FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 4, February, 2026, pp 1 – 3
DOI: https://doi.org/10.33003/fjs-2026-1004-4302

STRONGLY CONTINUOUS SEMIGROUP ON DECOHERENCE-FREE SUBALGEBRA (DFS) OF QUANTUM MARKOV SEMIGROUP (QMS)

***[1]Ezekiel Abiodun Oluwafemi, [2]Micheal Oluniyi Ogundiran, [1]Ubong Sam Idiong and [1]Bankole Vincent Akinremi**

[1]Department of Mathematics, Adeyemi Federal University of Education, Ondo State, Nigeria.
[2]Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria.

Corresponding author's email: oluwafemiabiodun1234@gmail.com

ABSTRACT

This study examines the strongly continuous semigroup on the decoherence-free subalgebra (DFS) of the QMS. Initially, we derived the infinitesimal generator (IG) for the DFS. A theorem characterizing a strongly continuous semigroup on the decoherence-free subalgebra was established. Strongly continuous semigroups offer a natural framework for modeling the time evolution of quantum systems influenced by a quantum Markov semigroup. When paired with the concept of decoherence-free subalgebras, they enable the modeling of systems that are immune to decoherence due to the specific nature of their interaction with the environment. The dynamics on these subalgebras are governed by the restricted semigroup, which can be analyzed through operator theory and the tools of quantum Markov processes. This approach is particularly relevant for applications in quantum error correction, quantum computing, and quantum control, where isolating parts of a system from decoherence effects is crucial.

Keywords: strongly continuous semigroup (SCS), decoherence-free subalgebra (DFS), quantum Markov semigroup (QMS), infinitesimal generator (IG), quantum dynamical semigroup (QDS), Noise

INTRODUCTION

A QMS as a non-abelian extension of Kolmogorov semigroups, models the memoryless evolution of microscopic systems. Originating in studies of open quantum systems, its foundation lies in earlier abstract concepts like the characterization of completely positive maps (Alick & Lendi, 1987). Quantum channels and quantum Markov semigroups (QMS) characterize open quantum systems that are affected by noise through their interaction with the environment (Agredo et al., 2021). Decoherence-free subalgebras represent observables unaffected by noise, essential for studying QMS structure (Engel & Nagel, 2006) and analyzing decoherence (Agredo et al., 2014).

Given $X=(X_t)_{t\geq 0}$and$A=B(H)$ on the Hilbert space H. A subalgebra of X that is decoherence-free is

$$N(X)=\{\xi\in B(H): X_t(\xi^*\xi)=X_t(\xi)^*X_t(\xi), X_t(\xi\xi^*) = X_t(\xi)X_t(\xi)^*\forall t\geq 0\}.$$

Haase (2006) studied embedding operators into C_0-semigroups. It analyzed the embeddability of positive real operators that are bounded and map real vectors in complexified Banach spaces. If every operator X_t is real, then a strongly continuous semigroup (SCS) $(X_t)_{t\geq 0}$ is considered real (Haase, 2006). Oluwafemi et al. (2024) developed an analog of GKSL for the infinitesimal generator of decoherence-free subalgebra of Quantum Markov semigroup. The GKSL developed was then applied on $M_n(\mathbf{c})$. The result showed that the matrix space $M_n(\mathbf{c}) = V_0 \oplus V_-$ will not decohere under N(T), i.e $V_0 =$ N(T) implies $M_n(\mathbf{c}) =$ N(T) since $V- = \{0\}$. Also, by introducing GKSL of N(T) on $M_n(\mathbf{c})$, $L(x^*x) = 0$.

Oluwafemi et al. (2023) also studied ergodic theorem for quantum dynamical semigroup on decoherence-free subalgebra of quantum Markov semigroup. A quantum dynamical semigroup of non-expansive maps which has at least one stationary point was established on decoherence-free subalgebra. The measurability of the semigroup was then used to ensure the weak convergence of the Cesaro means to a stationary point. Abdulkarim et al. (2025) extended classical fixed-point results by exploring hybrid fixed points within semigroups of transformation. Hybrid fixed points generalize standard fixed points by incorporating auxiliary functions, allowing for broader applications in iterative methods and computational mathematics. Key results on hybrid fixed points were established by considering contractive and nonexpansive mappings in semigroups using Banach's contraction principle and related fixed-point theorems.

The dynamics on decoherence-free subalgebras are governed by the restricted semigroup, which can be analyzed using the tools of operator theory and quantum Markov processes. This approach is important for applications in quantum error correction, quantum computing, and the study of quantum control, where it is necessary to isolate certain parts of the system from decoherence effects. The mathematical theory of strongly continuous semigroups and decoherence-free subalgebras is deeply tied to practical concerns in quantum technology and quantum infostructure. Understanding how quantum systems evolve under a quantum Markov semigroup and how certain subspaces or algebras remain immune to decoherence is key to building more robust, error-resistant quantum systems.

In real-world quantum systems, these concepts help in designing error correction strategies, controlling quantum states, and building fault-tolerant systems that can withstand environmental noise that is essential for the development of practical quantum computers and quantum communication networks. In Section 2, we will introduce the fundamental definitions, while our main result will be presented in Section 3.

Preliminaries

This section defines key terms for later discussion.

Definition 1: Given $X:\Psi\to\Psi$, C_0 -semigroup is an operator family that satisfies;

1. $X_{t+s}=X_tX_s\forall t,s\in\mathbb{R}^+$
2. $X_0=1$, the identity operator on Ψ, and
3. $\lim_{t\to 0} X_t h\to h$ for each $h\in\Psi$ with respect to the norm on Ψ.

Definition 2: Considering the laws of quantum mechanics, a QDS on a von Neumann algebra (VNA) is a one-parameter family of entirely positive, trace-preserving linear maps that depict the time evolution of a quantum system.

Definition 3: QDS on A is called a QMS (or sub-Markov) if $X_t(I) = I, (for I \geq X_t(I)) \forall t \geq 0$, where $I$is the unit operator on the algebra.

Definition 4: The infinitesimal generator (IG) of QMS is a linear (but may not be bounded) operator $L : D\,(L) \to A$such that

$$L(\upsilon) = \lim_{t\to 0} \frac{X_t(\upsilon) - \upsilon}{t}, \upsilon \in D(L). \tag{1}$$

The domain of L, represented by $D(L)$, is the cluster of $\upsilon \in A$where the limit can be obtained.

Definition 5: Given $X : N\,(X) \to N(X)$, A C_0 semigroup on a decoherence-free subalgebra will be defined as operator family that satisfies;

1. $X_{t+s} = X_t X_s \forall t, s \in \mathbb{R}^+$,
2. $X_0 = 1$, the unit operator on $N(X)$, and
3. $\lim_{t\to 0} X_t \xi^*\xi \to \xi^*\xi$ for each $\xi^*\xi \in N(X)$ considering the norm on $N(X)$.

Existence Results

Lemma 1**:** If given $\xi \in N(X)$, then the infinitesimal generator of $N(X)$ is

$$L(\xi^*\xi) = \xi^* \lim_{h\to 0} \frac{X_h(\xi) - \xi}{h} + \lim_{h\to 0} \frac{X_h(\xi^*) - \xi^*}{h} \xi.$$

Let $P = \lim_{h\to 0} \frac{X_h(\xi)-\xi}{h}$ and $\Xi = \lim_{h\to 0} \frac{X_h(\xi^*)-\xi^*}{h}$

Proof. The generator of QMS on Ψ is

$$L(\xi) = P, \xi \in D(L).$$

Where the domain of L, denoted by $D(L)$, is the cluster of $\xi \in \Psi$ where the limit can be obtained. Let $\xi \in N(X)$,

$$L(\xi^*\xi) = \lim_{h\to 0} \frac{X_h(\xi^*\xi) - \xi^*\xi}{h}$$
$$= \lim_{h\to 0} \frac{X_h\xi^* X_h\xi - \xi^*\xi}{h}$$
$$= \Xi X_t(\xi) + \xi^* P$$
$$= \xi^* P + \Xi\xi$$
$$= \xi^* L(\xi) + L(\xi^*)\xi.$$

Lemma 2**:** If given $\xi \in N(X)$ and $Z = \int_0^h X_s\,(\xi^*\xi)ds \in D(L) for t \geq 0$. Then,

$$L(Z) = \int_0^t X_s L\,(\xi^*\xi) ds$$

Proof. Defining the generator L,

$L(\xi) = P, \xi \in D(L)$ for $x \in N(X)$, and $Z \in D(L)$ we have

$$L(Z) = \lim_{h\to 0} \frac{X_h(Z) - Z}{h}$$
$$= \lim_{h\to 0} \frac{\int_0^h X_t X_s(\xi^*\xi)ds - Z}{h}$$
$$= \int_0^h X_s \lim_{h\to 0} \frac{X_h\xi^*\xi - \xi^*\xi}{h} ds$$

Since for $\xi \in N(X)$, $L(\xi^*\xi) = \lim_{h\to 0} \frac{X_h(\xi^*\xi)-\xi^*\xi}{h}$, we have $L(Z) = \int_0^h X_s\, L(\xi^*\xi)ds$.

Theorem 1**:** Given that X_t is a C_0 semigroup and L be its generator. Then

1. For $\xi \in N(X)$, $\lim_{t\to 0} \frac{1}{h} \int_t^{t+h} X_s\,(\xi^*\xi)ds = X_t(\xi^*\xi)$
2. For $\xi \in N(X)$, $\int_0^t X_s\,(\xi^*\xi)ds \in D(L) and L\left(\int_0^t X_s(\xi^*\xi)ds\right) = 2X_t\xi^*\xi - 2\xi^*\xi$.
3. For $\xi \in N(X)$, $X_t(\xi^*\xi) \in D(L) and \frac{d}{dt} X_t(\xi^*\xi)|_{t=0} = LX_t(\xi^*\xi) = X_t L(\xi^*\xi)$.
4. For $\xi \in N(\mathrm{X})$, $\int_s^t X_\tau\, L(\xi^*\xi)d\tau = 2X_t\xi^*\xi - 2X_s\xi^*\xi$.

Proof: (1) by definition (4) and for $h > 0$, $X'_t(\xi^*\xi) = \lim_{h\to 0} \frac{X_{t+h}(\xi^*\xi) - X_t(\xi^*\xi)}{h}$.

By definition (5), $\int X'_t(\xi^*\xi)dt = \int \lim_{h\to 0} \frac{X_{t+h}(\xi^*\xi) - X_t(\xi^*\xi)}{h} dt$.

We have $\int X'_t(\xi^*\xi)dt = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_s(\xi^*\xi)ds = X_t\,(\xi^*\xi) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_s\,(\xi^*\xi)ds$.

Hence, $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_s(\xi^*\xi)ds = X_t\,(\xi^*\xi)$.

(2) Let $\xi \in N(X)$ and $h > 0$

$L(Z) = \int_0^h X_s L(\xi^*\xi)ds$ and by lemma (1),

$L(\xi^*\xi) = \xi^* P + \Xi\xi$.

So,

$$\int_0^t X_s\, L(\xi^*\xi)ds = \int_0^t X_s\,(\xi^* L(\xi) + \xi L(\xi^*))ds$$
$$= \int_0^t (\xi^* \lim_{h\to 0} \frac{X_{s+h}\xi - X_s\xi}{h}$$
$$+ \lim_{h\to 0} \frac{X_{s+h}(\xi^*) - X_s\xi^*}{h} \xi)ds$$
$$= \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_s\,\xi^*\xi ds - \lim_{h\to 0} \frac{1}{h} \int_0^{0+h} X_s\,\xi^*\xi ds$$
$$+ \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_s\xi^*\xi ds - \lim_{h\to 0} \frac{1}{h} \int_0^{0+h} X_s\,\xi^*\xi ds.$$

So, by lemma (1) above, we have

$$X_t(\xi^*\xi) - X_0(\xi^*\xi) + X_t(\xi^*\xi) - X_0(\xi^*\xi)$$
$$= X_t(\xi^*\xi) - \xi^*\xi + X_t(\xi^*\xi) - \xi^*\xi$$
$$= X_t(\xi^*\xi) + X_t(\xi^*\xi) - 2\xi^*\xi$$
$$= 2X_t(\xi^*\xi) - 2\xi^*\xi.$$

(3) For $\xi \in N(X)$ and $h > 0$. By lemma (1)

$L(\xi^*\xi) = \xi^* P + \Xi\xi$.

From L.H.S $X_t L(\xi^*\xi) = X_t(\xi^* P + \Xi\xi)$

$$= X_t(\xi^* \lim_{h\to 0} \frac{X_{h+0}(\xi) - X_0\xi}{h} + \lim_{h\to 0} \frac{X_{h+0}(\xi^*) - X_0\xi^*}{h} \xi).$$

By definition (4), we have

$$X_t(\xi^* \frac{dX_t\xi}{dt}|_{t=0} + \frac{dX_t\xi^*\xi}{dt}|_{t=0})$$
$$= X_t\xi^* \frac{dX_t\xi}{dt}|_{t=0} + \frac{dX_t\xi^*}{dt} X_t\xi|_{t=0})$$
$$= \frac{d}{dt} X_t(\xi^*) X_t(\xi)|_{t=0} = \frac{d}{dt} X_t(\xi^*\xi)|_{t=0}.$$

Hence, $\frac{d}{dt} X_t(\xi^*\xi)|_{t=0} = LX_t(\xi^*\xi) = X_t L(\xi^*\xi)$.

(4) For $\xi \in N(X)$ and $h > 0$. By lemma (1)

$L(\xi^*\xi) = \xi^* P + \Xi\xi$.

So,

$\int_s^t X_\tau\,[\xi^* L(\xi) + \xi L(\xi^*)]d\tau = \int_0^t X_\tau\,[\xi^* P + \Xi]d\tau - \int_0^s X_\tau\,[\xi^* P + \Xi]d\tau$

$$= \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_\tau\xi^*\,\xi d\tau - \lim_{h\to 0} \frac{1}{h} \int_0^h X_\tau\,\xi^*\xi d\tau$$
$$+ \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} X_\tau\,\xi^*\xi d\tau - \lim_{h\to 0} \frac{1}{h} \int_0^h X_\tau\,\xi^*\xi d\tau$$
$$- \lim_{h\to 0} \frac{1}{h} \int_s^{s+h} X_\tau\xi^*\xi d\tau + \lim_{h\to 0} \frac{1}{h} \int_0^h X_\tau\xi^*\xi d\tau$$
$$- \lim_{h\to 0} \frac{1}{h} \int_s^{s+h} X_\tau\,\xi^*\xi d\tau + \lim_{h\to 0} \frac{1}{h} \int_0^h X_\tau\,\xi^*\xi d\tau.$$

So, by lemma (1) above, we have

$$X_t\xi^*\xi - X_0\xi^*\xi + X_t\xi^*\xi - X_0\xi^*\xi - X_s\xi^*\xi + X_0\xi^*\xi - X_s\xi^*\xi + X_0\xi^*\xi$$
$$= 2X_t\xi^*\xi - 2X_s\xi^*\xi + 2X_0\xi^*\xi - 2X_0\xi^*\xi$$
$$= 2X_t\xi^*\xi - 2X_s\xi^*\xi.$$

CONCLUSION

This paper explores the topic of strongly continuous semigroups on the decoherence-free subalgebra (DFS) of QMS. Initially, the infinitesimal generator of the subalgebra, which exhibits no coherence, was derived and verified. Following this, a theorem was formulated to characterize a strongly continuous semigroup on the DFS of QMS. These results extend and refine the findings presented by Agredo et al. (2021) by offering more mathematical detail and introducing important concepts such as strongly continuous semigroups**,** decoherence-free subalgebras**,** and restricted semigroups. These ideas help to refine the general framework of quantum Markov semigroups by showing how quantum systems can be structured to mitigate noise and preserve coherence. Moreover, the explicit connection to quantum error correction and quantum control provides a concrete real-world application of these concepts, making the theory not just a mathematical tool, but a blueprint for building fault-tolerant quantum systems in practice.

AUTHORS' CONTRIBUTIONS

EAO- concept formulation, original draft and resources, M.O.O-supervision, U.S.I-typesetting, final editing and resources, BVA-resources.

REFERENCES

Abdulkarim, A., Muhammad, A. A., & Abdullahi, A. (2025). A study of hybrid fixed points on semigroups of transformation. *fudma journal of sciences*, *9*(3), 77-79. https://doi.org/10.33003/fjs-2025-0903-3191

E. A. Oluwafemi, M. O. Ogundiran, I. O. Adisa; & J. I. Opadara (2024).Application of GKSL Master Equation of Infinitesimal Generator of Decoherence-free Subalgebra of Uniformly Continuous Quantum Markov Semigroup. In K. S. Adegbie, A. A. Akinsemolu, & B. N. Akintewe (Eds.), *Exploring STEM frontiers: A festschrift in honour of Dr. F. O. Balogun*. SustainE.

J. Agredo, F. Fagnola and D. Poletti, Gaussian Quantum Markov Semigroups on a One-Mode Fock Space: Irreducibility and Normal Invariant States. Open Sys. Information Dyn. **28** (1) 2150001 (2021).

J. Agredo, F. Fagnola and R. Rebolledo, Decoherence free subspaces of a quantum Markov semigroup, J. Math. Phys. **4** 1343-1362 (2014).

Oluwafemi, E. A., Ogundiran, M. O., & Fabelurin, O. (2023). Ergodic theorem for quantum dynamical semigroup on decoherence-free subalgebra of a quantum markov semigroup. *Unilag Journal of Mathematics and Applications*, *3*, 25-34.

R. Alicki and Lendi, K. (1987). Quantum Dynamical Semigroups and Applications. Lecture Notes in Physics (LNP, volume 286), Springer- Verlag.

P. Blanchard and Olkiewicz, R. (2006). Decoherence as irreversible dynamical process in open quantum systems. Open quantum systems, Recent developments Vol. III, Springer-Verlag Berlin, 117-159.

K. Engel, R. Nagel, One parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.

M. Haase, The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169, Birkhauser Verlag, Basel, 2006.