2.156548	7.718681	9.285230
LomD	-	1.894405	9.051699
InExD	2.484729		
DATASET II			
LomInExD	9.430712	4.263884	9.544674
LomD	-	0.3882904	8.0559476
InExD	5.712981		
DATASET III			
LomInExD	7.11601255	7.57096620	0.08453986
LomD	-	7.123845	9.760757
InExD	1.41041		

Table3: The statistics $\hat{\ell}$, AIC, CAIC, K-S and P-Value of K-S for datasets I, II &n III

Distribution	$\hat{\ell}$	AIC	CAIC	K-S	P-Value (K-S)	Ranks
DATASET I						
LomInExD	421.7843	849.5686	849.7621	0.20531	4.115e-05	1 st
LomD	425.8843	855.7686	855.8646	0.21032	2.415e-05	2 nd
InExD	460.3823	922.7646	922.7963	0.23156	2.185e-06	3 rd
DATASET II						
LomInExD	306.2361	618.4722	619.0722	0.35147	2.297e-05	1 st
LomD	307.8454	619.6907	619.9834	0.38919	1.636e-06	2 nd
InExD	353.13	708.2601	708.3553	0.76195	2.2e-16	3 rd
DATASET III						
LomInExD	24.98252	55.96505	56.36505	0.22247	0.003546	1 st
LomD	94.61429	193.2286	193.4253	0.4328	7.729e-11	2 nd
InExD	90.88183	183.7637	183.8282	0.48863	1.068e-13	3 rd



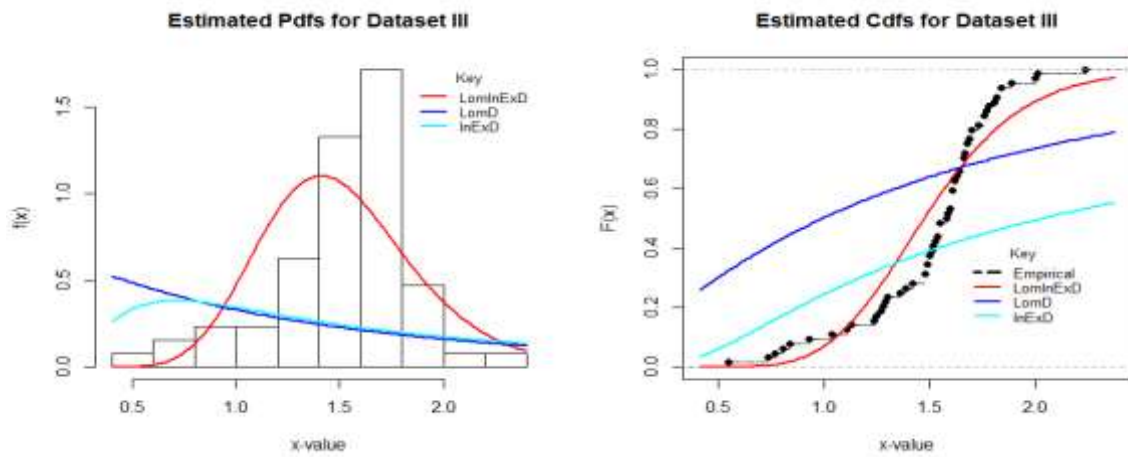
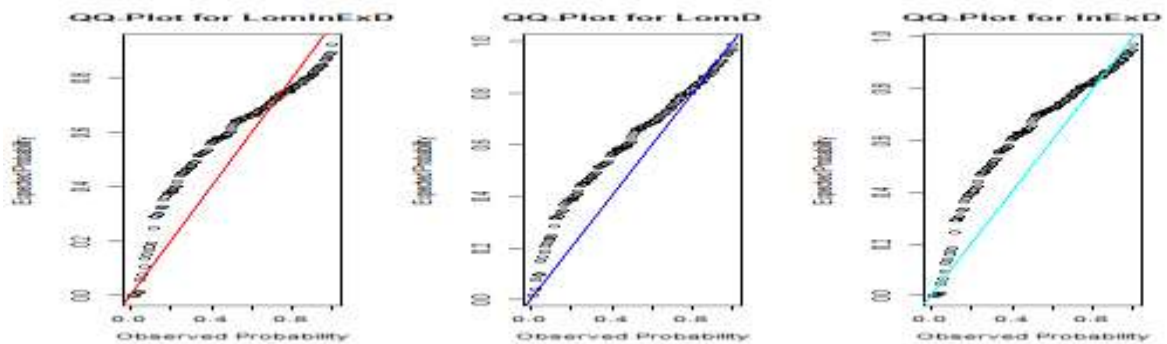
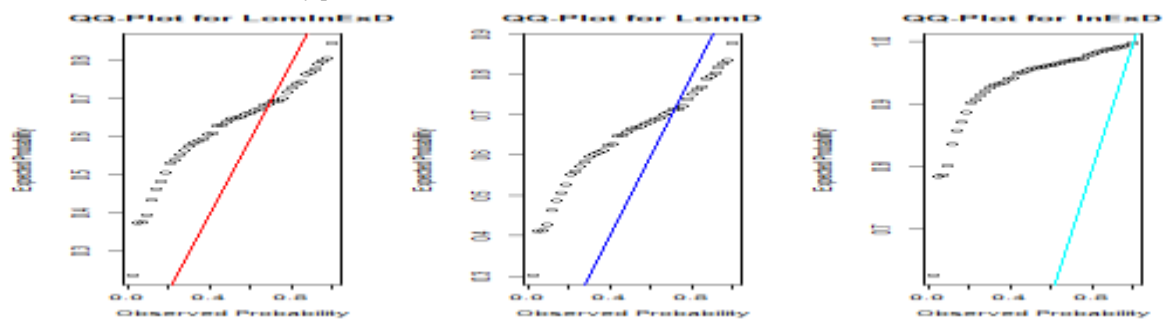


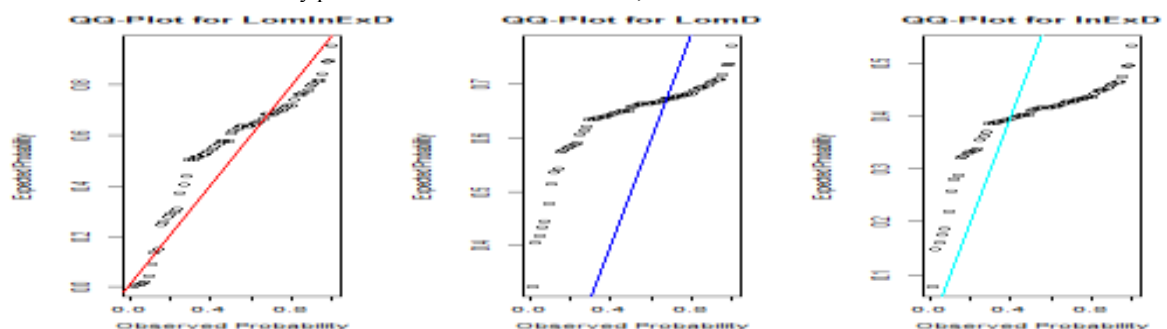
Figure 5: Histogram and plots of the estimated densities and CDFs of the fitted distributions to datasets I, II and III.



Probability plots for the fit of the LomInExD, LomD & InExD based on dataset I.



Probability plots for the fit of the LomInExD, LomD & InExD based on dataset II.



Probability plots for the fit of the LomInExD, LomD & InExD based on dataset III.

Figure 6: Probability plots for the fit of the LomInExD, LomD & InExD to datasets I,II and III.

Table 2 lists the MLEs of the parameters for the fitted models for both dataset I, II and III. The values of the statistics AIC, CAIC, K-S and P-Value (K-S) are presented in Table 3 for datasets I, II and III. For all the datasets, the proposed distribution, LOMINEXD provides the best fit compared to the LOMD and INEXD. Also, the estimated PDFs and CDFs displayed in Figure 5 clearly support the results in Tables 3. Similarly, the probability plots in Figures 5 and 6 for datasets I, II and III respectively also confirm the results in table 3 which agrees that the proposed LOMINEXD is more flexible than the Lomax and the inverse exponential distributions based on the datasets used. The results above show that the Lomax generator of distributions by Cordeiro et al. (2014) is useful for providing additional Skewness and flexibility in continuous distributions just as induced in the Lomax-inverse exponential distribution. The results are also in line with the results of Venegas et al., (2019), Omale et al. (2019), Ieren et al., (2020), Ieren and Kuhe (2018) as well as Ieren et al. (2019) which is an indication that the Lomax-G family by Cordeiro et al., (2014) should be considered in subsequent studies aiming to extend other continuous distributions.

CONCLUSION

This paper introduced a new extension of the inverse exponential distribution called Lomax-inverse exponential distribution. The properties of the distribution are discussed and graphs are used to demonstrate its appropriateness. The derivations of survival function, hazard function, quantile function and ordered statistics of the distribution are done effectively. The method of maximum likelihood estimation is used to estimate the parameters of LOMINEXD. All the plots for the survival function indicate that the Lomax-inverse exponential distribution could be used to analyze age-dependent or time dependent events or variables whose survival decreases as time grows or events where survival rate decreases with time. Also, the plots hazard function of the proposed distribution are decreasing for all parameter values indicating that the model would be useful for modeling events whose failure rates have decreasing shapes or variables whose failure rate increase shortly at initial stage and then decreases with time to the end. The results of the applications showed that the proposed distribution is more flexible compared to the Lomax and inverse exponential distributions and would gain application in many fields especially reliability and survival analysis.

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