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Das-Gupta-Type Contraction Mapping Theorems in (α, β) - Complex Valued b-Metric Spaces

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ABSTRACT
In this paper, we investigate fixed-point problems for a class of Das–Gupta-type rational contraction mappings defined on complete (α, β)-complex-valued b-metric spaces. We establish the existence and uniqueness of a fixed point. The proof is based on the Picard iterative sequence, estimates for successive iterates, the generalized triangle inequality, and completeness of the underlying space. The additional parameter restriction is essential for the geometric-series argument used to establish the Cauchy property of the iterative sequence. The result provides a sufficient fixed-point criterion that extends the study of rational contractions to the (α, β)-complex-valued (b)-metric setting.

Keywords: Fixed Point, b-Metric Space, Complex-Valued Metric Space, (α, β)-Complex Valued b-Metric Space, Das-Gupta Contraction, Rational Mapping

INTRODUCTION
The origin of modern metric fixed-point theory traces back to the seminal doctoral work of Stefan Banach (1922), who formulated the celebrated Banach Contraction Principle (BCP). Banach proved that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping satisfying

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, 1)$, then T possesses a unique fixed point. The analytical beauty of the BCP lies in its constructive nature, providing an iterative algorithm that guarantees convergence to the fixed point, a property extensively leveraged to solve existence criteria for initial value problems and integral equations.

Over the past century, researchers have progressively advanced this principle by executing two distinct types of generalizations: weakening the contractive condition itself, or broadening the underlying geometric structure of the topological space.

The Geometric and Structural Evolution
The structural expansion away from classical metric spaces began when Bakhtin (1989) and subsequently Czerwik (1993) introduced the concept of b-metric spaces. They relaxed the standard triangular inequality by introducing a real constant multiplier $s \geq 1$, such that

$$d(x, y) \leq s[d(x, z) + d(z, y)]$$

This modification accommodates a wider array of spaces where the standard triangle inequality fails, such as certain Lp spaces ($0 < p < 1$).

A parallel breakthrough occurred when Azam et al. (2011) introduced complex-valued metric spaces. Instead of mapping distances to non-negative real numbers R+, they mapped them to the complex plane C, introducing a partial order relation $\precsim$ governed by the Cartesian properties of complex coordinates ($Z1 \precsim Z2$ if and only if $Re(Z1) \leq Re(Z2)$ and $Im(Z1) \leq Im(Z2)$).

To unify these two separate topological leaps, Rao et al. (2013) combined the concepts to establish complex-valued b-metric spaces, which introduced the real multiplier $s \geq 1$ directly into the partially ordered complex plane inequalities. Most recently, the geometric horizon has been extended to (α, β)-complex valued b-metric spaces by Singh et al. (2021) as follows:

$$d(x, y) \leq \alpha d(x, z) + \beta d(z, y),$$

where $\alpha, \beta \geq 1$ are scaling factors which themselves are controlled by distinct structural parameters acting on the real and imaginary coordinate vectors, offering a highly robust framework for multi-dimensional operator structures.

The Evolution of Contractive Conditions
Concurrently, the algebraic boundaries of contraction inequalities were drastically shifted. Das and Gupta (1975) introduced rational contractive expressions of the form

$$d(Tx, Ty) \leq ad(x, y) + b\,\frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}$$

By integrating self-deviation terms into the numerator and placing the distance metric in the denominator, Das-Gupta contractions effectively bypass the strict, uniform shrinking constraints of the BCP. Subsequent authors have investigated related rational contractions in several generalized metric structures (see Auwalu and Hassan (2024) and references therein).

The Research Gap and Paper Objective
Despite these extensive parallel tracks of research, a critical gap exists in the functional analysis literature: a definitive exploration of Das-Gupta-type rational contractions within the rigid, parameter-driven environment of (α, β)-complex valued b-metric spaces has not yet been conducted. Because divisions in the complex plane do not mirror real numbers linearly and (α, β)-scaling influences the triangle inequality directly, proving convergence in this space introduces non-trivial analytical obstacles.

The primary objective of this paper is to bridge this gap. We establish unique fixed-point theorems for a newly defined class of Das-Gupta type contractions operating in (α, β)-complex valued b-metric spaces. Our results explicitly reconcile previous logical ambiguities in existing literature and generalize foundational outcomes established by Azam, Rao, and Singh.

MATERIALS AND METHODS
Preliminaries
Definition 2.1
(Azam et al., 2011) Let C be the set of complex numbers and $W1, W2 \in C$.

Define a partial order relation $\precsim$ on C as follows:

$W1 \precsim W2 \iff Re(W1) \leq Re(W2)$ and $Im(W1) \leq Im(W2)$

It follows that W1 ≾ W2 if one of the following conditions is satisfied:
i. Re(W1) = Re(W2) and Im(W1) < Im(W2),
ii. Re(W1) < Re(W2) and Im(W1) = Im(W2),
iv. Re(W1) < Re(W2) and Im(W1) < Im(W2),
v. Re(W1) = Re(W2) and Im(W1) = Im(W2).
Specifically, we write W1 ≾ W2 if W1≠ W2 and at least one of conditions (1), (2), or (3) is satisfied, and we write W1 < W2 if only condition (3) is satisfied. Consequently, the following properties hold:
$0 \precsim W1 \precsim W2 \Longrightarrow |W1| < |W2|$,
$W1 \precsim W2, W2 \precsim W3 \Longrightarrow W1 \precsim W3$,
If a, b ∈ R and a ≤ b, then aW ≾ bW for all W ∈ C with W ≥ 0.

Definition 2.2
(Azam et al., 2011) Let M be a non-empty set. Suppose the mapping $\rho : M \times M \to C$ satisfies the following properties for all u, v, r ∈ M:
$0 \precsim \rho(u, v)$,
$\rho(u, v) = 0 \Longleftrightarrow u = v$,
$\rho(u, v) = \rho(v, u)$,
$\rho(u, v) \precsim \rho(u, r) + \rho(r, v)$.
Then ρ is called a complex-valued metric on M, and the pair (M, ρ) is called a complex-valued metric space.

Definition 2.3
(Rao et al., 2013) Let M be a non-empty set and s ≥ 1 be a given real number.
Suppose the mapping $\rho : M \times M \to C$ satisfies the following properties for all u, v, r ∈ M:
$0 \precsim \rho(u, v)$,
$\rho(u, v) = 0 \Longleftrightarrow u = v$,
$\rho(u, v) = \rho(v, u)$,
$\rho(u, v) \precsim s[\rho(u, r) + \rho(r, v)]$.
Then ρ is called a complex-valued b-metric on M, and the pair (M, ρ) is called a complexvalued b-metric space.

Definition 2.4
(Singh et al., 2021) Let M be a non-empty set and α, β ≥ 1 be given real numbers. Suppose the mapping $\rho : M \times M \to C$ satisfies the following properties for all u, v, r ∈ M:
$0 \precsim \rho(u, v)$,
$\rho(u, v) = 0 \Longleftrightarrow u = v$,
$\rho(u, v) = \rho(v, u)$,
$\rho(u, v) \precsim \alpha\rho(u, r) + \beta\rho(r, v)$.
Then ρ is called an (α, β)-complex valued b-metric on M, and the pair (M, ρ) is called an (α, β)-complex valued b-metric space.

Example 2.1
(Singh et al., 2021) Let M = {1, 3}. Define a mapping $\rho : M \times M \to C$ for all u, v ∈ M by:

$$\rho(u, v) = e^{|u-v|} + ie^{|u-v|}, \text{ if } u \neq v \text{ and } \rho(u, v) = 0, \text{ if } u = v$$

Clearly, the pair (M, ρ) forms an (α, β)-complex valued b-metric space for appropriate coefficients α, β ≥ 1, but fails the standard triangle inequality, making it distinct from a classical complex-valued metric space.

Definition 2.5
(Singh et al., 2021) Let (M, ρ) be an (α, β)-complex valued b-metric space and let {xj} be a sequence in M. The sequence $\{x_j\}$ is said to converge to an element x ∈ M if and only if $\lim_{j\to\infty} |\rho(x_j, x)| = 0$.

Definition 2.6
(Singh et al., 2021) Let (M, ρ) be an (α, β)-complex valued b-metric space and let $\{x_j\}$ be a sequence in M. The sequence $\{x_j\}$ is called a Cauchy sequence in (M, ρ) if $\lim_{j,m\to\infty} |\rho(x_j, x_{j+m})| = 0$.

Definition 2.7
(Singh et al., 2021) Let (M, ρ) be an (α, β)-complex valued b-metric space. The space (M, ρ) is said to be complete if every Cauchy sequence $\{x_j\}$ in M converges to a point $x^* \in M$ such that $\lim_{j\to\infty} |\rho(x_j, x^*)| = 0$.

Lemma 2.1
(Rao et al., 2013) Suppose a real λ satisfies 0 ≤ λ < 1, then $\lim_{j\to\infty} \lambda_j = 0$.

RESULTS AND DISCUSSION
Main Results
Theorem 3.1
Let (M, ρ) be a complete (α, β)-complex-valued b-metric space with coefficients α, β ≥ 1. Suppose that the mapping *f* : M → M satisfies the rational contractive condition:

$$\rho(fu, fv) \precsim \lambda_1\rho(u, v) + \lambda_2\frac{\rho(v,fv)[1 + \rho(u,fu)]}{1 + \rho(u,v)} \quad (1)$$

for all u, v ∈ M, where λ_1, λ_2 are non-negative real numbers such that $\lambda_1 + \lambda_2 < 1$, $\beta\lambda_2 < 1$, and $\beta\lambda_1 < 1 - \lambda_2$. Then f has a unique fixed point in M.

Proof
We begin by constructing an iterative sequence $\{x_j\}$ in M. Let $x_0 \in M$ be an arbitrary initial point, and define the sequence recursively by:

$$x_j = fx_{j-1} \text{ for all } j = 1, 2, 3, \ldots \quad (2)$$

Substituting $u = x_{j-1}$ and $v = x_j$ into contractive condition (1), we obtain:

$$\rho(x_j, x_{j+1}) = \rho(fx_{j-1}, fx_j)$$
$$\precsim \lambda_1\rho(x_{j-1}, x_j) + \lambda_2 \frac{\rho(xj,fxj)[1 + \rho(xj-1,fxj-1)]}{1 + \rho(xj-1,xj)} \quad (3)$$

$$\precsim \lambda 1\rho(xj-1, xj) + \lambda 2 \frac{\rho(xj,xj+1)[1 + \rho(xj-1,xj)]}{1 + \rho(xj-1,xj)}$$

Simplifying the rational fraction yields:

$$\rho(x_j, x_{j+1}) \precsim \lambda_1\rho(x_{j-1}, x_j) + \lambda_2\rho(x_j, x_{j+1})$$

Rearranging the terms to isolate $\rho(x_j, x_{j+1})$ gives:

$$\rho(xj, xj+1) - \lambda 2\rho(xj, xj+1) \precsim \lambda 1\rho(x_{j-1}, x_j)$$
$$(1 - \lambda_2)\rho(x_j, x_{j+1}) \precsim \lambda 1\rho(x_{j-1}, x_j) \quad (4)$$
$$\rho(x_j, x_{j+1}) \precsim \frac{\lambda 1}{1 - \lambda 2}\rho(xj-1, xj)$$

Since λ1 + λ2 < 1, it follows that:

$$\lambda := \frac{\lambda 1}{1 - \lambda 2} < 1 \quad (5)$$

Applying this contractive relation (4) inductively for all j ∈ N leads to:

$$\rho(x_j, x_{j+1}) \precsim \lambda\rho(x_{j-1}, x_j) \precsim \lambda_2\rho(x_{j-2}, x_{j-1}) \precsim \cdots \precsim \lambda_j\rho(x_0, x_1) \quad (6)$$

$$|\rho(x_j, x_{j+1})| \leq \lambda_j|\rho(x_0, x_1)| \quad (7)$$

Taking the limit as j → ∞, and since 0 ≤ λ < 1, Lemma 2.9 ensures that $\lim_{j\to\infty} \lambda^j = 0$. Consequently, we obtain:

$$\lim_{n\to\infty} |\rho(xj, xj + 1)| = 0 \quad (8)$$

For n, m ∈ N with n ≥ m, applying the generalized triangle inequality of the (α, β)-complex valued b-metric space repeatedly yields:

$$\rho(x_n, x_{n+m}) \precsim \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m})$$
$$\precsim \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_{n+m})]$$
$$\precsim \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \beta 2\rho(x_{n+2}, x_{n+m})$$

Continuing this structural expansion inductively results in:

$\rho(x_n, x_{n+m}) \precsim \alpha\lambda^n\rho(x_0, x_1) + \alpha\beta\lambda^{n+1}\rho(x_0, x_1) + \cdots + \beta^{m-1}\lambda^{n+m-1}\rho(x_0, x_1) \precsim \alpha\lambda^n\rho(x_0, x_1)[1 + \beta\lambda + (\beta\lambda)^2 + \cdots + (\beta\lambda)^{m-2}] + \beta^{m-1}\lambda^{n+m-1}\rho(x_0, x_1)$
$\precsim \alpha\lambda^n\rho(x_0, x_1)[1 + \beta\lambda + (\beta\lambda)^2 + \ldots]$

Evaluating the geometric series under the convergence condition $\beta\lambda < 1$ leads to:

$$\rho(x_n, x_{n+m}) \precsim \frac{\alpha\lambda n}{1-\beta\lambda}\rho(x_0, x_1) \qquad (9)$$

$$|\rho(xn, xn+m)| \leq \frac{\alpha\lambda n}{1-\beta\lambda}|\rho(x_0, x_1)| \qquad (10)$$

Taking the limit as $n \to \infty$, and since $0 \leq \lambda < 1$, Lemma 2.9 ensures that $\lim_{n\to\infty} \lambda^n = 0$.

Consequently, we obtain:

$$\lim_{n\to\infty} |\rho(x_n, x_{n+m})| = 0 \qquad (11)$$

This demonstrates that $\{x_n\}$ is a Cauchy sequence in (M, ρ). Since the space (M, ρ) is complete, there exists an element $x^* \in M$ such that the sequence $\{x_n\}$ converges to x^*.

Existence of a Fixed Point

We now show that $fx^* = x^*$. Suppose on the contrary that $fx* \neq x*$, meaning there exists a non-zero complex value $Z \in C$ such that $\rho(x*, fx^*) = Z > 0$. By applying the generalized triangle inequality, we have:

$$Z = \rho(x^*, fx^*) \precsim \alpha\rho(x^*, x_{j+1}) + \beta\rho(x_{j+1}, fx^*) \qquad (12)$$

Substituting the contractive condition (1) into the second term yields:

$$Z \precsim \alpha\rho(x*, xj+1) + \beta\left[\frac{\lambda1\rho(xj,x*) + \lambda2\,\rho(x*,fx*)[1+\rho(xj,fxj)]}{1+\rho(xj,x*)}\right]$$

$$Z \precsim \alpha\rho(x,xj+1) + \beta\lambda1\rho(xj, x*) + \beta\lambda2\,\frac{Z[1+\rho(xj,xj+1)]}{1+\rho(xj,x*)} \qquad (13)$$

Taking the modulus on both sides results in:

$$|Z| \leq \alpha|\rho(x^*, x_{j+1})| + \beta\lambda_1|\rho(x_j, x^*)| + \beta\lambda_2\,\frac{|Z||1+\rho(xj,xj+1)|}{|1+\rho(xj,x*)|}$$

As $j \to \infty$, the convergence properties $\lim_{j\to\infty} \rho(x,xj+1) = 0$, $\lim_{j\to\infty} \rho(x_j, x^*) = 0$, and $\lim_{j\to\infty} \rho(x_j, x_{j+1}) = 0$ reduce the inequality strictly to:

$|Z| \leq \beta\lambda2|Z|$

Which can be rewritten as:

$(1 - \beta\lambda2)|Z| \leq 0$

Since we have explicitly established the hypothesis that $\beta\lambda_2 < 1$, the factor $(1-\beta\lambda_2)$ is strictly positive. This forces $|Z| \leq 0$, which directly contradicts our initial assumption that $|Z| > 0$. Thus, we must have $|Z| = 0$, implying $|\rho(x^*, fx^*)| = 0$, thus $\rho(x^*, fx^*) = 0$, which proves that $fx^* = x^*$. Hence, x^* is a fixed point of f.

Uniqueness of the Fixed Point

To establish the uniqueness of the fixed point, suppose that u^* and v^* are two distinct fixed points of f in M, such that $fu^* = u^*$ and $fv^* = v^*$. Using contractive condition (1) with $u = u^*$ and $v = v^*$, we have:

$$\rho(u^*, v^*) = \rho(fu^*, fv^*) \precsim \lambda_1\rho(u^*, v^*) + \lambda_2\,\frac{\rho(v*,fv*)[1+\rho(u*,fu*)]}{1+\rho(u*,v*)} \qquad (14)$$

Since $fu^* = u^*$ and $fv^* = v^*$, by the identity property of the metric, we know that $\rho(u^*, fu^*) = \rho(u^*, u^*) = 0$ and $\rho(v^*, fv^*) = \rho(v^*, v^*) = 0$. Substituting these values directly into the rational contractive expression eliminates the second term completely:

$$\rho(u^*, v^*) \leq \lambda_1\rho(u^*, v^*) + \lambda_2\frac{0\cdot[1+0]}{1+\rho(u*,v*)} \qquad (15)$$

$\rho(u^*, v^*) \leq \lambda_1\rho(u^*, v^*)$

Taking the modulus on both sides yields:

$|\rho(u^*, v^*)| \leq \lambda_1|\rho(u^*, v^*)|$

$$(1 - \lambda_1)|\rho(u,v^*)| \leq 0 \qquad (16)$$

Since $\lambda_1 + \lambda_2 < 1$ and $\lambda_1, \lambda_2 \geq 0$, it is guaranteed that $\lambda_1 < 1$, making the term $(1 - \lambda_1)$ strictly positive. Therefore, the inequality forces $|\rho(u^*, v^*)| = 0$, which implies $\rho(u^*, v^*) = 0$. By the axioms of the space, this yields $u^* = v^*$, concluding the uniqueness proof.

Special Cases and Remarks

Remark 3.2.

The condition

$\lambda_1 + \lambda_2 < 1$

ensures

$0 \leq \lambda := \frac{\lambda1}{1-\lambda2} < 1.$

However, this condition alone does not guarantee the convergence of the geometric series generated by the repeated b-metric inequality. The additional condition

$\beta\lambda_1 < 1 - \lambda_2$

which gives

$\beta\lambda < 1$

is therefore required by the proof.

Equivalently,

$\frac{\beta\lambda1}{1-\lambda2} < 1.$

Remark 3.3

The existence argument also requires

$\beta\lambda_2 < 1.$

Thus, a fully explicit set of assumptions for Theorem 3.1 is

$\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 < 1, \beta\lambda_2 < 1, \beta\lambda_1 < 1 - \lambda_2.$

Remark 3.4

When

$\lambda_2 = 0,$

condition (1) reduces to the ordinary contraction-type inequality

$\rho(f u, f v) \precsim \lambda_1\rho(u, v),$

and the theorem reduces to a Banach-type fixed-point result in the (α, β)-complex-valued bmetric framework, subject to $\beta\lambda_1 < 1.$

Remark 3.5

When

$\alpha = \beta = s,$

the underlying structure reduces to the usual complex-valued b-metric setting with coefficients, and the theorem provides a corresponding rational contraction result under the stated parameter restrictions.

CONCLUSION

In this paper, we established an existence and uniqueness theorem for a class of Das–Guptatype rational contractions on complete (α, β)-complex-valued b-metric spaces. The proof is based on a Picard iterative sequence. We first established a geometric estimate for successive iterates and then used the generalized (α, β)-triangle inequality to prove that the resulting sequence is Cauchy. Completeness of the space yields convergence to an element $x* \in M$.

Furthermore, foundational landmarks (Azam et al, 20211, Rao et al, 2013, and Singh et al, 2021) are generalized by our results. The contractive condition is then used to prove that $x*$ is a fixed point of f, while a direct comparison of two fixed points establishes uniqueness. A significant feature of the result is the explicit parameter condition

$\beta\lambda_1 < 1 - \lambda_2,$

which is required for the geometric-series argument in the Cauchy-sequence proof. The condition

$\beta\lambda_2 < 1$

is also retained for the fixed-point existence argument. The result therefore provides a rigorous sufficient condition for the existence and uniqueness of a fixed point for the considered class of rational contractions. Further work may investigate weaker parameter restrictions, alternative rational contractive conditions, common fixed points, multivalued mappings, and applications to nonlinear operator equations.

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