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Neumann Series-Based Adaptation of Stationary Iterative Methods for Solution of Linear Algebraic Systems of Equations

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ABSTRACT
Solution of linear algebraic systems of equations plays a major role in real life problems. The iteration matrices of both the Jacobi and Gauss-Seidel schemes are split into strictly lower triangular, diagonal and strictly upper triangular matrices. The resulting Jacobi and Gauss-Seidel schemes are then expressed in terms of Neumann series. Taking the initial approximation to be the zero-column vector of the same dimension as the right-hand side constant column vector, a new series is obtained in terms of each of the iteration matrices. A recursive scheme is developed through the application of Neumann series. This series was then approximated to give the new scheme which solves linear systems with various coefficient matrices. The new iterative schemes obtained were verified with examples.

Keywords: Linear Algebraic Systems, Iteration Matrix, Initial Approximation Jacobi Iteration, Gauss-Seidel Iteration, Neumann Series

INTRODUCTION

Linear algebraic systems of equations are equations of the form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n-1,n}x_n + a_{nn}x_n = b_n \end{array} \quad (1)$$

which can be expressed form

$$Ax = b \quad (2)$$

where A is called the coefficient matrix and b, the righthand side of the system (Brownson, 1989; Meyer, 2000; Sa'ad, 2003). A given linear system of equations may have a unique solution, no solution at all or infinitely many solutions (Young and Mohlenkamp, 2021). Broadly, (2) can be solved by either direct methods such as Gaussian elimination, Cholesky decomposition or matrix inversion methods (Brownson, et al 1989; Lipschutz and Lipson, 2009). On the other hand, stationary iterative methods such as Jacobi, Gauss-Seidel, the SOR methods and their variants can be employed as well for the solution of liner systems of algebraic equations (Davis and Thomson, 2000), taking into consideration the properties of the coefficient matrix of the linear system (Bamigbola and Ibrahim, 2014; Sa'ad, 2003). The successive approximation property of the Neumann series makes it attractive in application to convergent problems (Matebie, 2016).

MATERIALS AND METHODS

Stationary iterative methods for solving the linear system (1) involve the splitting of the coefficient matrix A into matrices P and Q which are of the same order as A i.e. $A = P - Q$ and where P is invertible (Meyer, 2000, Boyd *et al*, 2018). Thus, the linear system is transformed into

$$Px = Qx + b \quad (3)$$

This implies that P must be chosen in such a way that that it is invertible. Thus, we obtain the iterative scheme

$$x^{(k+1)} = P^{-1}Qx^{(k)} + P^{-1}b \quad (4)$$

which simplifies to

$$x^{(k+1)} = Tx^{(k)} + c \quad (5)$$

The Jacobi, forward and backward Gauss-Seidel methods are respectively $x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$ (6)

$$x^{(k+1)} = -(L+U)^{-1}Ux^{(k)} + (L+U)^{-1}b \quad (7)$$

$$x^{(k+1)} = -(D+U)^{-1}Lx^{(k)} + (D+U)^{-1}b \quad (8)$$

These schemes can be generalized as

$$x^{(k+1)} = T^{(k)}x + c$$

Taking $k = 0,1,2,\cdots,n$ we have

$$x^{(1)} = Tx^{(0)} + c$$

$$x^{(2)} = Tx^{(1)} + c = T\left(Tx^{(0)} + c\right) + c = T^2x^{(0)} + Tc + c$$

$$x^{(3)} = Tx^{(2)} + c = T\left(T^2x^{(0)} + Tc + c\right) + c = T^3x^{(0)} + T^2c + Tc + c$$

$$x^{(4)} = Tx^{(3)} + c = T\left(T^3x^{(0)} + T^2c + Tc + c\right) + c = T^4x^{(0)} + T^3c + T^2c + Tc + c$$

$$x^{(n)} = T^nx^{(0)} + T^{n-1}c + T^{n-2}c + T^{n-3}c + \cdots Tc + c$$

which simplifies as

$$x^{(n)} = T^nx^{(0)} + \sum_{k=0}^{n-1} T^kc$$

Choosing the initial approximation $x^{(0)} = 0$ implies that $T^nx^{(0)} = 0$, hence

$$x^{(n)} = (I + T + T^2 + T^3 + \cdots T^{n-4} + T^{n-3} + T^{k-2} + T^{n-1})c$$

$$x^{(n)} = \sum_{k=0}^{n-1} T^kc \quad (9)$$

Employing (10) in (6) to (8), we have the following schemes for the Jacobi and Gauss-Seidel iterations respectively:

$$x^{(n)} = \sum_{k=0}^{n-1}\left(-D^{-1}(L+U)\right)^kD^{-1}b \quad (10)$$

$$x^{(n)} = \sum_{k=0}^{n-1}(-(L+D)^{-1}U)^k(L+D)^{-1}b \quad (11)$$

RESULTS AND DISCUSSION

The examples used are hereby given in the table below. They are all diagonally dominant of different dimensions.

Table 1: Table of Numerical Examples

S/N	*A*	*b*	Dimension	Properties
1	$\begin{pmatrix} 5 & 2 & 1 \\ 2 & 9 & 4 \\ 1 & 4 & 9 \end{pmatrix}$	$\begin{pmatrix} 11 \\ 7 \\ 25 \end{pmatrix}$	(3×3)	Non-symmetric dense diagonally dominant
2	$\begin{pmatrix} 9 & 1 & 2 & 4 \\ 2 & 7 & -2 & 1 \\ 1 & 1 & 8 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$	$\begin{pmatrix} 34 \\ 12 \\ 13 \\ 11 \end{pmatrix}$	(4×4)	Non-symmetric dense diagonally dominant
3	$\begin{pmatrix} 4 & 3 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 8 \\ 8 \\ 8 \\ 5 \end{pmatrix}$	(5×5)	Non-symmetric sparse diagonally dominant
4	$\begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 7 \\ 7 \\ 7 \\ 6 \end{pmatrix}$	(5×5)	Symmetric sparse diagonally dominant

The results obtained for the Jacobi and Gauss-Seidel with their Neumann-based schemes. The Gauss-Seidel method converged faster than the Jacobi method in all the examples. The Neumann-based Gauss-Seidel method converged faster than its Jacobi method. While the standard Gauss-Seidel and its Neumann-based derivation had the same iteration counts, the Neuman-based method converged faster in examples 2 to 4. Overall, the Neumann-based schemes converged faster than the standard iterative schemes. Just as the standard forms, the Neumann-based Gauss-Seidel method converged faster than its Jacobi counterpart.

Table 2: Table of Results

S/N	Jacobi		Seidel		Neumann-Based Jacobi		Neumann-Based Gauss-Seidel	
	Solution	Iterations	Solution	Iterations	Solution	Iterations	Solution	Iterations
1	$\begin{pmatrix} 2.0007 \\ -0.9992 \\ 3.0006 \end{pmatrix}$	15	$\begin{pmatrix} 1.9998 \\ -0.9999 \\ 3.0000 \end{pmatrix}$	7	$\begin{pmatrix} 1.9998 \\ -1.0074 \\ 2.9912 \end{pmatrix}$	10	$\begin{pmatrix} 1.9998 \\ -0.9999 \\ 3.0000 \end{pmatrix}$	7
2	$\begin{pmatrix} 2.9997 \\ 0.9998 \\ 0.9998 \\ 0.9998 \end{pmatrix}$	10	$\begin{pmatrix} 2.9999 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	4	$\begin{pmatrix} 2.9999 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	10	$\begin{pmatrix} 3.0000 \\ 1.0003 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	2
3	$\begin{pmatrix} 1.0005 \\ 1.0006 \\ 1.0003 \\ 1.0002 \\ 1.0001 \end{pmatrix}$	29	$\begin{pmatrix} 0.9983 \\ 1.0013 \\ 0.9994 \\ 1.0002 \\ 0.9999 \end{pmatrix}$	13	$\begin{pmatrix} 0.9157 \\ 0.9367 \\ 0.9437 \\ 0.9789 \\ 0.9906 \end{pmatrix}$	10	$\begin{pmatrix} 0.9970 \\ 1.0023 \\ 0.9989 \\ 1.0004 \\ 0.9999 \end{pmatrix}$	10
4	$\begin{pmatrix} 0.9999 \\ 0.9998 \\ 0.9997 \\ 0.9998 \\ 0.9999 \end{pmatrix}$	8	$\begin{pmatrix} 0.9998 \\ 1.0001 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	5	$\begin{pmatrix} 1.0000 \\ 1.0001 \\ 1.0001 \\ 1.0001 \\ 1.0000 \end{pmatrix}$	7	$\begin{pmatrix} 0.9998 \\ 1.0001 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	3

CONCLUSION

The results obtained shows that, the Neumann-based schemes outperformed the standard Jacobi and Gauss-Seidel iterative methods for diagonally dominant linear systems with better convergence rates. It's noteworthy that the Neumann-based Gauss-Seidel method had the best convergence rate, especially when the magnitude of the diagonal entry is much bigger than the sum of the absolute values of the off-diagonal entries in a row or column as indicated in examples 1, 2 and 4.

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