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A REPLENISHMENT POLICY WITH TWO-STAGE CONSUMPTION RATES FOR PARTIALLY REPAIRED NON-INSTANT DECAYING GOODS AND VARYING STORAGE COST UNDER ALLOWABLE PAYMENT DELAY

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ABSTRACT

Many inventory models for deteriorating items are developed on the assumptions that deteriorated items are not fixed or replaced; instead, they are thrown away and inventory carrying costs are not considered while developing inventory policies. However, in real-world, deteriorated items such as roofs, ceilings, doors, locks, windows, plumbing pipes and taps, electronics, desks, seats, spare parts, and so on are repairs or replaced. Similarly, carrying charges affect inventory total cost, influence order quantity, determine the level of stock to keep, and influence profit, pricing, and budgeting decisions, thus their impact cannot be overlooked when designing inventory policies. In this work, a two-stage consumption rate replenishment policy with variable storage costs under allowable payment delays is examined for partially repaired non-instantaneous decaying goods. Prior to product deterioration, the rate of consumption is a quadratic function of time, and it remains constant thereafter after that. The main aim of this model is to identify optimal cycle length and order quantity amount that minimize the overall variable cost. It is established that solution exist and are unique solutions exist and are distinct. The model is validated using numerical test based on tests of some existing data, and a comparison with the current model reveals that the proposed model performs better in terms of cost minimization and turnover. Some suggestions for lowering the total variable cost of the inventory system are provided based on in light of the sensitivity analysis. The model could be used in inventory management and control of items such as doors, windows, plumbing pipes and taps, electronics, and so on.

Keywords: Replenishment, Two-Stage Consumption rates, Partially Repaired Non-Instant Decaying goods, Variable Storage Cost Varying Storage Cost, Allowable Payment Delay

INTRODUCTION

The majority of studies on inventory models have assumed assumed that holding costs were constant. However, in real-life settings, the holding cost of many products may fluctuate as the time value of money and price index change alter. The cost of storing decaying and perishable commodities when additional storage facilities and services are required can might always be expensive. The holding cost for some items in stock varies linearly with the amount of time they are stored. Typically, the cost of keeping commodities in stock, such as fruits, vegetables, fish, meat, and milk, is higher when better-preserving facilities are utilized to maintain freshness and prevent decomposition, resulting in a lower degradation rate. Furthermore, holding costs can rise due to inflation, bank interest, hiring charges, and so on. Thus, it is critical to examine an inventory model with time-varying holding costs. Baraya and Sani (2011) proposed an EPQ model for delayed deteriorating degrading items that includes a stock-dependent demand rate and a linear time-dependent holding cost. Musa and Sani (2012a) developed created an EOQ model for delayed deteriorating degrading products with a linear time-dependent holding cost. Tayal et al. (2015) proposed created an EPQ model for non-instantaneous deteriorating items in which the demand rate is exponential, the production rate is a function of the demand rate, the holding cost varies over time, partially deteriorated items are sold at a discount from the original price, and completely deteriorated items are superfluous.Sivashankari (2016) investigated an EPQ model for instantaneously decaying items with constant, linear, and quadratic holding costs, and conducted a comparative analysis of these three holding cost constant, linear, and quadratic holding costs. Selvaraju and Ghuru (2018) formulated created EOQ models for instantaneously decaying items with constant, linear, and quadratic holding costs and shortages, and conducted a comparative assessment of constant, linear, and quadratic holding costs. Singhal and Singh (2018) investigated an integrated replenishment model for deteriorating degrading items with multiple market demand rates under volume flexibility, in which the deterioration rate is determined by quality level and time and follows a two-parameter Weibull distribution. Holding cost is assumed supposed to increase linearly over time. Furthermore, Mishra and Singh (2011), Tyagi et al. (2014), and others have published relevant research on inventory models with time-varying holding costs.

When creating inventory policies for items like electronics, fashion, cars and their parts, seasonal goods, rice, beans, yam, maize, and so forth, it would be incorrect to assume that deterioration begins as soon as the items are placed in stock. Therefore, businesses may overestimate the total relevant inventory cost if they are unaware of this characteristic uninformed of the feature of these kinds of things, which could lead to poor decision-making. An inventory model for non-instantaneous deteriorating items with an allowable payment delay was examined by Ouyang et al. (2006). An optimal ideal replenishment strategy for non-instantaneous deteriorating items with stock-dependent demand rates was proposed created by Wu et al. (2006).Shortages are allowed and partially backlogged; the backlogging rate is variable and depends on the waiting time for the next replenishment. Chung (2009) established a thorough proof of the solution technique for non-instantaneous deteriorating products with allowable payment delays. Musa and Sani (2012b) developed created ordering strategies for deteriorating degrading commodities with an acceptable payment delay. The demand rates before and after deterioration occurs set in are different, and both are assumed to remain constant. Maihami and Abadi (2012) examined cooperative pricing and inventory control for non-instantaneous deteriorating products with an allowable payment delay. Shortages are permitted and may result in partial backlog. Wu et al. (2014) gave a remark on optimal replenishment policies for non-instantaneous deteriorating items with price and stock sensitive demand rates where a payment delay is acceptable.Chang et al. (2015) developed an EOQ model for non-instantaneously deteriorating products with order-size-dependent payment delays. The model calculates the appropriate pricing and ordering procedures to maximize total profit per unit time. Babangida and Baraya (2018) created an inventory model for non-instantaneous degrading items with time-dependent quadratic demand under a trade credit regime. The demand rate before degradation is considered to be a time-dependent quadratic, whereas the demand rate after deterioration is believed to be constant because some customers are eager to buy after deterioration has occurred. Muazu et al. (2023) developed an economic order quantity model for non-instant deteriorating decaying goods with three-stage demand rates, linear holding cost and linear reciprocal partial backlogging amount. The average annual demand rates before goods start deteriorating after deterioration decaying, after goods start decaying and during stockouts are not the same are and both taken as constant. The model determined the best time with positive inventory, cycle length and order quantity that reduce entire total variable cost. Ahmed et al. (2025) developed an order quantity model for non-instantaneous deteriorating items with two-level pricing strategies under trade credit policy, two-phase demand rates, linear holding costs, and time-dependent partial backlog rates. The model determines determine the optimal period for positive inventory, cycle length, and order quantity amount to maximize the inventory system's overall profit.

Babangida and Baraya (2019) established an inventory model for non-instantaneous deteriorating items with two components: demand and linear time varying time holding costs under trade credit.The things do not decay quickly while in stock and have a period of preserving their original qualities before deterioration occurs. However, degraded objects are not repaired fixed or replaced; instead, they are discarded thrown, and inventory carrying costs are not considered. Real-world repairs or replacements include roofs, ceilings, doors, locks, windows, plumbing pipes and taps, electronics, desks, seats, spare parts, and so on. Similarly, carrying charges affect inventory total cost, influence order quantity, determine the level of stock to keep, and influence profit, pricing, and budgeting decisions, thus their impact cannot be overlooked when designing inventory policies rules.

This model proposes a replenishment policy with two-stage consumption rates for partially repaired non-instant decaying products and time-varying changing storage costs within the permissible payment delay (see Table 1). The model determines will calculate the ideal cycle length and order quantity to minimize reduce the average total variable cost. Some numerical examples are have been provided to demonstrate the theoretical results of the model. Sensitivity analysis of some model parameters was performed to determine the effect of changing these parameters on the decision variables, and ideas for reducing the average total variable cost of the inventory system were also presented provided.

Table 1: Comparison of Some Existing Literatures Relevant to the Proposed Model

Authors and year	Non-Instant Decaying goods	Two-Stage Consumption rates	Varying Storage Cost	Payment Delay	Closed form solution	Carrying charges	Repairment of item
Ouyang *et al.* (2006)	Yes	No	No	Yes	Yes	No	No
Chung (2009)	Yes	No	Yes	No	Yes	No	No
Baraya and Sani (2011)	Yes	No	Yes	No	Yes	No	No
Musa and Sani (2012a)	Yes	yes	Yes	No	Yes	No	No
Musa and Sani (2012b)	Yes	yes	No	Yes	Yes	No	No
Maihami and Abadi (2012)	Yes	No	No	Yes	No	No	No
Wu *et al.* (2014)	Yes	No	No	Yes	No	No	No
Chang *et al.* (2015)	Yes	No	No	Yes	No	No	No
Babangida&Baraya (2018)	Yes	Yes	No	Yes	Yes	No	No
Babangida&Baraya (2019)	Yes	Yes	Yes	Yes	Yes	No	No
Zulkifilu M. *et al.* (2023)	Yes	Yes	Yes	No	Yes	No	No
Ahmed *et al.* (2025)	Yes	Yes	Yes	Yes	Yes	No	No
Proposed model	Yes	Yes	Yes	Yes	Yes	Yes	Yes

MATERIALS AND METHODS

Methodology Model Description and Formulation

This section describes the proposed model notation, assumptions and formulation. The inventory system is developed based on the following assumptions and notation.

Notation

O_C The ordering cost per order.

P_C The purchasing cost per unitper unit time ($/unit/ year).

S_P The selling price per unit per unit time ($/unit/ year).

I_p The Commission paid commissionpaid in stock by the supplier per Dollar per year ($/unit/year)($I_c \geq I_e$).
I_g The Commission gained commissiongained per Dollar per year ($/unit/year).
T The trade credit period (in year) for settling accounts.
θ Deterioration rate($0 < \theta < 1$).
π The rate at which goods are repaired
i Carrying charges
μ The length of time in which the product exhibits no deterioration.
P The length of the replenishment cycle time (time unit).
I_0 The number of items received at the beginning of the inventory system (units).
$I_1(t)$ The inventory level before deterioration sets in.
$I_2(t)$ The inventory level after deterioration begins.

Assumptions

i. The replenishment rate is infinite.
ii. The lead time is zero.
iii. A single non-instantaneous deteriorating item is considered.
iv. During the fixed period,l, there is no deterioration and at the end of this period, the inventory item deteriorates at the constant rateφ.
v. Deteriorated items are either replaced or repaired.
vi. Demand rate before deterioration begins is a quadratic function of time t and is given by
vii. $\alpha + bt + ct^2$where$\alpha \geq 0, b \neq 0, c \neq 0$.
viii. Demand rate after deterioration sets in is assumed to be constant and is given by$d$.
ix. Storage cost $H(t)$per unit time is linear time dependent and is assumed to be
x. $H(t) = h_1 + h_2 t$; where$h_1 > 0$and $h_2 > 0$.
xi. During the trade credit period$T(0 < T < 1)$, the account is not settled; generated sales revenue is deposited in acommission bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.
xii. Shortages are not allowed.

Formulation of the Model

The inventory system is designed as follows.I_0units of a single product from the manufacturer arrive at the inventory system at the start of each cycle (i.e., at time$t = 0$). During the time span $[0, \mu]$, the inventory level$I_1(t)$gradually depletes due to market demand and is assumed to be a quadratic function of time. At time interval $[\mu, P]$the inventory level$I_2(t)$ is depleting due to combined effects of demand and deterioration and the demand rate at this time is reduced to a constant$d$. At time $t = P$, the inventory level depletes to zero. The behaviour of the inventory system is described in figure 1 below.

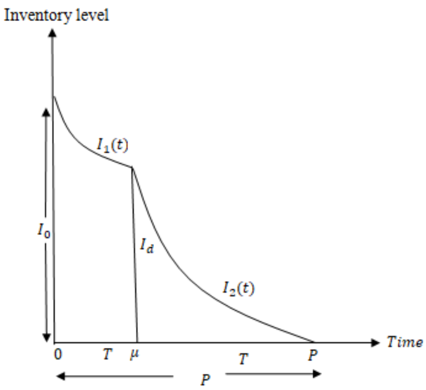

Figure 1: Graphical Representation of Inventory System

Based on the above description, the inventory level $I_1(t)$ at time$t \in [0, P]$ is given by

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2), \quad 0 \leq t \leq \mu \tag{1}$$

$$\frac{dI_2(t)}{dt} + (\theta - \pi)I_2(t) = -d, \quad \mu \leq t \leq P \tag{2}$$

with boundary conditions $I_1(0) = I_0$,$I_1(\mu) = I_2(\mu) = I_d$, and $I_2(P) = 0$ at $t = P$

The solutions of equations (1) and (2) are

$$I_1(t) = \frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(T-\mu)} - 1\right) + a(\mu - t) + \frac{b}{2}(\mu^2 - t^2) + \frac{c}{3}(\mu^3 - t^3) \quad 0 \leq t \leq \mu \tag{3}$$

$$I_2(t) = \frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-t)} - 1\right), \quad \mu \leq t \leq P \tag{4}$$

Also applying the condition$I_2(\mu) = I_d$ at $t = \mu$ into (4) and $I_1(0) = I_0$ at $t = 0$ into (3) to obtain

$$I_d = \frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)} - 1\right) \tag{5}$$

$$I_0 = \frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)} - 1\right) + \left(a\mu + b\frac{\mu^2}{2} + c\frac{\mu^3}{3}\right) \tag{6}$$

(i) *The total demand during the period* $[\mu, T]$ *is given by*

$$D_i = \int_\mu^P d\, dt = d(P - \mu) \tag{7}$$

The Total Number of Deteriorated Items Per Cycle is given by

$N_i = I_d - D_i$

Substituting I_d and D_i from (11) and (14) respectively intoN_i, we obtain

$$N_i = \frac{d}{(\theta-\pi)}\left[e^{(\theta-\pi)(P-\mu)} - 1 - (\theta - \pi)(P - \mu)\right] \tag{8}$$

The Deterioration Cost is given by

$$D_C = P_C\left[\frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)} - 1 - (\theta - \pi)(P - \mu)\right)\right] \tag{9}$$

(iv) The ordering cost per order is given by C_0

The Inventory Storage Cost During the Period [0, P] is given by

$$H(t) = i\left[\int_0^\mu (C_1 + C_2 t)I_1(t)dt + \int_\mu^P (C_1 + C_2 t)I_2(t)dt\right] \tag{10}$$

Substituting (3)and(4) into (10)to obtain

$$= iC_1\left(\frac{d\mu}{(\theta-\pi)}e^{(\theta-\pi)(P-\mu)} + \frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 + \frac{d}{(\theta-\pi)^2}e^{(\theta-\pi)(P-\mu)} - \frac{d}{(\theta-\pi)^2} - \frac{dP}{(\theta-\pi)}\right) +$$

$$iC_2\left(\frac{d\mu^2}{2(\theta-\pi)}e^{(\theta-\pi)(P-\mu)}+\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu}{(\theta-\pi)^2}e^{(\theta-\pi)(P-\mu)}-\frac{dP}{(\theta-\pi)^2}-\frac{d}{(\theta-\pi)^3}+\frac{d}{(\theta-\pi)^3}e^{(\theta-\pi)(P-\mu)}-\frac{dP^2}{2(\theta-\pi)}\right) \quad (11)$$

The Commission Payable

This is the commission paid for the inventory not being sold after the expiration of trade credit period which are categorised into Case 1($0<T\le\mu$and Case 2($\mu<T\le P$).

Case 1: ($0<T\le\mu$)

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate I_c for the items in stock. Thus, the commission payable is given by

$$I_{P1}=P_C I_p\left[\int_T^\mu I_1(t)dt+\int_\mu^P I_2(t)dt\right] \quad (12)$$

$$=P_C I_p\left[\frac{d(\mu-T)}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)}-1\right)+\frac{a}{2}(\mu-T)^2+\frac{b}{6}(2\mu+T)(\mu-T)^2+\frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2+\frac{d}{(\theta-\pi)^2}\left(e^{(\theta-\pi)(P-\mu)}-1-(\theta-\pi)(P-\mu)\right)\right] \quad (13)$$

Case 2: ($\mu<T\le P$)

This is when the end point of credit period is greater than the period with no deterioration but shorter than or equal to the length of period with positive inventory. The commission payable is

$$I_{P2}=P_C I_p\left[\int_T^P I_2(t)dt\right] \quad (14)$$

$$=P_C I_p\left[\frac{d}{(\theta-\pi)^2}\left(e^{(\theta-\pi)(P-T)}-1-(\theta-\pi)(P-T)\right)\right] \quad (15)$$

The Commission Gained

It is assumed that during the period when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and gains the commission with rate I_e. Therefore, the commissiongained per cycle for two different cases are given below

Case 1: ($0<T\le\mu$)

In this case, the retailer can gain commission on revenue generated from the sales up to the trade credit period T. Although, the retailer has to settle the accounts at period T, for that he has to arrange money at some specified rate of commission in order to get his remaining stocks financed for the period T to μ. The commissiongain is

$$I_{E1}=S_P I_g\left[\int_0^T(\alpha+bt+ct^2)tdt\right]=S_P I_g\left(a\frac{T^2}{2}+b\frac{T^3}{3}+c\frac{T^4}{4}\right) \quad \textbf{(16)}$$

Case 2: ($\mu<T\le P$)

In this case, the retailer can gain commission on revenue generated from the sales up to the trade credit period T. Although, the retailer has to settle the accounts at period T, for that he has to arrange money at some specified rate of commission in order to get his remaining stocks financed for the period T to P. The commissiongain is

$$I_{E2}=S_P I_g\left[\int_0^\mu(a+bt+ct^2)tdt+\int_\mu^T dtdt\right]=S_P I_g\left[\left(\alpha\frac{\mu^2}{2}+b\frac{\mu^3}{3}+c\frac{\mu^4}{4}\right)+\frac{dT^2}{2}-\frac{d\mu^2}{2}\right] \quad \textbf{(17)}$$

(viii) The Average Total Variable Cost per Unit Time

The average total variable cost per unit time for case 1 ($0<T\le\mu$) is given by

$Z_1(P)=\frac{1}{P}${Ordering cost +inventory storage cost + deterioration cost + commission payable during the permissible delay period – commissiongained during the cycle}

$$=\frac{1}{T}\left\{O_C+iC_1\left(\frac{d\mu}{(\theta-\pi)}e^{(\theta-\pi)(T-\mu)}+\frac{a}{2}\mu^2+\frac{b}{3}\mu^3+\frac{c}{4}\mu^4+\frac{d}{(\theta-\pi)^2}e^{(\theta-\pi)(T-\mu)}-\frac{d}{(\theta-\pi)^2}-\frac{dP}{(\theta-\pi)}\right)+iC_2\left(\frac{d\mu^2}{2(\theta-\pi)}e^{(\theta-\pi)(P-\mu)}+\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu}{(\theta-\pi)^2}e^{(\theta-\pi)(P-\mu)}-\frac{dP}{(\theta-\pi)^2}-\frac{d}{(\theta-\pi)^3}+\frac{d}{(\theta-\pi)^3}e^{(\theta-\pi)(P-\mu)}-\frac{dP^2}{2(\theta-\pi)}\right)+P_C\left[\frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)}-1-(\theta-\pi)(P-\mu)\right)\right]+P_C I_p\left[\frac{d(\mu-T)}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)}-1\right)+\frac{a}{2}(\mu-T)^2+\frac{b}{6}(2\mu+T)(\mu-T)^2+\frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2+\frac{d}{(\theta-\pi)^2}\left(e^{(\theta-\pi)(P-\mu)}-1-(\theta-\pi)(P-\mu)\right)\right]-S_P I_g\left(\alpha\frac{T^2}{2}+b\frac{T^3}{3}+c\frac{T^4}{4}\right)\right\} \quad \textbf{(18)}$$

The average total variable cost per unit time for case 2($\mu<T\le P$) is given by

$Z_2(P)=\frac{1}{P}${Ordering cost +inventory storage cost + deterioration cost + commission payable during the permissible delay period – commission gained during the cycle}

$$=\frac{1}{P}\left\{O_C+iC_1\left(\frac{d\mu}{(\theta-\pi)}e^{(\theta-\pi)(P-\mu)}+\frac{a}{2}\mu^2+\frac{b}{3}\mu^3+\frac{c}{4}\mu^4+\frac{d}{(\theta-\pi)^2}e^{(\theta-\pi)(P-\mu)}-\frac{d}{(\theta-\pi)^2}-\frac{dP}{(\theta-\pi)}\right)+iC_2\left(\frac{d\mu^2}{2(\theta-\pi)}e^{(\theta-\pi)(P-\mu)}+\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu}{(\theta-\pi)^2}e^{(\theta-\pi)(P-\mu)}-\frac{dP}{(\theta-\pi)^2}-\frac{d}{(\theta-\pi)^3}+\frac{d}{(\theta-\pi)^3}e^{(\theta-\pi)(P-\mu)}-\frac{dP^2}{2(\theta-\pi)}\right)+P_C\left[\frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P-\mu)}-1-(\theta-\pi)(P-\mu)\right)\right]+P_C I_p\left[\frac{d}{(\theta-\pi)^2}\left(e^{(\theta-\pi)(P-T)}-1-(\theta-\pi)(P-T)\right)\right]-S_P I_g\left[\left(a\frac{\mu^2}{2}+b\frac{\mu^3}{3}+c\frac{\mu^4}{4}\right)+\frac{dT^2}{2}-\frac{d\mu^2}{2}\right]\right\} \quad \textbf{(19)}$$

Since $0<(\theta-\pi)<1$,by utilizing a quadratic approximation for the exponential terms in equations (16) and (17) to obtain:

$$Z_1(T)=\frac{1}{T}\left\{O_C+iC_1\left(\frac{a}{2}\mu^2+\frac{b}{3}\mu^3+\frac{c}{4}\mu^4-\frac{d\mu^2}{2}+\frac{d\mu^3(\theta-\pi)}{2}+\frac{d\mu(\theta-\pi)P^2}{2}+\frac{dP^2}{2}-d\mu^2(\theta-\pi)P\right)+iC_2\left(\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu^4(\theta-\pi)}{4}+\frac{d\mu^2(\theta-\pi)P^2}{4}+\frac{d\mu P^2}{2}-\frac{d\mu^3(\theta-\pi)P}{2}-\frac{d\mu^2P}{2}\right)+\frac{P_C d(\theta-\pi)}{2}(\mu^2+P^2-2\mu P)+P_C I_p\left[\frac{a}{2}(\mu-T)^2+\frac{b}{6}(2\mu+T)(\mu-T)^2+\frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2-\frac{d\mu^2}{2}+dT\mu+\frac{d(\mu-T)(\theta-\pi)\mu^2}{2}+\frac{dT^2}{2}+\frac{d(\mu-T)(\theta-\pi)P^2}{2}-dTP-d(\mu-T)(\theta-\pi)\mu P\right]-S_P I_g\left(a\frac{T^2}{2}+b\frac{T^3}{3}+c\frac{T^4}{4}\right)\right\} \quad \textbf{(20)}$$

and

$$Z_2(T)=\frac{1}{T}\left\{O_C+iC_1\left(\frac{a}{2}\mu^2+\frac{b}{3}\mu^3+\frac{c}{4}\mu^4-\frac{d\mu^2}{2}+\frac{d\mu^3(\theta-\pi)}{2}+\frac{d\mu(\theta-\pi)P^2}{2}+\frac{dP^2}{2}-d\mu^2(\theta-\pi)P\right)+iC_2\left(\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu^4(\theta-\pi)}{4}+\frac{d\mu^2(\theta-\pi)P^2}{4}+\frac{d\mu P^2}{2}-\frac{d\mu^3(\theta-\pi)P}{2}-\frac{d\mu^2P}{2}\right)+\frac{P_C d(\theta-\pi)}{2}(\mu^2+P^2-2\mu P)+P_C I_p\frac{d}{2}[T^2+P^2-2TP]-S_P I_g\left(a\frac{\mu^2}{2}+b\frac{\mu^3}{3}+c\frac{\mu^4}{4}+\frac{dT^2}{2}-\frac{d\mu^2}{2}\right)\right\} \quad \textbf{(21)}$$

Optimal Decision

We define the necessary and sufficient conditions for determining the best ordering strategies that minimize the average total variable cost per unit time. The required condition for the average total variable cost per unit ime.$Z_i(P)$to be minimum is obtained by differentiating $Z_i(P)$ with respect P for $i=1,2$ and equates to zero. The optimum value of P for which the sufficient condition $\frac{d^2Z_i(P)}{dP^2}>0$ is satisfied gives a minimum for the average total variable cost per unit time$Z_i(P)$.

Optimality condition for Case 1: ($0<T\le\mu$)

The necessary and sufficient conditions that minimize $Z_1(P)$are respectively,$\frac{dZ_1(P)}{dP}=0$ and $\frac{d^2Z_1(P)}{dP^2}>0$

The first derivatives of the average total variable cost, in (18), with respect to P is as follows.

$$\frac{dZ_1(P)}{dP}=\frac{1}{P^2}\left\{\frac{P^2}{2}d\left[C_1(\mu(\theta-\pi)+1)+C_2\mu\left(\frac{\mu(\theta-\pi)}{2}+1\right)+P_C(\theta-\pi)+P_C I_p((\theta-\pi)(\mu-T)+1)\right]-\left[O_C+C_1\left(\frac{a}{2}\mu^2+\frac{b}{3}\mu^3+\frac{c}{4}\mu^4-\frac{d\mu^2}{2}+\frac{d\mu^3(\theta-\pi)}{2}\right)+C_2\left(\frac{a}{6}\mu^3+\frac{b}{8}\mu^4+\frac{c}{10}\mu^5+\frac{d\mu^4(\theta-\pi)}{4}\right)+\frac{P_C d(\theta-\pi)}{2}\mu^2+P_C I_p\left(\frac{a}{2}(\mu-T)^2+\frac{b}{6}(2\mu+T)(\mu-T)^2+\frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2-\frac{d\mu^2}{2}+dT\mu+\frac{d(\mu-T)(\theta-\pi)\mu^2}{2}\right)-S_P I_g\left(a\frac{T^2}{2}+b\frac{T^3}{3}+c\frac{T^4}{4}\right)\right]\right\} \quad \textbf{(22)}$$

Therefore, $\frac{dZ_1(P)}{dP} = 0$ gives the following nonlinear equation in terms P

$$\frac{1}{P^2}\left\{\frac{P^2}{2} d\left[C_1(\mu(\theta-\pi)+1) + C_2\mu\left(\frac{\mu(\theta-\pi)}{2}+1\right) + P_C(\theta-\pi) + P_C I_p\big((\theta-\pi)(\mu-T)+1\big)\right] C - \left[O_C + C_1\left(\frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 - \frac{d\mu^2}{2} + \frac{d\mu^3(\theta-\pi)}{2}\right) + C_2\left(\frac{a}{6}\mu^3 + \frac{b}{8}\mu^4 + \frac{c}{10}\mu^5 + \frac{d\mu^4(\theta-\pi)}{4}\right) + \frac{P_C d(\theta-\pi)}{2}\mu^2 + P_C I_p\left(\frac{a}{2}(\mu-T)^2 + \frac{b}{6}(2\mu+T)(\mu-T)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu T + T^2)(\mu-T)^2 - \frac{d\mu^2}{2} + dT\mu + \frac{d(\mu-T)(\theta-\pi)\mu^2}{2}\right) - S_P I_g\left(a\frac{T^2}{2} + b\frac{T^3}{3} + c\frac{T^4}{4}\right)\right]\right\} = 0 \quad \textbf{(23)}$$

From (29), let

$$X_1 = d\left[C_1(\mu(\theta-\pi)+1) + C_2\mu\left(\frac{\mu(\theta-\pi)}{2}+1\right) + P_C(\theta-\pi) + P_C I_p\big((\theta-\pi)(\mu-T)+1\big)\right] \quad (24)$$

and

$$X_2 = O_C + C_1\left(\frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 - \frac{d\mu^2}{2} + \frac{d\mu^3(\theta-\pi)}{2}\right) + C_2\left(\frac{a}{6}\mu^3 + \frac{b}{8}\mu^4 + \frac{c}{10}\mu^5 + \frac{d\mu^4(\theta-\pi)}{4}\right) + \frac{P_C d(\theta-\pi)}{2}\mu^2 + P_C I_p\left(\frac{a}{2}(\mu-T)^2 + \frac{b}{6}(2\mu+T)(\mu-T)^2 + \frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2 - \frac{d\mu^2}{2} + \lambda T\mu + \frac{d(\mu-T)(\theta-\pi)\mu^2}{2}\right) - S_P I_g\left(a\frac{T^2}{2} + b\frac{T^3}{3} + c\frac{T^4}{4}\right) \quad \textbf{(25)}$$

Substituting X_1 and X_2 into (21) to obtain

$$\frac{1}{P^2}\left\{\frac{P^2}{2}X_1 - X_2\right\} = 0 \quad (26)$$

which implies

$$P^2 X_1 - 2X_2 = 0 \quad (27)$$

Let

$$\Delta_1 = S_P I_g\left(a\frac{T^2}{2} + b\frac{T^3}{3} + c\frac{T^4}{4}\right) - \left[O_C + C_1\left(\frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 - d\mu^2\right) + C_2\left(\frac{a}{6}\mu^3 + \frac{b}{8}\mu^4 + \frac{c}{10}\mu^5 - \frac{d\mu^3}{2}\right) + P_C I_p\left(\frac{a}{2}(\mu-T)^2 + \frac{b}{6}(2\mu+T)(\mu-T)^2 + \frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2 - d\mu^2 + dT\mu\right)\right] \quad (28)$$

Lemma 1. For $0 < M \leq \mu$, we have

i. If $\Delta_1 \leq 0$, then the solution of $P \in [\mu, \infty)$ (say P_1^*) which satisfies (22) not only exists but also is unique
ii. If $\Delta_1 > 0$, then the solution of $P \in [\mu, \infty)$ which satisfies (22) does not exist.

Proof of (i). From (21), we define a new function $F_1(P)$ as follows

$$F_1(P) = \frac{P^2}{2} d\left[C_1(\mu(\theta-\pi)+1) + C_2\mu\left(\frac{\mu(\theta-\pi)}{2}+1\right) + P_C(\theta-\pi) + P_C I_p\big((\theta-\pi)(\mu-T)+1\big)\right] - \left[O_C + C_1\left(\frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 - \frac{d\mu^2}{2} + \frac{d\mu^3(\theta-\pi)}{2}\right) + C_2\left(\frac{a}{6}\mu^3 + \frac{b}{8}\mu^4 + \frac{c}{10}\mu^5 + \frac{d\mu^4(\theta-\pi)}{4}\right) + \frac{P_C\lambda(\theta-\pi)}{2}\mu^2 + P_C I_p\left(\frac{a}{2}(\mu-T)^2 + \frac{b}{6}(2\mu+T)(\mu-T)^2 + \frac{c}{12}(3\mu^2+2\mu T+T^2)(\mu-T)^2 - \frac{d\mu^2}{2} + dT\mu + \frac{d(\mu-T)(\theta-\pi)\mu^2}{2}\right) - S_P I_g\left(a\frac{T^2}{2} + b\frac{T^3}{3} + c\frac{T^4}{4}\right)\right], P \in [\mu, \infty) \quad (29)$$

Taking the first-order derivative of $F_1(T)$ with respect to $T \in [\mu, \infty)$, we have

$$\frac{F_1(P)}{dP} = Pd\left[C_1(\mu(\theta-\pi)+1) + C_2\mu\left(\frac{\mu(\theta-\pi)}{2}+1\right) + P_C(\theta-\pi) + P_C I_p\big((\theta-\pi)(\mu-T)+1\big)\right] \quad (30)$$

$$= PX_1 > 0$$

We obtain that $F_1(P)$ is an increasing function of P in the interval $[\mu, \infty)$. Moreover, we have

$$\lim_{P\to\infty} F_1(P) = \infty$$

and

$$F_1(\mu) = S_P I_g\left(a\frac{T^2}{2} + b\frac{T^3}{3} + \gamma\frac{T^4}{4}\right) - \left[O_C + C_1\left(\frac{a}{2}\mu^2 + \frac{b}{3}\mu^3 + \frac{c}{4}\mu^4 - d\mu^2\right) + C_2\left(\frac{a}{6}\mu^3 + \frac{b}{8}\mu^4 + \frac{c}{10}\mu^5 - \frac{d\mu^3}{2}\right) + P_C I_p\left(\frac{a}{2}(\mu-T)^2 + \frac{b}{6}(2\mu+M)(\mu-T)^2 + \frac{c}{12}(3\mu^2+2\mu T+M^2)(\mu-T)^2 - d\mu^2 + dT\mu\right)\right] \quad (31)$$

$$= \Delta_1 \leq 0$$

Now $F_1(\mu) \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique $P_1^* \in [\mu, \infty)$ such that $F_1(P_1^*) = 0$. Hence P_1^* isthe unique solution of (22). Thus, the value of P (denoted by P_1^*) can be found from (30) and is given by

$$P_1^* = \sqrt{\frac{2X_2}{X_1}} \quad (32)$$

Proof of (ii). If $\Delta_1 > 0$, then from (23), we have $F_1(P) > 0$. Since $F_1(P)$ is an increasing function of $P \in [\mu, \infty)$, then $F_1(P) > 0$ for all $P \in [\mu, \infty)$. Thus, we cannot find a value of $P \in [\mu, \infty)$ such that $F_1(P) = 0$. This completes the proof.

Theorem 1. When $0 < T \leq \mu$, we have

i. (i) If $\Delta_1 \leq 0$, then the average total variable cost $Z_1(P)$ is convex and reaches its global minimum at the point $P_1^* \in [\mu, \infty)$, where P_1^* is the point which satisfies (22).
ii. If $\Delta_1 > 0$, then the average total variable cost $Z_1(P)$ has a minimum value at the point $P_1^* = \mu$.

Proof of (i). When $\Delta_1 \leq 0$, we see that P_1^* is the unique solution of (22) from Lemma l(i). Taking the second derivative of $Z_1(P)$ with respect to P and then finding the value of the function at the point T_1^*, we obtain

$$\left.\frac{d^2 Z_1(P)}{dP^2}\right|_{P_1^*} = \frac{X_1}{P_1^*} > 0 \quad (33)$$

We thus conclude from (25) and Lemma 1 that $Z_1(P_1^*)$ is convex and P_1^* is the global minimum point of $Z_1(P)$. Hence the value of P in (24) is optimal.

Proof of (ii). When $\Delta_1 > 0$, then we know that $F_1(P) > 0$ for all $P \in [\mu, \infty)$. Thus, $\frac{dZ_1(P)}{dP} = \frac{F_1(P)}{P^2} > 0$ for all $P \in [\mu, \infty)$ which implies $Z_1(P)$ is an increasing function of P. Thus $Z_1(P)$ has a minimum value when P is minimum. Therefore, $Z_1(P)$ has a minimum value at the point $P = \mu$. This completes the proof.

Optimality condition for Case 2: $(\mu < T \leq P)$

Applying the same procedure as in case 1, the value of the optimal cycle length denoted by P_2^* is given by

$$P_2^* = \sqrt{\frac{2Y_2}{Y_1}} \quad (34)$$

Thus the EOQ corresponding to the best cycle length P^* will be computed as follows:

EOQ^* =Total demand before deterioration set in+total demand after deterioration set in+total number of deteriorated items

$$= \int_0^{\mu}(a + bt + ct^2)dt + \int_{\mu}^{P^*} d\,dt + \left[\frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P^*-\mu)} - 1\right) - d(P^* - \mu)\right] \quad (35)$$

$$= a\mu + b\frac{\mu^2}{2} + c\frac{\mu^3}{3} + \frac{d}{(\theta-\pi)}\left(e^{(\theta-\pi)(P^*-\mu)} - 1\right) \quad (36)$$

Numerical Examples

Example (Case 1)

The model was validated numerically by adopting parameter values from Babangida and Baraya (2019), with $\pi$and i added $i$added in this work, and their values were optimally estimated. The parameters' values are summarized in table 1 below:

Table 1: Parameters' Values

Parameter	Value
O_C	\$200/Order
P_C	\$45/unit/year
S_P	\$50/unit/year

Parameter	Value
C_1	\$4/unit/year
C_2	\$0.5/unit/year
θ	0.02 units/year
π	0.008 units/year
a	1500 units
b	400 units
c	50 units
i	\$0.5/unit/year
d	800 units
μ	0.2190 year (80days)
T	0.1971 year (72days)
I_p	0.12
I_g	0.08

RESULTS AND DISCUSSION

We first check the condition $\Delta_1 = -175.9230 < 0$. Substituting the above values into $(24), (18)$ and(27), we obtain as follows the values of the optimal cycle length, the optimal average total cost, and the economic order quantity respectively in table 2

Table 2: Decision Variables and Their Values

Decision Variables	Values
P_1^*	0.3129 year (114 days)
$Z_1(P_1^*)$	\$1127.3281
EOQ_1^*	413.3877 Units

Example (Case 2)

The data are same as in Example (Case 1) except that$T =$ 0.2327 year (85 days).

Results and Discussion Case 2

We first check the condition $\Delta_2 = -31.9858 < 0$. Substituting the above values into $(26), (19)$ and(27), we obtain as follows the values of the optimal cycle length, the optimal average total constant the economic order quantity respectively in table 3

Table 3: Decision Variables and Their Values

Decision Variables	Values
P_2^*	0.2456 year (90 days)
$Z_2(P_2^*)$	\$477.2484
EOQ_2^*	359.5421 Units

Therefore, $\boldsymbol{Z(P^*)} = \boldsymbol{min}\{Z_1(P_1^*), Z_2(P_2^*)\} = Z_2(P_2^*) = \$477.2484$ per year

Comparison

Since the proposed model and Babangida and Baraya (2019) both sought to determine the optimal cycle length order quantity that minimize minimized the average total variable cost per unit, and the proposed model is an extension of Babangida and Baraya (2019), the two models' results can be compared (see table 4). As the average total variable cost per unit in the proposed model (\$2.7270for case1 and\$1.3273for case 2) is lower than that of Babangida and Baraya (2019).(\$2.9285for case1 and \$2.3285 for case 2), the proposed model performs better. As the average total variable cost per unit in the proposed model(\$2.7270for case1 and\$1.3273for case 2) is lower than that of Babangida and Baraya (2019).(\$2.9285for case1 and \$2.3285 for case 2) the proposed model is more optimal to that of Babangida and Baraya (2019).

Table 4: Comparison Between the Proposed and Existing Model

Model	Average Total Variable Cost for Case 11	Average Total Variable Cost ForCase 2
Babangida and Baraya (2019)	\$2.9285	\$2.3285
Proposed Model	\$2.7270	\$1.3273

Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-10\%, -5\%$, $+5\%$ to $+10\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these changes on the decision variables are discussed.

Table 5: Effect of Changes of Some Parameters on Decision Variables

Parameter	% Change in Parameter	% Change in P^*	% Change in EOQ^*	% Change in $Z(P^*)$
θ	−10	0.1056	0.0576	-0.0180
	−5	0.0525	0.0286	-0.0089
	+5	-0.0520	-0.0284	0.0088
	+10	-0.1036	-0.0565	0.0176
P_C	−10	0.4581	0.25042	-0.0911
	−5	0.2202	0.1204	-0.0446
	+5	-0.2043	-0.1117	0.0430
	+10	-0.3944	-0.2156	0.0845
S_P	−10	3.8678	2.1144	14.1566
	−5	1.9523	1.0672	7.1453
	+5	-1.9912	-1.0884	-7.2873
	+10	-4.0236	-2.1993	-14.7254
π	−10	-0.0417	-0.0227	0.0071
	−5	-0.0209	-0.0114	0.0035
	+5	0.0210	0.0114	-0.0036
	+10	0.0420	0.0229	-0.0071

Discussion on Sensitivity Analysis

The managerial insights presented below are based on the computational results reported in Table 3.2.1.

i. As the rate of deterioration (θ) increases, the optimal cycle length(P^*) and economic order quantity (EOQ^*) decrease while total variable cost$(Z(P^*))$ increase.Hence the retailer will order less quantity to avoid the items being deteriorating when the deterioration rate increases.

ii. As the unit purchasing cost (P_C) increases, the optimal cycle length(P^*), and the economic order quantity (EOQ^*) decrease while the average total variable cost$(Z(P^*))$ increase. In real market situation the higher the cost of an item, the higher the average total variable cost. This result implies that the retailer will order a smaller quantity to enjoy the benefits of permissible delay in payments more frequently in the presence of an increased unit purchasing price and consequently shortening cycle length.

iii. As the unit selling price (S_P) increases, the optimal cycle length(P^*), the economic order quantity (EOQ^*) and the average total variable cost$(Z(P^*))$decrease. In real market situation the higher the selling price of an item, the lower the demand. This means that when the unit selling price is increasing, the retailer will order less quantity to take the benefits of the trade credit more frequently.

iv. As the number of repaired items (π) increases, the optimal cycle length(P^*), and the economic order quantity (EOQ^*) increase while the average total variable cost$(Z(P^*))$ decreases. Whenever the number of repaired items increases, the number of deteriorated items decreases, and lead to the decrease in the average total variable cost.

CONCLUSION

This paper presents a two-stage consumption rate replenishment policy for partially repaired non-instantaneous decaying commodities with variable storage costs under allowable acceptable payment delays. The rate of consumption is a quadratic function of time before product deterioration and remains constant thereafter. The optimal best cycle duration and order quantity that minimize the total variable cost are determined. The existence uniqueness of solutions have been established. of solutions and their uniqueness have been demonstrated. Tests on existing data were conducted are utilized to evaluate the model, and a comparison with the current model reveals that the proposed model outperforms beats the current model in terms of turnover and cost minimization. Based on the sensitivity analysis, recommendations are provided certain recommendations are made for reducing the inventory system's overall variable cost. Based on findings and sensitivity analysis, the total variables can be minimize by ordering less quantity when deterioration rate, unit purchasing price and unit selling price increase and the number of repaired items decreases.The model developed in this work is a generalisation of Babangida and Baraya (2019), i.e., if $\pi =$ 0and $i = 1$, The findings in this study are consistent with those reported in Babangida and Baraya (2019). The proposed model can be expanded to account for shortages, varying deterioration rates, and other practical considerations. and so on.

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