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AN EXTENSION OF SOME COMMON FIXED POINT THEOREMS IN COMPLEX-VALUED METRIC SPACES

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ABSTRACT

As a generalization of the ordinary metric spaces, complex-valued metric spaces have significantly influenced research in fixed point theory. The idea intends to develop rational expressions that are insignificant within the context of cone metric spaces since the latter banks on the underlying Banach space which is not a division ring. This paper demonstrates a common fixed point theorem applicable to a pair of mappings that fulfill a rational type contractive condition in the setting of complex-valued metric spaces. The mappings examined in this study are expected to comply with particular metric inequalities, which broadens and includes various results from other scholars that were established for mappings in complex-valued metric spaces.

Keywords: Common fixed point, Complex-valued metric space, Mappings, Rational type contraction

INTRODUCTION

The primary objective of fixed point theory is to determine the existence of one or more fixed points within a mapping. One of the key and foundational results in this discipline is the Banach contraction principle, which confirms that a unique fixed point is present under particular circumstances. A mapping $H: X \to X$, where (X, d) is a metric space, is said to be a contraction mapping if for all $u, v \in X, \lambda \in (0,1)$,

$d(Hu, Hv) \leq \lambda\, d(u, v)$. (1)

For decades now, different authors have improved, changed, and generalized the contraction (1) by focusing on the broadening of distinct contractive conditions in metric spaces and its generalizations.

The complex-valued metric space, a newly introduced concept by Azam et al. (2011), is recognized for its generality compared to established metric spaces. This notion presented by the authors provides a notable improvement to the Banach fixed point theorem, stressing the existence of fixed points in the context of complex-valued metric spaces.

Along this direction, Rouzkard and Imdad (2012) expanded and improved upon the fixed point theorems, which possess a greater generality than the conclusions drawn by Azam et al. (2011). In response to this development, there has been a considerable increase in research pertaining to fixed point theorems that are associated with rational inequalities in the established concept. For further reading, one can refer (Ahmad et. al., 2013; Ahmad et. al., 2014; Dubey et. al., 2015; Kutbi et. al., 2013; Nashine & Fisher, 2015; Nigam & Shukla, 2015; Saluja, 2017; Shinde & Pathak, 2024; Singh et. al, 2016; Sintunavarat & Kumam, 2012).

On the basis of this premise, this paper furthers the work started by Azam et al. (2011) and that of Rouzkard and Imdad (2012) by validating a common fixed point theorem for two mappings that meet a more extensive contraction condition related to rational inequalities specifically using the contraction by Reich (1971) in complex-valued metric spaces.

MATERIALS AND METHODS

Preliminaries

This segment presents the basic ideas and notation structures that will be utilized in later sections.

The following is due to Azam et.al (2011).

Let $\mathbb{C}$ be the set of complex numbers and $w_1, w_2 \in \mathbb{C}$. For $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$, define a partial order $\preccurlyeq$ on $\mathbb{C}$by

$w_1 \preccurlyeq w_2 \Leftrightarrow u_1 \leq u_2, v_1 \leq v_2$

We can therefore conclude that $w_1 \preccurlyeq w_2$if any of (i)-(iv) holds:

i. $u_1 = u_2,\ v_1 < v_2$;
ii. $u_1 < u_2,\ v_1 = v_2$;
iii. $u_1 < u_2,\ v_1 < v_2$;
iv. $u_1 = u_2,\ v_1 = v_2$.

Specifically, we will write $w_1 \precnsim w_2$if $w_1 \neq w_2$and one of (i), (ii) and (iii) is satisfied and $w_1 \prec w_2$ if only (iii) is satisfied.

We also notice that

$0 \preccurlyeq w_1 \precnsim w_2 \Longrightarrow |w_1| < |w_2|andw_1 \preccurlyeq w_2, w_2 \prec w_3 \Longrightarrow w_1 \prec w_3$

Definition 1 (Azam et. al. 2011). Let X_c be a nonempty set. Suppose that $d_c: X_c \times X_c \to \mathbb{C}$ satisfies:

$0 \preccurlyeq d_c(u, v)$ and$d_c(u, v) = 0$ if and only if $u = v$;

$d_c(u, v) = d_c(u, v)$;

$d_c(u, v) \preccurlyeq d_c(u, z) + d_c(z, v)$, for all $u, v, z \in X_c$.

Then d_c is called a complex valued metric on X_c and the pair (X_c, d_c) is called a complex-valued metric space.

Definition 2 (Azam et. al., 2011). Let (X_c, d_c) be a complex-valued metric space.

(i) A sequence $\{u_n\}$ inX_c is convergent if for $c \in \mathbb{C}$, $0 \preccurlyeq c$, there exist $n_0 \in \mathbb{N}$ such that for $n \geq n_0, d_c(u_n, u) \preccurlyeq c$. We write $u_n \to u$ as $n \to \infty$

(ii) A sequence $\{u_n\}$ inX_c is said to be Cauchy if for $c \in \mathbb{C}$, $0 \preccurlyeq c$, there exist $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0, d_c(u_n, u_m) \preccurlyeq c$.

(iii) $(X_c, d_c)$is said to be complete if every sequence in (X_c, d_c) converges.

The existence of a common fixed point for a pair of mappings that adhere to a contractive condition in a complete complex-valued metric space was established by Azam et al. (2011) as follows.

Theorem 1 (Azam et. al., 2011). Let (X_c, d_c) be a complete complex-valued metric space and let $G, H: X_c \to X_c$satisfy:

$$d_c(Gu, Hv) \preccurlyeq \lambda\, d_c(u, v) + \frac{\mu\, d_c(u, Gu) d_c(v, Hv)}{1 + d_c(u, v)}$$

for all $u, v \in X_c$, where $\lambda,\ \mu$ are nonnegative reals, with $\lambda + \mu < 1$. Then G, H have a unique common fixed point.

The work of Rouzkard and Imdad(2012) built upon Theorem 2.1 by applying additional contractive condition to demonstrate some common fixed point theorems in the framework of complex valued metric spaces as follows.

Theorem 2 (Rouzkard and Imdad, 2012). Let (X_c, d_c) be a complete complex-valued metric space and let $G, H: X_c \to X_c$ satisfy:

$$d_c(Gu, Hv) \preccurlyeq \lambda\, d_c(u,v) + \frac{\mu\, d_c(u,Gu)d_c(v,Hv) + \gamma d_c(v,Gu)d_c(u,Hv)}{1 + d_c(u,v)}$$

for all $u, v \in X_c$, where λ, μ, γ are nonnegative reals, with $\lambda + \mu + \gamma < 1$. Then G, H have a unique common fixed point.

RESULTS AND DISCUSSION

This section details the establishment of a unified fixed point for two mappings that conform to a broader contraction of Reich type, which includes rational inequality.

Theorem 3 Let (X_c, d_c) be a complete complex-valued metric space and let $G, H: X_c \to X_c$ satisfy:

$$d_c(Gu, Hv) \preccurlyeq \mu_1 d_c(u,v) + \mu_2 d_c(u,Gu) + \mu_3 d_c(v,Hv) + \frac{\mu_4 d_c(u,Gu)d_c(v,Hv)}{1 + d_c(u,v)} + \frac{\mu_5 d_c(v,Gu)d_c(u,Hv)}{1 + d_c(u,v)}$$

for all $u, v \in X_c$, where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are nonnegative reals, with $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 < 1$. Then G, H have a unique common fixed point.

Proof

Let u_0 be arbitrary point in X_c and define $u_n = Gu_{n+1}$, if n is odd and $u_{n+2} = Hu_{n+1}$, if n is even.

Then,

$$\begin{aligned} d_c(u_1,u_2) &= d_c(Gu_0, Hu_1) \\ &\preccurlyeq \mu_1 d_c(u_0,u_1) + \mu_2 d_c(u_0,Gu_0) + \mu_3 d\_c(u_1,Hu_1) + \mu_4 \frac{d_c(u_0,Gu_0)d_c(u_1,Hu_1)}{1 + d_c(u_0,u_1)} + \mu_5 \frac{d_c(u_1,Gu_0)d_c(u_0,Hu_1)}{1 + d_c(u_0,u_1)} \\ &= \mu_1 d_c(u_0,u_1) + \mu_2 d_c(u_0,u_1) + \mu_3 d_c(u_1,u_2) + \mu_4 \frac{d_c(u_0,u_1)d_c(u_1 u_2)}{1 + d_c(u_0,u_1)} + \mu_5 \frac{d_c(u_1,u_1)d_c(u_0,u_2)}{1 + d_c(u_0,u_1)} \\ &\preccurlyeq \mu_1 d_c(u_0,u_1) + \mu_2 d_c(u_0,u_1) + \mu_3 d_c(u_1,u_2) + \mu_4 d_c(u_1,u_2) \\ &\preccurlyeq \frac{\mu_1 + \mu_2}{1 - \mu_3 - \mu_4} d_c(u_0,u_1). \end{aligned}$$

Again,

$$\begin{aligned} d_c(u_2,u_3) &= d_c(Hu_1, Gu_2) \\ &\preccurlyeq \mu_1 d_c(u_2,u_1) + \mu_2 d_c(u_2,Gu_2) + \mu_3 d\_c(u_1,Hu_1) + \mu_4 \frac{d_c(u_2,Gu_2)d_c(u_1,Hu_1)}{1 + d_c(u_1,u_2)} + \mu_5 \frac{d_c(u_1,Gu_2)d_c(u_2,Hu_1)}{1 + d_c(u_1,u_2)} \\ &= \mu_1 d_c(u_1,u_2) + \mu_2 d_c(u_2,u_3) + \mu_3 d_c(u_1,u_2) + \mu_4 \frac{d_c(u_2,u_3)d_c(u_1 u_2)}{1 + d_c(u_1,u_2)} + \mu_5 \frac{d_c(u_1,u_3)d_c(u_2,u_2)}{1 + d_c(u_1,u_2)} \\ &\preccurlyeq \mu_1 d_c(u_1,u_2) + \mu_2 d_c(u_2,u_3) + \mu_3 d_c(u_1,u_2) + \mu_4 d_c(u_2,u_3) \\ &\preccurlyeq \frac{\mu_1 + \mu_3}{1 - \mu_2 - \mu_4} d_c(u_1,u_2). \end{aligned}$$

By induction, letting $\rho = max\left\{\frac{\mu_1+\mu_2}{1-\mu_3-\mu_4}, \frac{\mu_1+\mu_3}{1-\mu_2-\mu_4}\right\}$, we get

$$d_c(u_n,u_{n+1}) \preccurlyeq \rho\, d_c(u_{n-1},u_n) \preccurlyeq \rho^2 d_c(u_{n-2},u_{n-1}) \preccurlyeq \cdots \preccurlyeq \rho^n d_c(u_0,u_1),$$ for all $n \geq 1$.

Now for $m > n$,

$$\begin{aligned} d_c(u_n,u_m) &\preccurlyeq d_c(u_n,u_{n+1}) + d_c(u_{n+1},u_{n+2}) + \cdots + d_c(u_{m-1},u_m) \\ &\preccurlyeq \rho^n d_c(u_0,u_1) + \rho^{n+1} d_c(u_0,u_1) + \cdots + \rho^{m-1} d_c(u_0,u_1) \\ &\preccurlyeq [1 + \rho + \rho^2 + \cdots + \rho^{m-n-1}]\rho^n d\_c(u_0,u_1) \\ &\preccurlyeq \left[\frac{\rho^n}{1-\rho}\right] d_c(u_0,u_1). \end{aligned}$$

$$|d_c(u_n,u_m)| \leq \left[\frac{\rho^n}{1-\rho}\right] |\, d_c(u_0,u_1)|.$$

As $n \to \infty$, $|d_c(u_n,u_m)| \to 0$, which shows that $\{u_n\}$ is a Cauchy sequence.

Since X_c is complete, there exist $u^* \in X_c$ such that $u_n \to u^*$. It follows that $d_c(u^*, Gu^*) = 0$ otherwise $d_c(u^*, Gu^*) > 0$ and we have

$$\begin{aligned} d_c(u^*, Gu^*) &\preccurlyeq d_c(u^*, u_{n+2}) + d_c(u_{n+2}, Gu^*) \\ &\preccurlyeq d_c(u^*, u_{n+2}) + d_c(Hu_{n+1}, Gu^*) \\ &\preccurlyeq d_c(u^*, u_{n+2}) + \mu_1 d_c(u^*, u_{n+1}) + \mu_2 d_c(u^*, Gu^*) + \mu_3 d_c(u_{n+1}, Hu_{n+1}) \\ &\quad + \mu_4 \frac{d_c(u^*, Gu^*)d_c(u_{n+1}, Hu_{n+1})}{1 + d_c(u^*, u_{n+1})} + \mu_5 \frac{d_c(u_{n+1}, Gu^*)d_c(u^*, Hu_{n+1})}{1 + d_c(u^*, u_{n+1})} \\ &\preccurlyeq d_c(u^*, u_{n+2}) + \mu_1 d_c(u^*, u_{n+1}) + \mu_2 d_c(u^*, Gu^*) + \mu_3 d_c(u_{n+1}, u_{n+2}) \\ &\quad + \mu_4 \frac{d_c(u^*, Gu^*)d_c(u_{n+1}, u_{n+2})}{1 + d_c(u^*, u_{n+1})} + \mu_5 \frac{d_c(u_{n+1}, Gu^*)d_c(u^*, u_{n+2})}{1 + d_c(u^*, u_{n+1})} \end{aligned}$$

As $n \to \infty$, in the above inequality, we get

$|d_c(u^*, Gu^*)| \leq \mu_2 |d_c(u^*, Gu^*)| < |d_c(u^*, Gu^*)|$,

which is a contradiction since $\mu_2 < 1$. Hence $u^* = Gu^*$. Similarly, $u^* = Hu^*$.\\

In what follows, we show that G and H have a unique common fixed point.

Suppose that $v^* \neq u^*$ is another fixed point of $G$and H. Then,

$$d_c(u^*, v^*) \preccurlyeq \mu_1 d_c(u^*, v^*) + \mu_2 d_c(u^*, Gu^*) + \mu_3 d_c(v^*, Hv^*) + \mu_4 \frac{d_c(u^*, Gu^*)d_c(v^*, Hv^*)}{1 + d_c(u^*, v^*)} + \mu_5 \frac{d_c(v^*, Gu^*)d_c(u^*, Hv^*)}{1 + d_c(u^*, v^*)},$$

Which yields,

$|d_c(u^*, v^*)| < \mu_1 |d_c(u^*, v^*)|$.

Since $\mu_1 < 1$, we get a contradiction and $d_c(u^*, v^*) = 0$ implies $u^* = v^*$ since $d_c(u^*, v^*) = 0$ if and only if $u^* = v^*$.

Remark 1 It is interesting to note that from Theorem 3 one can obtain Theorem 2 if we set $\mu_2 = \mu_3 = \mu_5 = 0$ and Theorem 2 if we put$\mu_2 = \mu_3 = 0$.

Corollary 1 Let (X_c, d_c) be a complete complex-valued metric space. Suppose that $H: X_c \to X_c$ satisfy

$$d_c(Hu, Hv) \preccurlyeq \mu_1 d_c(u,v) + \mu_2 d_c(u,Hu) + \mu_3 d_c(v,Hv) + \frac{\mu_4 d_c(u,Hu)d_c(v,Hv)}{1 + d_c(u,v)} + \frac{\mu_5 d_c(v,Hu)d_c(u,Hv)}{1 + d_c(u,v)}$$

for all $u, v \in X_c$, where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are nonnegative reals, with $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 < 1$. Then Hhas a unique fixed point.

Proof

The proof follows from Theorem 3 by setting $G = H$.

Remark 2 By equating $\mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$ and $\mu_4 = \mu_5 = 0$ in Corollary 1, we obtain the Banach fixed point theorem and Reich fixed point theorem respectively in complex-valued metric spaces.

CONCLUSION

The idea put forth by Azam et al. (2011) and Rouzkard and Imdad (2012) offers a substantial advancement to the Banach fixed point theorem, highlighting the existence of fixed points in the realm of complex-valued metric spaces.

In a similar way, this research significantly enhanced Reich fixed point theorem, emphasizing the presence of common fixed point within complex-valued metric spaces. It was clearly shown that the main results of Theorem 1 and Theorem 2 can be obtained from Theorem 3. The findings presented in this paper can be explored and enhanced via different rational contractions, and other generalization of metric spaces.

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