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MONOTONICITY OF HAZARD RATE FUNCTION WITH APPLICATION TO THE NEW MIXTURE EXPONENTIAL-GAMMA (NMEG) DISTRIBUTION

***[1]Lawrence C. Nzei, [2]Jacob C. Ehiwario and [3]Joseph T. Eghwerido**

[1]Department of Statistics, University of Benin, Benin city, Nigeria.
[2]Department of Mathematics and Statistics, University of Delta, Agbor, Delta State, Nigeria.
[3]Department of Mathematics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria.

*Corresponding authors' email: lawrencedumebi@gmail.com

ABSTRACT

In this study, a review of the monotonicity of the hazard rate function of a probability distribution for survival models is presented. The conditions for monotonicity were discussed. The concept of monotonicity can be analytically derived and tested using the knowledge of elementary differential calculus. The New Mixture of Exponential-Gamma distribution was considered as a case study to illustrate the usefulness of these conditions on survival data analysis.

Keywords: Monotonicity, Survival Probability Model, Monotone Hazard Rate, Bathtub Hazard Rate

INTRODUCTION

Survival analysis is a unit of statistics that is concerned with the study of the time to the occurrence of a particular event. Several disciplines have interest in the study of the time to the occurrence of certain event in excess of a given threshold and each field of application determine the name it is called. For instance, in Engineering, it is called reliability; in Actuaries science, it is called force of mortality; in Economics, it is called duration of event; in Sociology, it is called event of history; amongst others. Several probability distribution models such as exponential, Weibull, gamma, Logistic, Gompertz, Rayleigh, Lindley distributions etc., have been applied in the study and analysis of survival (reliability) data in literature. Due to the inefficiency of some of these classical distributions, researchers have presented their generalized forms, thus leading to corresponding flexible hazard rate function for analysing survival data. Some of these generalizations include the works of Eugene *et al*. (2002), Ristic and Balakrishnan (2012), Bourguignon *et al.* (2014), Ekhosuehi *et al*. (2016), Ehiwario et al. (2023), Edeme and Okwonu (2024) etc.

Hazard rate (HR) function is one of the probability measures that is of great importance in survival analysis. Its importance is related to the interpretation, which is concerned with the probability that an event will occur in a time interval $(t, t+\delta t)$ given that it survived up to time t. It indicates how risk of failure varies with time. Being informed about the nature (monotonicity) of the hazard rate function, can help a researcher in terms of model selection for survival analysis. Some of this importance can be seen in the application of hazard rate (HR) concept to solve real life problems as used by Kiefer (1988), Bean (2001), Cleves (2008), Milly et al (2015), Laura and Read (2016), Alam and Almalki (2021), Turkson (2022), amongst others.

MATERIALS AND METHODS

Monotonicity of A Function

In this section, the monotonicity of a function is discussed and considered to be the family of both monotone and non-monotone functions. This idea has been studied extensively in literature, reference can be found in Glaser (1980), Desai *et al* (2011), Ekhosuehi *et al* (2019), Shalki et al. (2021), Hornik (2024), Najafi and Marassaei (2025), Schulz and Genest (2025), only to mention a few.

Monotone and Non-Monotone Functions

A function f is said to be monotonic in a given interval say $I \in \Re$ if the curve (graph) of the function is either completely increasing (non-decreasing) or decreasing (non-increasing) in I.

Definition 2.1

Let f be the function of a random variable T, but not necessarily a probability density function (pdf) defined and differentiable in the interval $I \in \Re$ where $t_1, t_2 \in I$ such that $t_1 < t_2$, the followings holds

(i) $f(t)$ is monotone increasing (non-decreasing) function if and only if $f(t_1) \leq f(t_2)$ for all $t_1 < t_2$.

(ii) $f(t)$ is monotone decreasing (non-increasing) function if and only if $f(t_1) \geq f(t_2)$ for all $t_1 < t_2$.

(iii) $f(t)$ is non-monotonic if its curve (graph) is either increasing-decreasing, decreasing-increasing or constant in the interval I

Figures 1 and 2 are graphical illustrations of monotone decreasing and monotone increasing functions respectively.

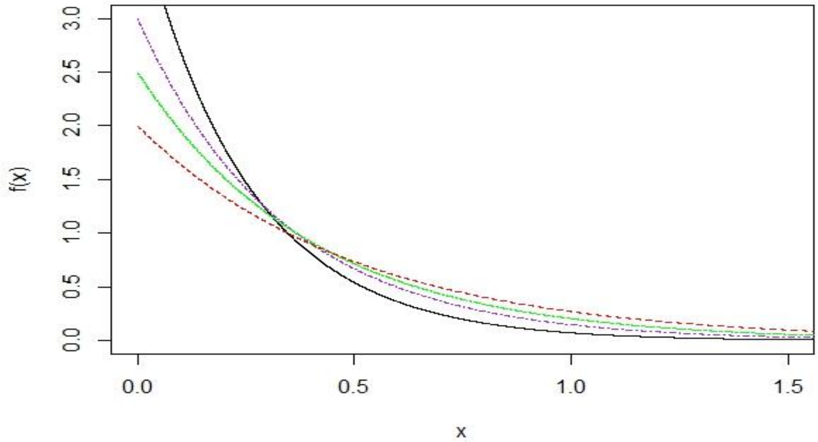

Figure 1: Monotone Decreasing Function

Figure 1 shows the density function of the exponential distribution, as an illustration of monotone decreasing function

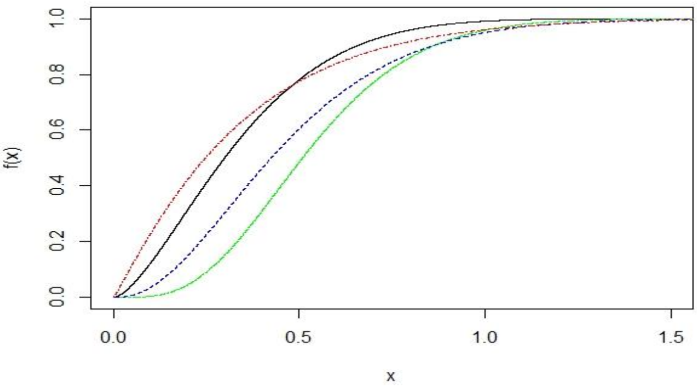

Figure 2: Monotone Increasing Function

Figure 2 shows the cumulative distribution function of the Topp-Leone Lindley distribution, as an illustration of monotone increasing function.

Monotonicity and Derivative

The concept of monotonicity can easily be tested using the knowledge of elementary differential calculus. The test for monotonicity of a function can be explained as follows.

Suppose f is a function of a random variable T, continuous and differentiable on the interval $I \in \Re$ where $t_1, t_2 \in I$ such that $t_1 < t_2$ then

(i) $f(t)$ Is increasing in $I \in \Re$ if $f'(t) > 0$ for all t. This implies monotone increasing function.

(ii) $f(t)$ Is decreasing in $I \in \Re$ if $f'(t) < 0$ for all t. This implies monotone decreasing function.

(iii) $f(t)$ Is non-monotonic in the interval $I \in \Re$ if $\exists\ t_0 \in I$ such that at $t = t_0$, $f'(t_0) = 0$. This means that the function is either at maximum turning point, minimum turning point or inflexion (constant) point at $t = t_0$.

Hazard Rate Function

As stated in the introduction, the hazard rate is of great practical interest in survival analysis. The hazard rate of distribution function is denoted by $h(t)$ and it is defined as:

$$h(t) = \underset{\delta t \to 0}{Limit}\left\{\frac{\Pr(t \le T < t + \delta t / T \ge t)}{\delta t}\right\} = \frac{f(t)}{S(t)} = \frac{f(t)}{1 - F(t)} \qquad (1)$$

where $S(t)$ is the survival function, $F(t)$ is the cumulative distribution function (cdf) and $f(t)$ is the probability density function (pdf).

The hazard rate of a random variable t, is the instantaneous rate of failure given that the individual item (system) survived up to time t. $h(t)\delta t$ is the approximate probability of failure in the interval $[t, t + \delta t)$ given that it survived up to time t. This interpretation has drawn the interest of researchers in literature. Some of these can be found in Barlow *et al* (1963), Thomas *et al* (1971), Glaser (1980), Alam and Almalki (2021), Turkson (2022), and host an of others.

Monotone Hazard Rate Function

Given that $f(t)$ and $F(t)$ are the pdf and the cdf of random variable T respectively. From (1), if $h(t) = f(t)/[1 - F(t)]$ is increasing for all values of $t\in(0,\infty)$ then $f(t)$ or $F(t)$ has a monotone-increasing hazard rate (IHR). In the same way, if $h(t)=f(t)/[1-F(t)]$ is decreasing for all values of $t\in(0,\infty)$ then $f(t)$ or $F(t)$ has a monotone decreasing hazard rate (DHR).

In survival analysis, probability distribution with IHR are useful models for fitting data from a system without improvement over the time. For instance, wearing of machine parts, aging in life organs etc. while the probability distribution with DHR represents a system with an improvement over the time. For instance, maintenance of a device helps to reduce failure, reduction of polio patients because of vaccination, and so on.

As an illustrative example, we consider the Lindley Exponentiated-Exponential (LEE) distribution proposed by Nzei and Ekhosuehi (2017). The pdf and cdf of LEE are given by:

$$f(t)=\frac{\beta\theta^2}{1+\theta}(1+\beta t)e^{-\beta\theta t},\ t>0;\ (\beta,\theta)>0 \quad (2)$$

and

$$F(t)=1-\left(1+\frac{\beta\theta t}{1+\theta}\right)e^{-\beta\theta t},\ t>0;\ (\theta,\beta)>0 \quad (3)$$

The hazard rate function which is of great interest in this study is given by:

$$h(t)=\frac{\beta\theta^2(1+\beta t)}{1+\theta+\beta t} \quad (4)$$

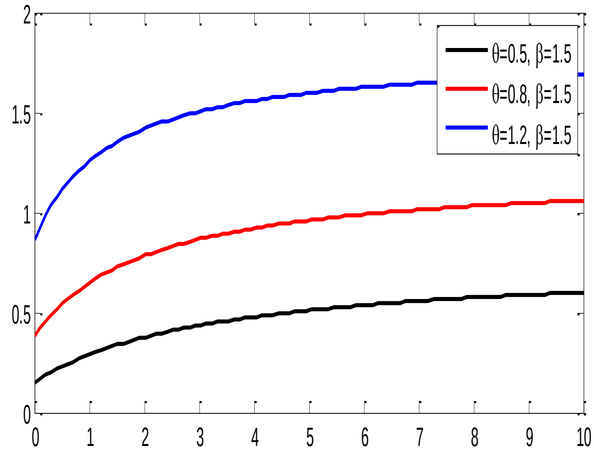

Figure 3: Monotone Increasing Hazard Rate of LEE Distribution

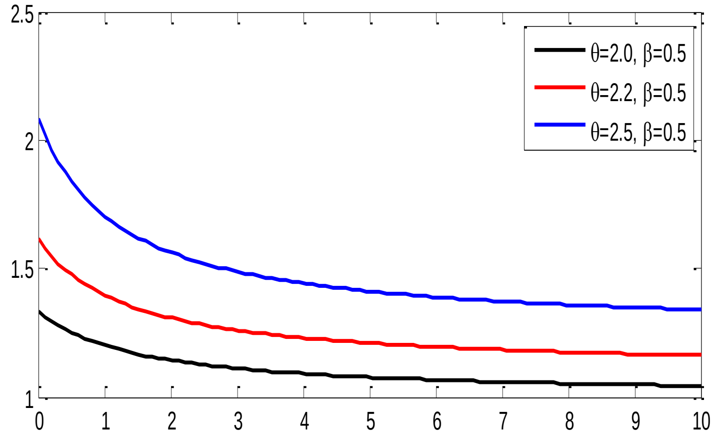

Figure 4: Monotone Decreasing Hazard Rate of LEE Distribution

Figures 3 and 4 shows the IHR and the DHR functions respectively of the LEE distribution. The hazard rate function of the LEE distribution exhibit both increasing hazard rate (IHR) and decreasing hazard rate (DHR) for some fixed values of the parameters. Hence, the LEE distribution has a monotone hazard function.

Non-Monotone Hazard Rate

The development of more flexible distribution model to solve real life problems, gave rise to a corresponding more flexible hazard rate in survival analysis. In this case, the hazard rate $h(t)$ is not limited at some points $t_i\in(0,\infty)$, $i=0,1,2,3,\ldots$ from either increasing continuously or decreasing continuously which forms turning points. Hence, $f(t)$ or $F(t)$ has a non-monotone hazard rate. The nature of the Non-Monotone hazard rate function includes the Bathtub hazard rate (BTHR), upside down Bathtub hazard rate (UBTHR) and constant hazard rate (CTHR).

Monotonicity of HRF

The analytical determination of the conditions for monotonicity of hazard rate was considered in this section. The nature of the hazard rate includes IHR, DHR, BTHR, UBTHR and CTHR. We make the following assumptions for better understanding.

i. $f(t)$ is defined and positive in $(0,\infty)$ i.e. $f(t)>0$ for all $t>0$
ii. $f(t)$ is continuous in $(0,\infty)$
iii. $f(t)$ is twice differentiable in $(0,\infty)$

Conditions for Monotonicity of HRF

In this section, we discuss the conditions for the monotonicity of hazard rate of a probability distribution for modelling in survival analysis following the methodology of Glaser (1980). To obtain these conditions analytically, we define $Z(t)$ as the reciprocal of the hazard rate function, see Desai *et al* (2011) for more detail.

$$Z(t)=\frac{1}{h(t)}=\frac{S(t)}{f(t)} \quad (5)$$

where $Z(t)>0$ and twice differentiable on the interval $(0,\infty)$, so that

$$Z'(t)=\frac{-[f(t)]^2-f'(t)S(t)}{[f(t)]^2}=-1-\frac{f'(t)S(t)}{[f(t)]^2}=Z(t)\phi(t)-1 \quad (6)$$

where $\phi(t)=-\frac{f'(t)}{f(t)}$

To study the monotonicity of hazard rate of any survival time distribution model, it is sufficient to examine the behaviour of $\phi(t)$ which determine the shape of the hazard rate

$$\phi(t)=-\frac{f'(t)}{f(t)}=-\eta'(t)$$

Since $\frac{d}{dt}\ln[f(t)]=\frac{f'(t)}{f(t)}=\eta'(t)$

Now,

$$\phi'(t)=-\eta''(t) \quad (7)$$

where $\eta(t)=\ln[f(t)]$ and $\phi(t)=-\eta'(t)$. Hence the following conditions holds for the hazard rate monotonicity:

(i) If $\phi'(t)>0$ for all $t>0$, then we have IHR
(ii) If $\phi'(t)<0$ for all $t>0$, then we have DHR
(iii) Given that there exist $t_0>0$ such that $\phi'(t)<0$ for all $t\in(0,t_0)$, $\phi'(t_0)=0$ and $\phi'(t)>0$, $t>t_0$, then we have BTHR with minimum turning point at t_0
(iv) Given that there exist $t_0>0$ such that $\phi'(t)>0$ for all $t\in(0,t_0)$, $\phi'(t_0)=0$ and $\phi'(t)<0$, $t>t_0$, then we have UBTHR with maximum turning point at t_0
(v) If $\phi'(t_0)=0$ for all $t>0$, then we have CTHR.

Remark: it follows from the above conditions that a researcher can determine the nature of hazard rate function of a probability distribution with the pdf provided the that it is defined and twice differentiable in the given interval *I*.

RESULTS AND DISCUSSION

Application to New Mixture of Exponential-Gamma (Nmeg) Distribution.

In this section, the application of the hazard rate monotonicity condition is illustrated with the New Mixture of Exponential-Gamma (NMEG) Distribution proposed by Ekhosuehi *et al* (2020).

The density function of the new mixture distribution is given as

$$f(t;\beta,\lambda)=\frac{\lambda}{1+\beta}\left\{1+\frac{\beta(\lambda t)^{\beta-1}}{\Gamma(\beta)}\right\}e^{-\lambda t};\ t>0,\lambda>0,\ \beta>0 \quad (8)$$

Equation (5) is a mixture of exponential, $f_1(t)$ and gamma $f_2(t)$ distributions respectively of a random variable T, with the mixture density given by:

$$f(t)=wf_1(t)+(1-w)f_2(t)$$

where $0\le w\le 1$ such that $w=\frac{1}{1+\beta}$, is the mixing proportion.

The corresponding cumulative distribution function is given by:

$$F(t,\beta,\lambda)=\frac{1}{1+\beta}\left\{1-e^{-\lambda t}+\frac{\beta\gamma(\beta,\lambda t)}{\Gamma(\beta)}\right\} \quad t>0,\ \lambda>0,\ \beta>1 \quad (9)$$

where $\gamma(\beta,\lambda t)$ is the lower case incomplete gamma function and $\Gamma(\beta)$ is the complete gamma function.

Special Case of The NMEG Distribution

The NMEG distribution includes some existing distributions as special cases:

i. If the shape parameter $\beta=1$, then the NMEG reduces to exponential distribution with density function

$$f_E(t;\lambda)=\lambda e^{-\lambda t};\ t>0,\lambda>0$$

ii. If $\lambda=1$, the NMEG reduces to standardized one parameter Lindley distribution with pdf

$$f_{SL}(t)=\frac{1}{1+\beta}\left(1+\frac{\beta t^{\beta-1}}{\Gamma(\beta)}\right)e^{-t};\ t>0,\beta>0$$

The Survival and Hazard Rate Function

The survival (Reliability) and the hazard rate functions of the NMEG distribution are defined respectively as:

$$S(t;\lambda,\beta)=1-F(t;\lambda,\beta)=1-\frac{1}{1+\beta}\left\{1+\frac{\beta\gamma(\beta,\lambda t)}{\Gamma(\beta)}-e^{-\lambda t}\right\} \quad (10)$$

and

$$h(t;\lambda,\beta)=\frac{f(t)}{S(t)}=\frac{\lambda\left\{1+\frac{\beta(\lambda t)^{\beta-1}}{\Gamma(\beta)}\right\}e^{-\lambda t}}{1+\beta-\left\{1+\frac{\beta\gamma(\beta,\lambda t)}{\Gamma(\beta)}-e^{-\lambda t}\right\}} \quad (11)$$

Monotonicity of NMEG Hazard Rate

In this section, the monotonicity of the NMEG distribution hazard rate functions in (8) is discussed using theorem 1 below.

Theorem 1: The monotonicity of the hazard rate function of the NMEG distribution can be summarized as follows:

(i) Constant for $\beta=1$, for all $\lambda,t>0$
(ii) Decreasing when $\beta<1$, for all $\lambda,t>0$
(iii) Increasing when $1<\beta<3$, for all $\lambda,t>0$
(iv) Bathtub if $\beta\ge 3$, $\lambda>1$ for all $t>0$

Proof.

Given that $f(t)$ is the pdf of the NMEG distribution, we examine the behaviour of $h(t)$ by considering the behaviour of $\phi'(t)$ defined in (7) as:

$$\phi'(t)=-\eta''(t)$$

where

$$\eta(t)=\ln f(t)=\ln\left[\frac{\lambda}{(1+\beta)\Gamma(\beta)}\right]+\ln\left[\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}\right]-\lambda t$$

$$\eta'(t)=\frac{\beta(\beta-1)\lambda^{\beta-1}t^{\beta-2}}{\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}}-\lambda \quad (12)$$

$$\eta''(t) = \frac{\left[\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}\right]\beta(\beta-1)(\beta-2)\lambda^{\beta-1}t^{\beta-3} - \left[\beta(\beta-1)\lambda^{\beta-1}t^{\beta-2}\right]^2}{\left[\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}\right]^2} \quad (13)$$

Thus

$$\phi(t) = -\eta'(t) = \lambda - \frac{\beta(\beta-1)\lambda^{\beta-1}t^{\beta-2}}{\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}} \quad (14)$$

and

$$\phi'(t) = -\eta''(t) = \frac{\left[\beta(\beta-1)\lambda^{\beta-1}t^{\beta-2}\right]^2 - \left[\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}\right]\beta(\beta-1)(\beta-2)\lambda^{\beta-1}t^{\beta-3}}{\left[\Gamma(\beta)+\beta\lambda^{\beta-1}t^{\beta-1}\right]^2} \quad (15)$$

From equation (15), using the shape parameter β as the determining parameter, we obtain the results as follows:

(i) At $\beta = 1$ and $\lambda > 0$, $\phi'(t) = 0$ for all values of t, hence the NMEG has a constant hazard rate (CTHR) function.

(ii) If $\beta < 1$ and $\lambda > 0$, we can clearly see that $\phi'(t) < 0$ for all t. hence the NMEG hazard rate is decreasing (DHR).

(iii) If $1 < \beta < 3$, for all $\lambda, t > 0$, we have $\phi'(t) > 0$ for all values of t, this means that NMEG has increasing hazard rate (IHR).

(vi) For $\beta \geq 3$ and $\lambda > 1$, there exist $t_0 > 0$ defined in NMEG distribution as $t_0 = \left(\frac{(\beta-2)\beta!}{\beta\lambda^{\beta-1}}\right)^{\frac{1}{\beta-1}}$ such that $\phi'(t) < 0$ for all $t \in (0, t_0)$, $\phi'(t_0) = 0$ and $\phi'(t) > 0$ for $t > t_0$. This means that $\phi'(t)$ changes its sign from negative to positive as t increases. Thus, bathtub hazard rate (BTHR) with minimum point at t_0

These results are demonstrated using Figure 5.

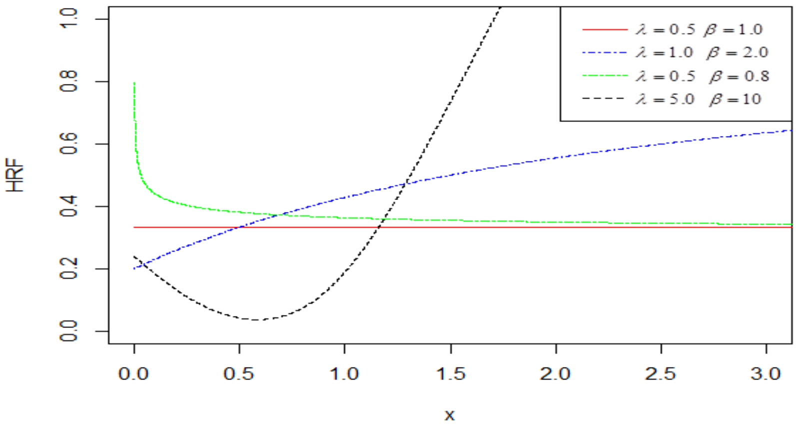

Figure 5: The Hazard Rate Function of NMEG distribution

Figure 5 above is the graphical illustration of the monotonicity of the NMEG distribution as proved above theorem.

CONCLUSION

In this study, we discussed the useful conditions for determining the nature (monotonicity) of hazard rate of probability distribution analytically. It was established that the monotonicity of hazard rate function of probability distribution can be determine when the density function is defined and twice differentiable. The NMEG distribution was used to illustrate the application of these conditions to HRF of Lifetime models.

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