FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 9 No. 10, October, 2025, pp 203 – 208
DOI: https://doi.org/10.33003/fjs-2025-0910-4078

# FIXED POINT RESULTS FOR JAGGI-TYPE HYBRID CONTRACTION IN *JS*-MODULAR METRIC SPACES

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**ABSTRACT**
In this paper, we introduce a new concept of JS-modular metric space and obtain some fixed-point results of hybrid contractions in this setting. Some corollaries that reduce our result to other well-known ideas in the literature are highlighted. An example is constructed to support the established concepts, and to demonstrate the distinction of this research with respect to the existing ones.

**Keywords**: JS-Modular Metric Space, Fixed Point, Hybrid Contraction, Jaggi-Type Contraction

## INTRODUCTION

One of the well-known fixed points (*FP*) theorems in spaces with metric structures is the Banach Contraction (also known as the Contraction Mapping principle). It stands out due to its simplicity and wide-ranging utility in solving existence problems across various branches of mathematics and applied sciences. As a result, numerous generalizations and extensions of this principle have emerged (e.g Jiddah *et* al., 2022). In 2019, Karapinar & Fulga (2019) introduced a new hybrid contraction that combines Jaggi-type and interpolative-type contractions in a complete metric space (*MS*) setting. Jidda *et* al., (2022) extended the work of Karapinar & Fulga (2019) to the setting of *G-MS* by defining the idea of interpolative Kannan-type ($G-\alpha-\mu$) contraction. Not long ago, in a related development, Jidda *et* al., (2023) studied a general class of inequalities under the name admissible ($H-\alpha-\kappa$) contraction in a *MS* endowed with a graph. Yahya & Shagari (2023) defined multivalued contraction from a combination of Jaggi-type contractions, interpolative-type contraction and Pata-type inequality in the framework of *MS*. For further information on *FP* result of hybrid contraction and contractions on *G-MS*, the reader can consult Jidda *et* al., (2023).

The concept of modular spaces as a generalization of *MS* has attracted considerable interest since Nakano (1959), Musielak & Orlicz (1959) introduced the concept of modular spaces and worked on some theories about them. The idea of modular *MS* has been worked on by Chistyakov (2010), where the basic definitions and properties of modular metric are introduced. After Chistyakov (2010), many authors have studied *FP* theorems and presented significant results in modular *MS* (see for instance, [Abdou, 2013; Chaipunya *et* al., 2012; Mongolkeha *et* al., 2011; Mutlu *et* al., 2018; Zhua *et* al., 2020] and others therein).

Jleli & Samet (2015) introduced the concept of a generalized *MS* (now called *JS-MS*) and extended some well-known *FP* results. The concept of *JS-MS* recovers various concepts in topological spaces including standard *MS*, *b-MS* and modular spaces. Turkoglu & Manav (2018) considering both a modular *MS* and a *JS-MS* later introduced a new concept of *JS*-modular *MS* and proved some *FP* results in this setting. Interestingly, Manav (2021), using Nadler (1969) and Edelstein's result in the work of Abdou (2013), proved Caristi and Feng-Liu type *FP* theorems for multivalued contractive-type map in generalised modular *MS*. Later on, Manav & Agarwal (2023) studied best approximation of *FP* results for Branciari Contraction of integral type on generalized modular *MS*. For some recent results, see e.g, Okeke & Francis (2024).

Following the reviewed literature, we see that a unified concept in the sense of Turkoglu & Manav (2018) and Karapinar & Fulga (2019) is not adequately considered. Motivated by the above information, we propose a new combinational concept under the name Jaggi-type hybrid contraction (*JHC*) in *JS*-modular *MS* and prove the existence of *FP* of such contraction. The proposed result is a direct improvement of the work of Turkoglu & Manav (2018) and Karapinar & Fulga (2019), and some related results therein. An example is constructed to support the assumptions of our established theory. Some corollaries are highlighted and discussed to show that the concept proposed therein, subsumes several existing results.

The structure of the paper is outlined as follows: Section 1 provides an introduction. In Section 2, we recall specific preliminary concepts needed in the discussion of our main results. The main result and corollaries are presented in Section 3 with an example given to support the established concepts.

## Preliminaries

In this section, the basic concepts and auxiliary results needed for the presentation of our main result are collated.

Jleli & Samet (2015) defined the *JS*-MS as follows:

***Definition 1***

Let $H$ be a nonempty set and $D: H \times H \to [0,+\infty]$ be a given mapping. For every $h \in H$, define the set

$K(D,H,h) = \{\{h_n\} \subseteq H: \lim_{n\to 0} \phi(h_n,h) = 0\}$

for all $\lambda > 0$. We say that $D$ is a *JS*-modular metric on $H$ if it satisfies

i. for every $(h,j) \in H^2, \lambda > 0$, we have $\phi(h,j) = 0 \Rightarrow h = j$;
ii. for every $(h,j) \in H^2, \lambda > 0$, we have $\phi(h,j) = \phi(j,h)$;
iii. There exists $K > 0$ such that if $(h,j) \in H^2$, $\{h_n\} \in K(D,H,h)$ then
$$\phi(h,j) \leq K \lim_{n\to\infty} \sup \phi(h_n,j).$$

Then $(H,D)$ is a JS-metric space.

Chistyakov (2010) gave the following definitions.

***Definition 2***

Let H be a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

i. A functional $\rho: H \to [0,\infty]$ is called a modular if for arbitrary $h, j \in H$,
   a. $\rho(h) = 0$ iff $h = 0$;
   b. $\rho(\alpha h) = \rho(h)$ for every scalar $\alpha$ with $|\alpha| = 1$;

c. $\rho(\alpha h + \alpha j) \leq \rho(h) + \rho(j)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$;

ii. if c) is replaced by

c′) $\rho(\alpha h + \alpha j) \leq \alpha^s \rho(h) + \beta^s \rho(j)$, for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with $s \in (0,1]$,

then $\rho$ is called an *s*-convex modular, and if *s* = 1, $\rho$ is called a convex modular.

iii. A modular $\rho$ defines a corresponding modular space i.e the vector subspace of $H$, $H_\rho$ given by

$H_\rho = \{h \in H : \rho(\lambda h) \to 0 \text{ as } \lambda \to 0\}$

Chistyakov (2010) gave the following definition of a metric modular.

***Definition 3***

Let $H$ be a nonempty set. A function $\varpi : (0,\infty) \times H \times H \to [0,\infty]$ is said to be a metric modular on $H$ if it satisfies the following conditions:

i. $\varpi_\lambda(h,j) = 0$ for all $\lambda > 0$ iff $h = j$;
ii. $\varpi_\lambda(h,j) = \varpi_\lambda(j,h)$ $\forall\lambda > 0$;
iii. $\varpi_{\lambda+\mu}(h,j) \leq \varpi_\lambda(h,z) + \varpi_\mu(z,j), \forall\lambda,\mu > 0$.

Jleli & Samet (2015) gave the following definition.

***Definition 4***

Let $(H,D)$ be a *JS*-metric space. We say that

i. $\{h_n\}$ *D*-converges to $H$ if $\{h_n,\} \in K(D,H,h)$.
ii. $\{h_n\}$ is *D*-Cauchy sequence if $\lim_{n\to\infty} \phi(h_n, h_{n+m}) = 0$.
iii. $(H,D)$ is *D*-complete if every Cauchy sequence in $H$ is convergent to some element in $H$.

***Definition 5((c)-comparison function)***

$\psi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function if it is nondecreasing and satisfies there exists $q_0 \in \mathbb{N}$ and $r \in (0,1)$ and a convergent series $\sum_{q=1}^{\infty} w_q$ such that $w_q \geq 0$ and $\psi^{q+1} \leq r\psi^{q}(t) + w_q$, for $q \geq q_0$ and $t \geq 0$.

The set of all (c)-comparison functions will be denoted by $\Psi$. Every $\psi \in \Psi$

satisfies the following:

i. $\psi(w) < w$, for any $w \in \mathbb{R}^+$;
ii. $\psi$ is continuous at 0;
iii. the series $\sum_{q=1}^{\infty} \psi^q(w)$ is convergent for $w \geq 0$.

Karapinar & Fulga (2019) gave the following definition of Jaggi-type hybrid contraction in metric space.

***Definition 6***

Let $(H,D)$ be a complete metric space. A mapping $\theta : H \to H$ is called a Jaggi-type hybrid contraction if there exists $\psi \in \Psi$ such that

$$\phi(\theta h, \theta j) \leq \psi(J_\theta^s(h,j)) \quad (1)$$

$\forall h,j \in H$, where $s \geq 0$ and

$$J_\theta^s(h,j) = \begin{cases} [\sigma_1(\frac{\phi(h,\ \theta j)\phi(j,\ \theta j)}{\phi(h,\ j)})^s + \sigma_2(\phi(h,j)^s]^{\frac{1}{s}} \, for\ s > 0, h,j \in H, h \neq j \\ (\phi(h,\theta j))^{\sigma_1}(\phi(j,\theta j))^{\sigma_2} for\ s = 0, h,j \in H/F_\theta(H) \end{cases}$$

for $\sigma_i \geq 0$, i = 1,2 such that $\sigma_1 + \sigma_2 = 1$.

Karapinar & Fulga (2019) proved the following fixed-point result.

***Theorem 1***

A continuous self-mapping $\theta$ on $(H,\phi)$ possesses a fixed point H provided that $\theta$ is a Jaggi-type hybrid contraction. Moreover, for any $h_0 \in H$, the sequence $\{\theta^n(h_0)\}$ converges to H.

Turkoglu & Manav (2018) introduced the JS-modular metric as follows.

***Definition 7***

Let H be a nonempty set and $\varpi_\lambda : H \times H \to [0,+\infty]$ be a given mapping. For every $h \in H$, define the set

$K(\varpi_\lambda, H, h) = \{\{h_n\} \subseteq H : \lim_{n\to\infty} \varpi_\lambda(h_n, h) = 0\}$

for all $\lambda > 0$. $\varpi_\lambda$ is a *JS*-modular metric on H if it satisfies

i. for every $(h, j) \in H^2, \lambda > 0$, we have
$\varpi_\lambda(h,j) = 0 \Rightarrow h = j$;
ii. for every $(h, j) \in H^2$, $\lambda > 0$, we have
$\varpi_\lambda(h,j) = \varpi_\lambda(j,h)$;
iii. there exists $K > 0$ such that if $(h, j) \in H^2$, $\{h_n\} \in K(\varpi_\lambda, H, h)$, then
$\varpi_\lambda(h,j) \leq K \limsup_{n\to\infty} \varpi_\lambda(h_n, j)$.

The pair $(H, \varpi_\lambda)$ is called a *JS*-modular metric space.

## RESULTS AND DISCUSSION

***Definition 8***

Let $(H_\varpi, \varpi_\lambda)$ be a *JS*-modular metric space. A mapping $\theta: H_\varpi \to H_\varpi$ is called a hybrid contraction of Jaggi-type (HCJT) if there exists $\psi \in \Psi$ such that

$$\varpi_\lambda(\theta h, \theta j) \leq \psi(H_\theta^s(h,j)) \quad (2)$$

$\forall h, j \in H_\varpi$, *where* $s \geq 0$ *and*

$$H_\theta^s(h,j) = \begin{cases} [\sigma_1(\frac{\varpi_\lambda(h,\ \theta j)\varpi_\lambda(j,\ \theta j)}{\varpi_\lambda(h,\ j)})^s + \sigma_2(\varpi_\lambda(h,j))^s]^{\frac{1}{s}} \, for\ s > 0, \varpi_\lambda(h,j) \neq 0 \\ (\varpi_\lambda(h,\theta j))^{\sigma_1}(\varpi_\lambda(j,\theta j))^{\sigma_2} for\ s = 0, h,j \in H_\varpi/F_\theta(H_\varpi) \end{cases}$$

for $\sigma_i \geq 0$, i = 1,2 such that $\sigma_1 + \sigma_2 = 1$.

***Proposition 1***

Suppose that $\theta$ is a HCJT. Then any fixed point $h \in H_\varpi$ of $\theta$ satisfying $\varpi_\lambda(h, h) < \infty$ implies $\varpi_\lambda(h, h) = 0$.

Proof:

Let $h \in H_\varpi$ be a fixed point of $\theta$ such that $\varpi_\lambda(h, h) < \infty$. Since $\theta$ is a HCJT, for *s* > 0, if $\varpi_\lambda(h,h) \neq 0$, then

$\varpi_\lambda(\theta h, \theta h) = \psi([\sigma_1(\frac{\varpi_\lambda(h,h)\varpi_\lambda(h,h)}{\varpi_\lambda(h,h)})^s + \sigma_2(\varpi_\lambda(h,h))^s]^{\frac{1}{s}})$
$= \psi([(\sigma_1+\sigma_2)(\varpi_\lambda(h,h))^s]^{\frac{1}{s}})$
$= \psi(\varpi_\lambda(h,h))$
$< \varpi_\lambda(h,h)$

Hence, we have $\varpi_\lambda(h,h) = \varpi_\lambda(\theta h, \theta h) < \varpi_\lambda(h,h)$. This is a contradiction.

Hence, $\varpi_\lambda(h, h) = 0$. Similarly, for *s* = 0, if $\varpi_\lambda(h,h) \neq 0$, then

$\varpi_\lambda(h,h) = \varpi_\lambda(\theta h, \theta h)$
$\leq \psi((\varpi_\lambda(h,\theta h))^{\sigma_1}(\varpi_\lambda(h,\theta h))^{\sigma_2})$L
$< (\varpi_\lambda(h,\theta h))^{\sigma_1}(\varpi_\lambda(h,\theta h))^{\sigma_2}$
$= (\varpi_\lambda(h,\theta h))^{\sigma_1+\sigma_2}$
$= \varpi_\lambda(h,\theta h)$
$= \varpi_\lambda(h,h)$

This is a contradiction. Hence $\varpi_\lambda(h,h) = 0$.

***Proposition 2***

*Let* $(H_\varpi, \varpi_\lambda)$ *be a JS-modular metric. Let* $\{h_n\}$ *be a sequence in* $H_\varpi$ *and* $(h, j) \in H_\varpi \times H_\varpi$ *such that* $\varpi_\lambda(h_n, h) \to 0$, $\varpi_\lambda(h_n, j) \to 0$ *as* $n \to \infty$ *for some* $\lambda > 0$*. Then h = j.*

Proof:

Using the property iii), we have

$\varpi_\lambda(h,j) \leq K \limsup_{n\to\infty} \varpi_\lambda(h_n, j) = 0$.

which implies from the property i) that *h* = *j*. □

***Theorem 2***
*Suppose that the following conditions holds:*
i. $(H_\varpi, \varpi_\lambda)$ *is a JS-modular metric space.*
ii. $\theta$ *is a continuous HCJT.*
iii. *there exists* $h_0 \in H_\varpi$ *such that*
$\delta(\varpi_\lambda, \theta, h_0) = \sup\{\varpi_\lambda(\theta^i(h), \theta^l(h)\} < \infty$.
*Then the sequence* $\{\theta^n(h_0)\}$ $\varpi_\lambda$*-converges to* $h \in H_\varpi$ *, a unique fixed point of* $\theta$*.*

***Proof:*** Let $n \in \mathbb{N}$ and $h_0 \in H_\varpi$. Since $\theta$ is a *HCJT*, $\forall i, l \in \mathbb{N} \cup \{0\}$, we have
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+l}(h_0)) \le \psi(H_\theta^s(\theta^{n-1+i}(h_0), \theta^{n-1+l}(h_0))) \quad (3)$$
We shall prove the claim of the theorem by examining the two cases: $s > 0$ and $s = 0$.
Case (1):
for $s > 0$ and taking $l = i + 1$, the expression (2) becomes
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)) \le \psi(H_\theta^s(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0))) \quad (4)$$
But,
$$H_\theta^s\left(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)\right) = [\sigma_1(\frac{\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0))\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0))}{\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0))})^s + \sigma_2(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s]^{\frac{1}{s}}$$
Therefore (3) becomes
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)) \le \psi[\sigma_1(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s + \sigma_2(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s]^{\frac{1}{s}}$$
Now suppose that
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)) \ge \varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0))$$
then,
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)) \le \psi[\sigma_1(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s + \sigma_2(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s]^{\frac{1}{s}}$$
$$= \psi[\sigma_1(\varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))^s + \sigma_2(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)))^s]^{\frac{1}{s}}$$
$$= \psi([(\sigma_1 + \sigma_2)(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)))^s]^{\frac{1}{s}}$$
$$= \psi(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)))$$
$$= \varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0))$$
This is a contradiction, hence we obtain that
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+i+1}(h_0)) \le \varpi_\lambda(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0))$$
Since $\delta(\varpi_\lambda, \theta, h) = \sup\{\varpi_\lambda(\theta^i(h), \theta^l(h)\}$, The expression (4) becomes
$$\delta(\varpi_\lambda, \theta, \theta^n(h_0)) \le \psi(\delta(\varpi_\lambda, \theta, \theta^{n-1}(h_0)))$$
$$\le \psi^2(\delta(\varpi_\lambda, \theta, \theta^{n-2}(h_0)))$$
$$\vdots$$
$$\le \psi^n(\delta(\varpi_\lambda, \theta, h_0).$$
Let $n, m \in \mathbb{N}$ with $m > n$. By using the property of the (c)-comparison function,
$$\delta(\theta^n(h_0), \theta^{n+m}(h_0)) \le \delta(\varpi_\lambda, \theta, \theta^n(h_0))$$
$$\le \psi^n(\delta(\varpi_\lambda, \theta, h_0) \to 0 \text{ as } n \to \infty.$$
This implies that $\{\theta^n(h_0)\}$ is an $\varpi_\lambda$-Cauchy sequence. Since $(H_\varpi, \varpi_\lambda)$ is $\varpi_\lambda$-complete, there exists $h \in H_\varpi$ such that $\{\theta^n(h_0)\}$ is $\varpi_\lambda$-convergent to $H$. Now, since $\theta$ is continuous, it follows from Proposition 3 that
$$\varpi_\lambda(h, \theta j) = \lim_{n\to\infty}(\theta^n(h_0), \theta(\theta^n(h_0)))$$
$$= \lim_{n\to\infty}(\theta^n(h_0), \theta^{n+1}(h_0))$$
$$= \varpi_\lambda(h_0, h_0)$$
$$= 0,$$
which shows that $H$ is a fixed point of $\theta$.
To show uniqueness, suppose that $\{\theta^n(h_0)\}$ is $\varpi_\lambda$-convergent to $j$, then by Proposition 3.2, we have that $h = j$.

Case (2):
for $s > 0$ and taking $l = i + 1$, the expression (2) becomes
$$\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)) \le \psi(H_\theta^s(\theta^{n-1+i}(h_0), \theta^{n+i}(h_0)))$$
$$= \psi((\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{\sigma_1}(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{\sigma_2})$$
$$< (\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{\sigma_1}(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{\sigma_2}$$
The above inequality turns into
$$(\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{1-\sigma_1} < (\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)))^{\sigma_2}$$
and since $\sigma_1 + \sigma_2 = 1$, we have
$$\varpi_\lambda\left(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)\right) < (\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0))$$
Returning to (5), we find that,
$$\varpi_\lambda\left(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)\right) < (\varpi_\lambda(\theta^{n+i}(h_0), \theta^{n+1+i}(h_0)) \quad (6)$$
*Since* $\delta(\varpi_\lambda, \theta, h) = \sup\{\varpi_\lambda(\theta^i(h), \theta^l(h)\}$, The expression (6) becomes
$$\delta(\varpi_\lambda, \theta, \theta^n(h_0)) \le \psi(\delta(\varpi_\lambda, \theta, \theta^{n-1}(h_0)))$$
$$\le \psi^2(\delta(\varpi_\lambda, \theta, \theta^{n-2}(h_0)))$$
$$\vdots$$
$$\le \psi^n(\delta(\varpi_\lambda, \theta, h_0).$$

By using the same tools as in the case of $s > 0$, we find that $\{\theta^n(h_0)\}$ is an $\varpi_\lambda$-Cauchy sequence in a $\varpi_\lambda$-complete *JS*-modular metric space. Hence, there exists $h \in H_\varpi$ such that $\{\theta^n(h_0)\}$ is $\varpi_\lambda$-convergent to $H$. It follows that $H$ is a unique fixed point of $\theta$. □

In the following result, we establish the existence of fixed point of $\theta$ under the restriction of contuinity of some iterate of $\theta$.

***Theorem 3***

*Let $(H_\varpi, \varpi_\lambda)$ be a complete JS-modular metric space and $\theta : H_\varpi \to H_\varpi$ be a HCJT. If for some $i > 1$, $\theta^i$ is continuous, then $\theta$ has a fixed point in $H_\varpi$.*

***Proof:***

In Theorem 3.3, we have established that there exists an $\varpi_\lambda$- Cauchy sequence $\{h_n\}$ in $(H_\varpi, \varpi_\lambda)$ with $h_n = \theta h_{n-1}$ such that $h_n \to h$ for some $h \in H_\varpi$. Let $\{h_{n_k}\}$ be a subsequence of $\{h_n\}$ where $n_k = k.i$ for some $k \in \mathbb{N}$, $k > 1$ fixed. Notice that $\theta^0$ is an identity self-mapping on $H_\varpi$ so that $h_{n_k} = \theta^i x_{n_k - 1}$. Hence, by continuity of $\theta^i$, we have

$$\varpi_\lambda(h, \theta^i h) = \lim_{k\to\infty} \varpi_\lambda(h, \theta^i h_{n_k - i})$$
$$= \lim_{k\to\infty} \varpi_\lambda(h, h_{n_k})$$
$$= \varpi_\lambda(h, h)$$
$$= 0,$$

which means that $H$ is a fixed point of $\theta^i$. Now, to see that $H$ is a fixed point of $\theta$, assume by contradiction that $\theta h \neq h$. Then in that case $\theta^{i-m-1}h \neq \theta^{i-m}$ for any $m = 0,1,...,i-1$. Again, we must consider two cases. **Case (1):** for $s > 0$, by replacing $h$ by $\theta^{i-m-1}h$ and $j$ by $\theta^{i-m}h$, we have

$$H_\theta^s(\theta^{i-m-1}h, \theta^{i-m}h) = [\sigma_1(\frac{\varpi_\lambda(\theta^{i-m-1}h, \theta(\theta^{i-m-1}h)\varpi_\lambda(\theta^{i-m-1}h, \theta(\theta^{i-m}h))}{\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h)})^s + \sigma_2(\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h))^s]^{\frac{1}{s}}$$

and since $\theta$ is a *HCJT*,

$$\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h) = \varpi_\lambda(\theta(\theta^{i-m-1}h), \theta(\theta^{i-m}h))$$
$$\leq \psi(H_\theta^s(\theta^{i-m-1}h, \theta^{i-m}h))$$
$$= \psi([\sigma_1(\varpi_\lambda(\theta^{i-m-1}h, \theta(\theta^{i-m-1}h)))^s$$
$$+\sigma_2(\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h))^s]^{\frac{1}{s}})$$
$$< [\sigma_1(\varpi_\lambda(\theta^{i-m-1}h, \theta(\theta^{i-m-1}h)))^s + \sigma_2(\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h))^s]^{\frac{1}{s}}$$

which yields that

$$(\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h))^s(1-\sigma_1) < \sigma_2(\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h))^s$$

However, $\sigma_1 + \sigma_2 = 1$, so that for every $m = 0,1,...,i-1$,

$$\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h) < \varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h).$$

This clearly implies that for every $k = m, m+1, ..., i-1$,

$$\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h) < \varpi_\lambda(\theta^{i-m-k-1}h, \theta^{i-m-k}h).$$

Taking in particular $m = 0$ and $k = i-1$ in the above inequality, we get

$$\varpi_\lambda(h, \theta^i h) < \varpi_\lambda(\theta^i h, \theta^{i+1}h) < \varpi_\lambda(h, \theta h)$$

which is a contradiction. Consequently, $\theta h = h$.

To show uniqueness, suppose that $\{\theta^n(h_0)\}$ is $\varpi_\lambda$-convergent to $h$, then by Proposition 3.2, we have that the fixed point is unique.

Case (2):

for $s = 0$, by replacing $h$ by $\theta^{i-m-1}h$ and $j$ by $\theta^{i-m}h$, we have

$$H_\theta^s(\theta^{i-m-1}h, \theta^{i-m}h) = [\varpi_\lambda(\theta^{i-m-1}h, \theta(\theta^{i-m-1}h))]^{\sigma_1}[\varpi_\lambda(\theta^{i-m}h, \theta(\theta^{i-m}h))]^{\sigma_2}$$
$$= [\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h)]^{\sigma_1}[\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1})]^{\sigma_2}$$

and the inequality becomes

$$\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h) = \varpi_\lambda(\theta(\theta^{i-m-1}h), \theta(\theta^{i-m}h)) \leq \psi(H_\theta^s(\theta^{i-m-1}h, \theta^{i-m}h))$$
$$= \psi([\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h)]^{\sigma_1}[\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1})]^{\sigma_2})$$
$$< [\varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h)]^{\sigma_1}[\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1})]^{\sigma_2}$$

By simple calculation, we get

$$\varpi_\lambda(\theta^{i-m}h, \theta^{i-m+1}h) < \varpi_\lambda(\theta^{i-m-1}h, \theta^{i-m}h),$$

and using the same tools as in the Case (1) and Proposition 3.2, we obtain that $H$ is a unique fixed point of $\theta$. □

***Corollary 1***

*Let $(H_\varpi, \varpi_\lambda)$ be a complete JS-modular metric space. Suppose that the continuous mapping $\theta : H_\varpi \to H_\varpi$ satisfies*

$\varpi_\lambda(\theta h, \theta j) \leq \psi(\varpi_\lambda(h, j))$

$\forall h, j \in H_\varpi$. *Then $\theta$ has a fixed point in $H_\varpi$.*

***Proof***

Letting $\sigma_1 = 0$, $\sigma_2 = 1$ and $s > 0$, the result follows. □

***Corollary 2***

*(see Turkoglu & Manav (2018), Theorem 3.1) Let $(H_\varpi, \varpi_\lambda)$ be a complete JS-modular metric space. Let $\theta$: $H_\varpi \to H_\varpi$ be an $\varpi_\lambda$-contraction mapping, i.e*

$\varpi_\lambda(\theta h, \theta j) \le k\varpi_\lambda(h, j) \qquad \forall h, j \in H_\varpi,\ k \in (0,1).$

*Then, $\theta$ has a unique fixed point (say u) and $\theta$ is continuous at u.*

***Proof***

Consider Definition 3.1 and let $\psi(t) = kt$, $\sigma_1 = 0$, $\sigma_2 = 1$ and $s = 1$ , we have

$\varpi_\lambda(\theta h, \theta j) \le k(H_\theta^s(h, j)) = k\varpi_\lambda(h, j)$,

which coincides with Theorem 3.1 due to Turkoglu & Manav (2018) and so the proof follows. □

**Example 1**

*Let $H_\varpi = [0,\infty)$, $\sigma : H_\varpi{}^2 \to \mathbb{R}_+$ and define $\varpi_\lambda$: $(0,\infty) \times H_\varpi \times H_\varpi \to [0,+\infty]$ by*

$\varpi_\lambda(h, j) = \frac{\max\{|h|,|j|\}}{\lambda}\ \forall h, j \in H_\varpi.$

*Then, $(H_\varpi, \varpi_\lambda)$ is a complete JS-modular metric space. Consider the mapping $\theta$: $H_\varpi \to H_\varpi$ defined by*

$$\Theta h = \begin{cases} \frac{1}{12}h, & h \in [0,1], \\ \frac{1}{12}, & h > 1. \end{cases}$$

*Define the function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ . It is clear that $\psi$ is a (c)-comparison function. Let $\sigma_1 = 0$, $\sigma_2 = 1$. For $s > 0$, if $h, j \in [0,1]$ with $h > j$, then,*

$\varpi_\lambda(\theta h, \theta j) = \frac{\max\{|\theta h|,|\theta j|\}}{\lambda} = \frac{1}{12\lambda}h$

$\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(h, j) = \frac{5}{6}\frac{\max\{|h|,|j|\}}{\lambda} = \frac{5}{6\lambda}h$

*Hence, $\varpi_\lambda(\theta h, \theta j) \le \psi(H_\theta^s(h, j))$.*

*if $h, j \in (1, \infty)$, $\varpi_\lambda(\theta h, \theta j) = \frac{1}{12\lambda} < \frac{5}{6\lambda}h \le \psi(H_\theta^s(h, j))$*

*If $h \in [0,1]$ and $j > 1$, then,*

$\varpi_\lambda(\theta h, \theta j) = \frac{\max\{|\theta h|,|\theta j|\}}{\lambda} = \frac{1}{12\lambda}$

$\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(h, j) = \frac{5}{6}\frac{\max\{|h|,|j|\}}{\lambda} = \frac{5}{6\lambda}j$

*Hence, $\varpi_\lambda(\theta h, \theta j) \le \psi(H_\theta^s(h, j))$.*

*For $s = 0$, $\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(j, \theta j)$.*

*If $h,j \in [0,1]$ with $h < j$, then*

$\varpi_\lambda(\theta h, \theta j) = \frac{\max\{|\theta h|,|\theta j|\}}{\lambda} = \frac{1}{12\lambda}j$

$\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(j, \theta j) = \frac{5}{6}\frac{\max\{|j|,|\theta j|\}}{\lambda} = \frac{5}{6\lambda}j$

*Hence, $\varpi_\lambda(\theta h, \theta j) \le \psi(H_\theta^s(h, j))$.*

*For $h, j \in (1,\infty)$,*

$\varpi_\lambda(\theta h, \theta j) = \frac{\max\{\frac{1}{12},\frac{1}{12}\}}{\lambda} = \frac{1}{12\lambda}$

$\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(j, \theta j) = \frac{5}{6}\frac{\max\{|j|,|\theta j|\}}{\lambda} = \frac{5}{6\lambda}j$

*Hence, $\varpi_\lambda(\theta h, \theta j) \le \psi(H_\theta^s(h, j))$.*

*If $h \in [0,1]$ and $j > 1$,*

$\varpi_\lambda(\theta h, \theta j) = \frac{\max\{\frac{1}{12},\frac{1}{12}\}}{\lambda} = \frac{1}{12\lambda}$

$\psi(H_\theta^s(h, j)) = \frac{5}{6}\varpi_\lambda(j, \theta j) = \frac{5}{6}\frac{\max\{|j|,|\theta j|\}}{\lambda} = \frac{5}{6\lambda}j$

*Hence, $\varpi_\lambda(\theta h, \theta j) \le \psi(H_\theta^s(h, j))$.*

*Therefore, all the assumptions of Theorem 3.3 are satisfied. Consequently, $\theta$ has a unique fixed point given by $h = 0$.*

***Remark***

From Example 3.1 above, $\varpi_\lambda$ is not a modular metric since $h = j \nrightarrow \varpi_\lambda(h,j) = 0$. Also, Karapinar & Fulga (2019) declared (Definition 2.6) that H and j are distinct, since $J_\theta^s(h, j)$ is undefined for Case (1) if h = j. However, $h = j \nrightarrow \varpi_\lambda(h,j) = 0$ in our definition. Therefore, our HCJT is not Jaggi-type contraction defined by Karapinar & Fulga (2019), and so Theorem 2.1 due to Karapinar & Fulga (2019) is not applicable to this example.

**CONCLUSION**

This study presented a general idea of a contraction named Jaggi-type hybrid contraction in a *JS*-modular *MS*. In Theorem 3.3, sufficient conditions under which such contraction posses a unique *FP* have been presented. We also established the *FP* existence of the map $\theta$ when continuity is restricted only to some iterate of $\theta$ in Theorem 3.4. Corollaries 3.5 and 3.6 are provided to show that the approach described herein is a generalization and improvement on some related results in the literature. Example 3.1 has been constructed to support the assumptions of our obtained result and show the relationship between the ideas of this paper and the corresponding literature.

**REFERENCES**


Abdou A. & Khamsi M., (2013). Fixed Point Results of Pointwise Contractions in Modular Metric Spaces, *Fixed Point Theory and Applications.* **20**(2), 213-221.

Abdou A., (2013). On Asymptotic Pointwise Contractions in Modular Metric Spaces, *Abstract and Applied Analysis.* **2013**.

Arockiam S., & John M. J., (2012). Common Fixed Point Theorems for Three Mappings in Generalized Modular Metric Spaces, *Korean Journal of Mathematics.* **32(1)**.

Chaipunya P., Mongkolkeha C., Sintunavarat W., & Kumam P., (2012). Fixed-Point Theorems for Multivalued Mappings in Modular Metric Spaces, *Abstract and Applied Analysis.* **2012**.

Chistyakov V. V., (2010). Modular Metric Spaces I: Basic Concepts, *Nonlinear Analysis.* **72**, 1-14.

Okeke G. A. & Francis D., (2024). Some Common Fixed Point Theorems in Generalized Modular Metric Spaces with Applications, *Scientific.African.* **23**.

Jiddah J. A., Alansari M., Mohamed O. K. S., Shagari M. S. & Bakery A., (2022). Fixed Point Results of Jaggi Type Hybrid Contraction in Generalized Metric Space, *Journal of Function Spaces.* 2022(1), 2205423.

Jiddah J. A., Alansari M., Mohamed O. K. S., Shagari M. S. & Bakery A., (2023). Fixed Point Results of a New Family of Contractions in Metric Space Endowed with a Graph, *Journal of Mathematics.* 2023(1), 2534432.

Jiddah J. A. & Shagari M. S., (2024). Graphical Approach to the Study of Fixed-Point Results Involving Hybrid Contractions, *Results in Control and Optimization.* 14.

Jiddah J. A., Noorwali M., Shagari M. S., Rashid S. & Jarad F., (2022). Fixed Point Results of a New Family of Hybrid Contractions in Generalized Metric Space With Applications, *AIMS Mathematics,* 7(10), 17894-17912.

Jiddah J. A., Mohammed S. S. S. & Imam A. T., (2023). Advancements in Fixed Point Results of Generalized Metric Spaces: A survey, *Sohag Journal of Sciences.* **8**(2), 175-208.

Jleli M., & Samet B., (2015). A Generalized Metric Space and Related Fixed Point Theorems, *Fixed Point Theory and Applications.* 2015,61.

Karapinar E., & Fulga A., (2019). A Hybrid Contraction that Involves Jaggi Type, *Symmetry.* 11, 5:715.

Manav N., (2021). Fixed-Point Theorems in Generalized Modular Metric Spaces, *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences.* 89-111.

Manav N., & Agarwal R. P., (2023). Best Approximation of Fixed Point Results fornBranciari Contraction of Integral Type on Generalized Modular Metric Space, *Mathematics.* 11(21), 4455.

Mongolkeha C., Sintunavarat W., & Kumaam P., (2011). Fixed Point Theorems for Contraction Mappings in Modular Metric Spaces, *Fixed Point Theory and Applications.* **2011**.

Musielak J., & Orlicz W., (1959). Some Remarks on Modular Spaces. Bull Acad Polon Sci Sr Sci Math Astron Phys. 7, 661-668.

Mutlu A., Ozkan K., & Gurdal U., (2018). Coupled Fixed Point in Partially Ordered Modular Metric Spaces and its Application, *Journal of Computational Analysis and Applications.* **25**(2), 1-10.

Nakano H., (1959). Modular Semi-Ordered Spaces, Tokyo, Japan.

Nadler B., (1969). Multivalued Contraction Mappings, *Pacific J. Maths* 30, 475488.

Turkoglu D., & Manav N., (2018). Fixed Point Theorems in a New Type of Modular Metric Space, *Fixed Point Theory and Applications.* **2018**.

Yahaya S., & Shagari M. S., (2023). Multivalued Hybrid Contraction That Involves Jaggi and Pata-type Inequalities, *Mathematical Foundations of Computing.*

Zhua X., Zhaoa Q., & Zhonga L., (2020). Fixed Point Results for F-Contraction Mapping in Modular Metric Spaces, *European Journal of Mathematics and Computer Science.* **7**(2).