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A LAGUERRE-PERTURBED GALERKIN METHOD FOR NUMERICAL SOLUTION OF HIGHER-ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT
This study presents a novel Laguerre-Perturbed Galerkin (LPG) method for the numerical solution of higher-order nonlinear integro-differential equations. The method integrates Laguerre polynomials as primary basis functions with shifted Chebyshev polynomial perturbations to improve approximation precision. Nonlinear terms are handled via quasilinearization, converting the problem into a sequence of linear systems solvable within a Galerkin projection framework. The LPG approach is tested on benchmark nonlinear Volterra and Fredholm integro-differential equations, exhibiting superior convergence rates and accuracy compared to existing techniques such as decomposition methods and wavelet collocation. Testing on classic Volterra and Fredholm examples shows LPG pulling ahead, errors drop from about at N=5 to a tiny at N=10, with faster exponential convergence than methods like Sharif et al.'s (2020) decomposition or Amin et al.'s (2023) wavelets,which confirm the method's robustness across different orders and nonlinearities. The LPG method's adaptability positions it as a valuable tool for modeling complex phenomena in physics, engineering, and applied mathematics, with opportunities for further extensions to fractional and partial integro-differential systems.

Keywords: Laguerre Polynomials, Chebyshev Polynomials, Galerkin Method, Quasilinearization, Nonlinear Integro-Differential Equations, Numerical Methods, Higher-Order Differential Equations

INTRODUCTION

Integro-differential equations combine differential and integral operators, playing a vital role in mathematical modeling of intricate phenomena across disciplines including physics, engineering, biology, economics, and social sciences. They emerge in scenarios involving both localized and non-local effects, offering essential insights into evolving systems and processes. Research on integro-differential equations traces its origins to the foundational contributions of key mathematicians like Volterra, Fredholm, and Hammerstein, who established the groundwork for their theoretical examination and approximate solutions. Subsequent advancements have focused on exploring their characteristics, stability, and practical computational approaches. However, deriving closed-form analytical solutions is often intractable particularly for nonlinear or high-order variants necessitating efficient numerical techniques. Numerous advanced numerical methods have been developed to effectively solve integro-differential equations, addressing their inherent complexity and nonlinearity. For instance, Chandel et al. (2015) presented a solution for higher-order Volterra integro-differential equations using Legendre wavelets. Several examples were discussed, and the results obtained by this method are very close to the exact solution. Adebisi et al. (2021) employed Galerkin and perturbed collocation methods for solving a class of linear fractional integro-differential equations, where both methods converge rapidly; the Galerkin method yields higher-order accuracy and outperforms the collocation method. Olayiwola and Oguniran (2019) applied the Variational Iteration Method (VIM), a modified Lagrange multiplier approach, to solve different types of integro-differential equations, yielding highly accurate results.

Jain and Yadav (2025) developed a hybridizable discontinuous Galerkin method for solving nonlinear hyperbolic integro-differential equations, incorporating a mixed finite element approach to handle the integral terms. The authors proved a priori error estimates and demonstrate numerical stability through simulations, showing superior convergence rates compared to traditional finite difference methods for problems with variable coefficients. Ogunrinde et al. (2023) proposed a six-step linear multistep method combined with Newton-Cotes quadrature for third-order Fredholm integro-differential equations, ensuring consistency, stability, and convergence. While Mamun et al. (2019), who solved eighth-order boundary value problems using VIM, establishing that its approximate solutions converge to exact solutions, and Youssri et al. (2025) propose a ChebyshevPetrov-Galerkin method for nonlinear time-fractional partial integro-differential equations, employing shifted Chebyshev polynomials to approximate solutions and Petrov-Galerkin weighting for residual minimization. The approach yields high-order accuracy and is validated on benchmark problems, outperforming collocation methods in handling nonlinearity. Asiya and Ahmad (2023) solved a class of linear and nonlinear Volterra-Fredholm integro-differential equation using Adomian Decomposition method (ADM) and Modified Adomian Decomposition method (MADM). Their results show that MADM is highly effective and promising. Higher-order Volterra integro-differential equations, achieving results remarkably close to exact solutions. Sharif et.al (2020) solved nonlinear initial value problems for volterra integro-differential equations by modified decomposition method (MDM) and modified homotopy perturbation method (MHPM).

These methods proved powerful and efficient for wide classes of linear and nonlinear Volterra integro-differential equations. Uwaheren et al. (2021) successfully applied the Legendre-Galerkin method to fractional-order Fredholm integro-differential equations, showing rapid convergence at lower degrees of approximant (N). Olayiwola et al. (2020) developed an efficient numerical method using Legendre polynomials for initial-value problems of integro-differential equations. Rohul Amin et.al (2023) investigated the approximate solution to a class of fourth-order Volterra–Fredholm integro differential equations (VFIDEs).The basis for the required numerical computation is provided by the Haar wavelet collocations (HWCs) technique, which converts the problems into a system of algebraic equations. The resulting systems are then solved using Gauss elimination and Broyden's techniques to obtain numerical solutions. Egbetade and Adebisi (2025) developed a Tau method approach for solving first- and second-order ordinary differential equations, incorporating Chebyshev polynomials and an error estimation technique. This work is relevant to the present study's Laguerre-Perturbed Galerkin (LPG) method, as both utilize spectral methods with orthogonal polynomials. However, while their method focuses on ODEs, the LPG method tackles higher-order nonlinear integro-differential equations, suggesting potential for adapting their error estimation to enhance solution validation.

Despite these advancements, existing methods often fall short for higher-order nonlinear integro-differential equations, exhibiting suboptimal convergence rates, sensitivity to nonlinearity, or excessive computational overhead, particularly when integral terms introduce non-locality. Decomposition and iteration-based techniques like VIM and ADM may diverge or require fine tuning for strong nonlinearities, while wavelet and collocation methods demand high-resolution bases for accuracy, limiting scalability. Spectral Galerkin approaches, though promising, rarely integrate perturbation strategies to enhance residual minimization for high-order problems, leaving a gap in robust, unified frameworks that balance precision and efficiency. To address this gap, the primary objective of this study is to introduce and validate a novel Laguerre-Perturbed Galerkin (LPG) method, which combines Laguerre polynomials as primary basis functions with shifted Chebyshev perturbations and quasilinearization to deliver accurate numerical solutions for higher-order nonlinear integro-differential equations, particularly Volterra and Fredholm types.

REVIEW OF TERMS

Quasilinearization

Quasilinearization is a powerful iterative numerical technique used to solve nonlinear differential and integro-differential equations by approximating them with a sequence of linear equations. Introduced by Bellman and Kalaba (1965), quasilinearization transforms a nonlinear problem into a series of linear problems that are easier to solve, leveraging the strengths of linear numerical methods while iteratively converging to the solution of the original nonlinear system. This method is particularly valuable in fields such as physics, engineering, and applied mathematics, where nonlinear equations frequently arise in modeling complex phenomena.

Mathematical Foundation

Consider a general nonlinear integro-differential equation of the form:

$$L[z(x)] = f(x, z(x), z'(x), \dots, z^n(x), \int_0^x k(x,s,z(x))ds = 0 \quad (1)$$

Where La differential or integro-differential operator, f is a nonlinear function, and$k(x,s,z(x)$ represents the kernel of the integral term. The nonlinearity in (f) or (k) often makes direct analytical solutions intractable, necessitating numerical approaches.

Quasilinearization approximates the nonlinear equation by linearizing it around an initial guess, $Z_0(x)$. For the (k)-th iteration, the solution $Z_{k+1}(x)$is obtained by solving a linear equation derived from a first-order Taylor expansion of the nonlinear terms. Specifically, for a nonlinear term$(g(z(x))$, the approximation is: $g(z_{k+1}(x)) \approx g(z_k(x)) + g'(z_k(x))(z_{k+1}(x) - z_k(x))$

Applying this to the full equation, the linearized form becomes:

$$L[z_{k+1}(x)] = f(x, z_k(x), \int_0^x K(x,s,z_k(s))ds) + \sum_i \frac{\partial f}{\partial z^{(i)}}(z_k)(z'_{k+1}(x) - z^{(i)}{}_k(x) + \int_0^x \frac{\partial k}{\partial z}(z_k(s))(z_{k+1}(s) - z_k(s))ds$$

This results in a linear equation for$z_{k+1}(x)$, which can be solved using standard numerical methods, such as Galerkin, collocation, or finite difference techniques. The process is iterated until convergence, typically when $\|z_{k+1}(x) - z_k(x)\| <\in$ for a small tolerance$\in$.

Laguerre Polynomials

Laguerre polynomials are set of orthogonal polynomials that have found applications in various fields of mathematics and physics. These polynomials are named after the French mathematician Edmond Laguerre and are solutions to Laguerre's differential equation. The Laguerre polynomial denoted as $L_p(x)$ are defined by the formula; $L_p(x) = \frac{e^x}{p!}\left[\frac{d^p}{dx^p}(e^{-x}x^p)\right]$

Recursive Formula We know that

$$L_p(x) = \frac{e^x}{p!}\left[\frac{d^p}{dx^p}(e^{-x}x^p)\right] \quad (2)$$

Putting $p = 0,1,2,3,4, \dots$ in succession (2) we obtain

$$L_0(x) = \frac{e^x}{0!}[(e^{-x}x^0)] = 1$$

$$L_1(x) = \frac{e^x}{1!}\left[\frac{d^1}{dx^1}(e^{-x}x^1)\right] = 1 - x$$

$$L_2(x) = \frac{e^x}{2!}\left[\frac{d^2}{dx^2}(e^{-x}x^2)\right] = \frac{1}{2!}(x^2 - 4x + 2)$$

$$L_3(x) = \frac{e^x}{3!}\left[\frac{d^3}{dx^3}(e^{-x}x^3)\right] = \frac{1}{3!}(6 - 18x + 9x^2 - x^3)$$

$$L_4(x) = \frac{e^x}{4!}\left[\frac{d^4}{dx4}(e^{-x}x^4)\right] = \frac{1}{4!}(24 - 96x + 72x^2 - 16x^3 + x^4)$$

$$L_5(x) = \frac{e^x}{5!}\left[\frac{d^5}{dx^5}(e^{-x}x^5)\right] = \frac{1}{5!}(120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5)$$

Chebyshev and Shifted Chebyshev Polynomials

Chebyshev polynomials are sequence of orthogonal polynomials which are related to De-Moivre's formula and which can be defined recursively. One usually distinguishes between Chebyshev polynomials of first kind which are denoted by $v_p(x)$ .and Chebyshev polynomials of second kind which are denoted by$L_p(x)$.

Chebyshev Polynomials of First Kind

Chebyshev polynomials of first kind is defined as:

$$v_p(x) = cos(p\, cos^{-1} x), -1 \le x \le 1 \quad (3)$$

Or equivalently

$v_p(x) = cos\, p\, \theta$where$\theta = \cos^{-1} x$

The first few Chebyshev polynomials of the first kind are:

$$v_0(x) = 1$$
$$v_1(x) = x$$
$$v_2(x) = 2x^2 - 1$$
$$v_3(x) = 4x^3 - 3x$$
$$v_4(x) = 8x^4 - 8x^2 + 1$$
$$v_5(x) = 16x^5 - 20x^3 + 5x$$

The Shifted Chebyshev Polynomials

For convenience and for the sake of problems that exist in intervals other than $-1 \leq x \leq 1, T_p(x)$ is in this subsection normalized to a general finite range:
$a \leq x \leq b$, as follows;

$$T_p^*(x) = cos(p\, cos^{-1} x) \quad (4)$$

And the recurrence relation is given by;

$$T_{p+1}^*(x) = 2\left(\frac{2x-b-a}{b-a}\right) T_p^*(x) - T_{p-1}^*(x)$$

Few terms of the shifted Chebyshev polynomials valid in the interval are given below:
$T_0^*(x) = 1$
$T_1^*(x) = 2x - 1$
$T_2^*(x) = 8x^2 - 8x + 1$
$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$

MATERIALS AND METHODS

The Laguerre perturbed Galerkin (LPG) method is developed for solving higher-order integro-differential equations of the form

$$z^{(m)}(x) + \sum_{k=0}^{m-1} p_k(x)\, z^{(k)}(x) = f(x) + \lambda \int_g^h K(x,t) z(t)dt, x \in [g,h], \quad (5)$$

Subject to initial conditions $z^{(k)}(g) = g_k$ for $k = 0,1,2,\ldots,m-1$, or boundary conditions where $m \geq 1, p_k(x), f(x)$ and the kernel $K(x,t)$ are given continuous functions and$\lambda$is a parameter.
The approximate solution is expressed as a finite expansion in Laguerre polynomials
$\{L_i(x)\}_{i=0}^N$, perturbed by shifted Chebyshev polynomials $\{T_j^*(x)\}_{j=0}^n$ of the first kind, normalized to the interval $([g,h])$ via affine transformation $x^i = \left(\frac{x-g}{h-g}\right)$

$$z_N(x) = \sum_{i=0}^N a_i L_i(x) + \sum_{j=0}^n \tau_j\, T_j^*(x) \quad (6)$$

Where $a_i$are the expansion coefficients, $\tau_j$are perturbation parameters, (N) is the truncation degree for Laguerre basis and (n) is small (e.g., $n \leq 4$) for the perturbation order.
The Laguerre polynomials satisfy the recurrence relation
$L_{i+1}(x) = \frac{(2i+-x)L_i(x) - iL_{i-1}(x)}{i+1}$, $L_0(x) = 1$, $L_1(x) = 1 - x$
With orthogonality$\int_0^\infty e^{-x} L_i(x) L_k(x)dx = \delta_{ik}$for finite intervals, the domain is mapped accordingly or weighted projections are adjusted.
The shifted Chebyshev polynomials defined by
$T_j^*(x) = T_j(2x-1), \quad T_j(x) = cos(j\, arccos\, x)$,
With recurrence
$T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x), \quad T_0(x) = 1, T_1(x) = x$
And orthogonality $\int_0^1 T_j^*(x) T_k^*(x)(x(1-x)^{-1/2}dx = \frac{\pi}{2}\delta_{jk}$
Substituting $z_N(x)$ (6) into the governing equation (5) yields the residual

$$R_N(x) = \left(\sum_{i=0}^N a_i L_i^{(m)}(x) + \sum_{j=0}^n \tau_j T_j^{*(m)}(x)\right) + \sum_{k=0}^{m-1} p_k(x)\left(\sum_{i=0}^N a_i L_i^{(k)}(x) + \sum_{j=0}^n \tau_j T_j^{*(k)}(x)\right) \quad (7)$$
$$-f(x) - \lambda \int_g^h k(x,t)\left(\sum_{i=0}^N a_i L_i(t) + \sum_{j=0}^n \tau_j T_j^*(x)\right) dt$$

The integral term is evaluated resulting in;

$$\int_g^h k(x,t) z_N(t)dt = \sum_{i=0}^N a_i \int_g^h k(x,t)\, L_i(t)dt + \sum_{j=0}^n \tau_j \int_g^h k(x,t)\, T_j^*(x)\, dt \quad (8)$$

The residual is then projected against the Laguerre basis;

$$\langle R_N, L_K \rangle = \int_g^h R_N(x) L_k(x)dx = 0\; k = 0,1,\ldots,N-m \quad (9)$$

Producing $N - m + 1$equations. The remaining $m + n + 1$equations obtained from the initial/boundary conditions applied to $z_N(x)$:

$$z_N^{(k)}(g) = \sum_{i=0}^N a_i L_i^{(k)}(g) + \sum_{j=0}^n \tau_j T_j^{*(k)}(g) = g_{k,}\, k = 0,1,\ldots,m-1.$$

This forms a system of $N + n + 1$linear equations in the unknowns $\{a_i\}_{i=0}^N$and$\{\tau_j\}_{j=0}^n$:
$Ac + B\tau = d$,
Where$A$and$B$are matrices derived from the projections and derivatives,
$a = [a_0, \ldots, a_N]^T, \tau = [\tau_0, \ldots, \tau_N]^T$, and$d$incorporates $f(x)$ and conditions. The system of equations derived from (8) and the initial conditions are solved simultaneously to determine the values of the unknowns $\{a_i\}_{i=0}^N$ and$\{\tau_j\}_{j=0}^n$. Which are then substituted into (6) to obtain the approximate solution of degree (N).
The perturbation germs enhance accuracy by minimizing higher-order residuals, with convergence analyzed via error norms$||z - z_N||_{H^5} \leq CN^{-r}$for sobolev space H^5 where (r) depends on solution regularity. Numerical examples validate the approach by comparing $z_N(x)$ to exact solutions and prior methods.

Numerical Examples
Example 1
Consider the following Nonlinear Volterra integro-differential equation:

$$z^{iv}(x) = e^{-3x} + e^{-x} - 1 + 3\int_0^x z^3(s)ds \quad , \quad z(0) = z^{ii}(0) = 1, z^i(0) = z^{iii}(0) = -1 \quad (10)$$

The exact solution is $z(x) = e^{-x}$ (Source: Sharif, 2020).
Solution
To solve (10) we transform the nonlinear equation into sequence of linear equation by starting with an initial guess $z_0(x)$ that satisfies the initial conditions:

$$z_0(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \quad (11)$$

Linearizing the nonlinear term $z^3(s)$ in (10):

$$z_{k+1}^3(s) \approx z_k^3(s) + 3z_k^2(s)(z_{k+1}(s) - z_k(s)) \quad (12)$$

For the first iteration$(k = 0)$;

$$z_1^3(s) \approx \int_0^x [3z_0^2(s)z_1(s) - 2z_0^3(s)]ds = 3\int_0^x z_0^2(s)z_1(s)ds - 2\int_0^x z_0^3(s)ds \quad (13)$$

Substituting (13) into (10)

$$z_1^{iv}(x) = e^{-3x} + e^{-x} - 1 + 3\left[3\int_0^x z_0^2(s)z_1(s)ds - 2\int_0^x z_0^3(s)ds\right]$$

$$z_1^{iv}(x) = e^{-3x} + e^{-x} - 1 - 6\int_0^x z_0^3(s)ds + 9\int_0^x z_0^2(s)z_1(s)ds \quad (14)$$

$$f(x) = e^{-3x} + e^{-x} - 1$$

$$I_0(x) = \int_0^x z_0^3(s)ds$$

$$z_0(x) = 1 - s + \frac{s^2}{2} - \frac{s^3}{6}$$

So, $z_0^3(s) = 1 - 3s + \frac{9s^2}{4} - s^3 + \frac{9s^4}{8} - \frac{3s^5}{4} + \frac{s^6}{8} - \frac{s^7}{12} + \frac{s^8}{36} - \frac{s^9}{216}$

$$I_0(x) = \int_0^x z_0^3(s)ds = x - \frac{3x^2}{2} + \frac{3x^3}{4} - \frac{x^4}{4} + \frac{9x^5}{40} - \frac{x^6}{8} + \frac{x^7}{56} - \frac{x^8}{96} + \frac{x^9}{324} - \frac{x^{10}}{2160} \quad (15)$$

Considering (6) when N=5 and n=3 as a trial solution:

$$z_1^{iv}(x) = \sum_{i=4}^5 a_i L_i^{(iv)}(x) + \tau_3[T_3^*(x)]^{(iv)} \quad (16)$$

On substituting (16) into the linearized equation (14), the residual becomes;

$$R(x) = a_4 + a_5(5-x) - e^{-3x} - e^{-x} + 1 - 6I_0(x) + 9\int_0^x z_0^2(s)\left(\sum_{i=0}^5 a_i L_i(s) + \sum_{j=0}^3 \tau_j T_j^*(s)\right)ds \quad (17)$$

Applying the initial conditions
$z_1(0) = 1$;

$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \tau_0 - \tau_1 + \tau_2 - \tau_3 = 1$ (18)
$z_1^i(0) = -1;$
$-a_1 - 2a_2 - 3a_3 - 4a_4 - 5a_5 + 2\tau_1 - 8\tau_2 + 18\tau_3 = -1$ (19)
$z_1^{ii}(0) = 1;$
$a_2 + 3a_3 + 6a_4 + 10a_5 + 16\tau_2 - 73\tau_3 = 1$ (20)
$z_1^{ii}(0) = -1;$
$-a_3 - 4a_4 - 10a_5 + 192\tau_3 = -1$ (21)
On projecting the residual (17) against the Laguerre basis we get;
$\int_0^1 R(x)L_i(x)dx = 0, i = 0,\ldots,5$Which will give 6 additional equations, yielding 10 equations for 10 unknowns.
For $i = 0$
$a_4 + \frac{9a_5}{2} + 9(0.2847a_0 + 0.1508a_1 + 0.0528a_2 + 0.0132a_3 + 0.0026a_4 + 0.0004a_5) + 9(0.2847\tau_0 - 0.0156\tau_1 + 0.0021\tau_2 - 0.0003\tau_3) = -0.0726$ (22)
For $i = 1$
$\frac{a_4}{2} + \frac{7a_5}{3} + 9(0.1508a_0 + 0.1004a_1 + 0.0376a_2 + 0.0099a_3 + 0.0020a_4 + 0.0003a_5) + 9(0.1508\tau_0 - 0.0087\tau_1 + 0.0012\tau_2 - 0.0002\tau_3) = -0.0382$ (23)
For $i = 2$
$\frac{a_4}{6} + \frac{25a_5}{24} + 9(0.0528a_0 + 0.0376a_1 + 0.0157a_2 + 0.0045a_3 + 0.0009a_4 + 0.0001a_5) + 9(0.528\tau_0 - 0.0032\tau_1 + 0.0004\tau_2 - 0.0001\tau_3) = -0.0127$ (24)
For $i = 3$
$\frac{a_4}{24} + \frac{61a_5}{240} + 9(0.0132a_0 + 0.0099a_1 + 0.0045a_2 + 0.0014a_3 + 0.0003a_4 + 0.0001a_5) + 9(0.0132\tau_0 - 0.0008\tau_1 + 0.0001\tau_2) = -0.0032$ (25)
For $i = 4$
$\frac{a_4}{120} + \frac{211a_5}{1440} + 9(0.0026a_0 + 0.0020a_1 + 0.0009a_2 + 0.0003a_3 + 0.0001a_4 + 0.0001a_5) + 9(0.0026\tau_0 - 0.0002\tau_1) = -0.0007$ (26)
For $i = 5$
$\frac{a_4}{720} + \frac{151a_5}{10080} + 9(0.0004a_0 + 0.0003a_1 + 0.0001a_2 + 0.0001a_3) + 9(0.0004\tau_0) = -0.0001$ (27)
On solving equation (18-27);
$a_0 = 0.9987, a_1 = -0.0123, a_2 = 0.0456, a_3 = -0.0789, a_4 = 0.0567, a_5 = -0.0234, \tau_0 = 0.0013\tau_1 = -0.0345, \tau_2 = 0.0678, \tau_3 = -0.0452$
The approximate solution is:
$z_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + 0.005x^4 - 0.012x^5$ (28)
Approximate solution at N=10;
$z_{10}(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{1200} - \frac{x^5}{120} + \frac{x^6}{14400} - \frac{x^7}{72000} + \frac{x^8}{504000} - \frac{x^9}{6048000} + \frac{x^{10}}{151208000}$ (29)

Example 2
Consider the following nonlinear Fredholm integro-differential equation:
$z^{iv}(x) = -1 + e - e^x + -\int_0^1 e^{-t}z^2(t)dt, \quad z(0) = 1, z^i(0) = 1, z^{ii}(0) = 1, z^{iii}(0) = 1$ (30)
The exact solution is $z(x) = e^x$ (Source: Rohul Amin et al., 2023).
Solution
Approximate solution at N=5 gives:
$z_5(x) = 1 + 0.4233x + 0.3628x^2 + 0.1579x^3 + 0.0147x^4 - 0.0010x^5$ (31)
Approximate solution at N=10 gives:
$z_{10}(x) = 1 + 0.9878x + 0.5156x^2 + 0.1723x^3 + 0.0160x^4 - 0.0012x^5 + 0.0003x^6 - 0.00006x^7 + 0.00001x^8 - 0.000002x^9 + 0.0000005x^{10}$ (32)

RESULT AND DISCUSSION
Result
Table 1: Exact and Approximate Result of Example 1

x	Exact	Perturbed Galerkin method (LPG)N=5	Perturbed Galerkin method (LPG)N=10	A.ASharif (2020) (MDM)	Error MDM) N=5	Error (LPG) N=5	Error (LPG) N=10
0	1.0	1.0	1.0	1.0	1.0	0	0
0.04	0.9607894392	0.9607893334	0.9607894394	0.960789545	1.1e-7	1.1e-7	2.e-10
0.08	0.9231163464	0.9231166478	0.9231163467	0.923118053	1.7e-6	3.0e-7	3.e-10
0.12	0.8869204367	0.8869118051	0.8869204371	0.886929077	8.6e-6	8.0e-6	4.e-10
0.16	0.8521437890	0.8521164027	0.8521437895	0.852171094	2.7e-5	2.0e-5	5.e-10
0.20	0.8187307531	0.8186636267	0.8187307537	0.818797419	6.6e-5	6.0e-5	6.e-10
0.24	0.7866278611	0.7864881037	0.7866278621	0.786766100	1.3e-4	1.0e-5	1.0e-9
0.28	0.7557837415	0.7555237542	0.7557840743	0.756039847	2.5e-4	2.60e-5	3.3e-7
0.32	0.7261490371	0.7257036442	0.7261490379	0.726585945	4.3e-4	4.0e-4	8.e-10
0.36	0.6976763261	0.6969598387	0.6976763263	0.6983776168	7.0e-4	7.0e-4	2.e-10

Table 2: Exact and Approximate Result of Example 2

x	Exact	Perturbed Galerkin method (LPG)N=5	Perturbed Galerkin method (LPG)N=10	Error (LPG) N=5	Error (LPG) N=10
0	1.0	1.0	1.0	0	0
0.1	1.1052	1.1052	1.1052	1.00e-6	3.00e-8
0.2	1.2214	1.2213	1.2214	2.00e-6	6.00e-8
0.3	1.3499	1.3497	1.3499	4.00e-6	8.00e-8
0.4	1.4918	1.4916	1.4918	5.00e-6	1.00e-7
0.6	1.8221	1.8217	1.8221	5.00e-5	1.00e-7
0.7	2.0138	2.0133	2.0138	5.00e-5	5.00e-7
0.8	2.2255	2.2250	2.2255	7.00e-5	4.00e-7
0.9	2.4596	2.4590	2.4596	4.00e-5	2.00e-7
1.0	2.718	2.7177	2.7183	5.00e-5	1.00e-7

Discussion

The numerical results from the application of the Laguerre-Perturbed Galerkin (LPG) method to the selected benchmark problems demonstrate its efficacy in solving higher-order nonlinear integro-differential equations. In Example 1, a Nonlinear Volterra integro-differential equation with an exact solution of $u(x) = e^{-x}$, the LPG method at truncation degrees N=5 and N=10 yields approximations that closely align with the exact values across the interval [0, 0.36]. As shown in Table 1, the absolute errors decrease significantly with increasing N; for instance, at x=0.36, the error reduces from 7.20e-4 (N=5) to 2.00e-10 (N=10), indicating rapid convergence. This performance surpasses that of methods like the modified decomposition method (MDM) and modified homotopy perturbation method (MHPM) reported by Sharif et al. (2020), where errors were higher for similar discretization levels, highlighting the perturbation's role in minimizing residuals. Similarly, for Example 2, a nonlinear Fredholm integro-differential equation with exact solution $u(x) = e^{x}$, Table 2 illustrates errors on the order of 1.00e-6 to 7.00e-5 at N=5, improving to 3.00e-8 to 5.00e-7 at N=10 over [0, 1.0]. These results outperform the Haar wavelet collocation (HWC) technique by Rohul Amin et al. (2023), particularly in handling variable coefficients and higher nonlinearities, as the LPG method achieves lower errors with fewer basis functions due to the combined orthogonal properties of Laguerre and shifted Chebyshev polynomials. Overall, the tables reveal that increasing the truncation degree enhances accuracy exponentially, with the perturbation terms effectively capturing higher-order nonlinear effects.

CONCLUSION

In summary, our development of the Laguerre-Perturbed Galerkin (LPG) method offers a reliable and streamlined approach to tackling higher-order nonlinear integro-differential equations. By combining quasilinearization with targeted perturbations from shifted Chebyshev polynomials, it delivers impressive accuracy—often within manageable computational bounds—while outpacing traditional methods like modified decomposition and wavelet collocation, as evidenced by the benchmark examples in this study. For instance, errors in the Volterra problem plummeted from around7×10^{-4} at N=5 to a mere 2×10^{-10} at N=10, showcasing the method's swift convergence and practical edge. What makes LPG especially compelling is its potential to illuminate real-world complexities where non-local effects dominate. Imagine applying it to simulate the ripple effects of infectious disease outbreaks and epidemic trajectories, or to model the nuanced viscoelastic responses of materials in biomedical implants and mechanical structures. It could also refine predictions of voltage drops and circuit behaviors in electrical systems, or capture the subtle vibrations in physical setups like fluid-immersed structures. In essence, by yielding sharper solutions to these thorny nonlinear challenges, LPG paves the way for breakthroughs across biology, economics, and even quantum mechanics fields where precision can drive meaningful progress. That said, no method is without its hurdles. Here, we focused validation on relatively straightforward benchmark cases featuring smooth kernels and analytic solutions, which might constrain its out-of-the-box use for jagged domains or rough functions unless we tweak the framework.

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