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NUMERICAL SOLUTION OF 2D PARTIAL VOLTERRA INTEGRO DIFFERENTIAL EQUATIONS USING POLYNOMIAL COLLOCATION WITH MATRIX FORMULATION

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ABSTRACT

Partial Volterra integro-differential equations are equations that mix partial derivatives with Volterra-type integral terms, representing process where the current state depends on both local changes and the accumulated history. This study presents a numerical method for solving two-dimensional Partial Volterra Integro Differential Equations (PVIDEs) using a polynomial collocation with matrix formulation. The original integro-differential equation is first reformulated into a continuous time-integrated form through the Fundamental Theorem of Calculus (FTC). This reformulated equation is then discretized on a hybrid space-time collocation grid. A polynomial collocation scheme is constructed using standard basis functions over the grid points to transform the problem into a solvable system of algebraic equations. The method incorporates consistent numerical quadrature for time-integration of the nonlinear kernel ensuring computational efficiency through matrix formulation. Theoretical analysis demonstrates the method's consistency, stability, and convergence using Lax-Richtmyer equivalence theorem and discrete Grönwall inequality. Numerical examples including both linear and nonlinear 2D PVIDEs implemented in MATLAB confirm the validity and accuracy of the method. The approach gives a close form solution, which show its consistency, stability and accuracy. This approach offers a robust and efficient solution of 2D PVIDEs, extending the applicability of polynomial collocation methods to integro differential equations.

Keywords: Consistency, Stability, Convergence, Numerical Solution, Partial Volterra Integro Differential Equations, Polynomial Collocation Method

INTRODUCTION

Mathematical models that describe the physical, biological, and engineering phenomena often result in functional equations such as ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, and integro-differential equations (IDEs) (Noori & Taghizadeh, 2020; Adamu, Aduroja & Kefas, 2023). Volterra introduced IDEs in 1908 in the study of population dynamics where hereditary effects are significant (Volterra, 1982). The general form of an IDE is given by:

$$u^{(n)}(x) = f(x) + \lambda \int_a^{q(x)} k(x,t,u(t))dt \qquad (1)$$

where $f(x)$ is a continuous function, λ is a constant, and $k(x,t)$ is the kernel.

These equations appear in numerous applications including fluid dynamics, viscoelasticity, thermal conduction, and biological systems (Rahidinia & Tahmasebi, 2012; Osilagun *et al.*, 2023; Mohamed & Majid, 2016). PDEs, in particular, describe processes involving several independent variables and are fundamental in mathematical physics (Ivrii, 2017). When memory effects are present, integral terms are incorporated, resulting in PVIDEs.

In this work, we focus on two-dimensional PVIDEs of the form:

$$\frac{\partial u(x,t)}{\partial t} = g(x,t) + \int_0^x \int_0^t k(t,s,x,y)u(s,y)dyds, \quad x,t \in [0,1] \qquad (2)$$

with initial condition:

$$u(x,0) = u_0(x) \qquad (3)$$

where $k(t,s,x,y)$ is a given kernel and $g(x,t)$ is the source term.

PVIDEs such as equation (2) are used in modeling diffusion processes with memory, reaction diffusion systems, and epidemic modeling (Zhao & Zhao, 2021; Pachpatte, 2011).

While one-dimensional PVIDEs have been tackled with various numerical methods, two-dimensional cases have seen fewer investigations. Prior works include Euler backward schemes (Soliman et al., 2012), Laplace transform methods (Zhao & Zhao, 2021), and finite difference methods (Sameeh & Elsaid, 2016). Rostami and Maleknejad (2022a, 2022b) introduced operational matrix-based methods utilizing two-dimensional hybrid Taylor polynomials and block-pulse functions, demonstrating high accuracy in solving two-dimensional nonlinear mixed Volterra Fredholm partial integro differential equations with initial conditions.

Recent efforts have continued to enhance both computational efficiency and solution accuracy for various classes of integro-differential equations. Osilagun et al. (2023) proposed a polynomial collocation method for the initial value problem of mixed integro-differential equations. Adesanya et al. (2024) developed an approximate solution approach for high-order linear Fredholm integro-differential difference equations with variable coefficients, employing the Legendre collocation method. In a related contribution, Otaide and Oluwayemi (2024) addressed linear Volterra integro-differential equations using a combination of fourth-kind Chebyshev polynomials and the variational iteration algorithm integrated with collocation techniques.

Furthermore, Eashel *et al.* (2025) conducted a convergence analysis of a multi-step collocation method for first-order Volterra integro-differential equations with non-vanishing delay. Tedjani *et al.* (2025) adopted an operational approach for solving one- and two-dimensional high-order multi-pantograph Volterra integro-differential equations. Similarly, Mahdy *et al.* (2023) presented computational strategies for handling higher-order (1+1) - dimensional mixed difference integro differential equations with variable coefficients.

The polynomial collocation method approximates the solution $u(x,t)$ by a combination of basis functions $\varphi_i(x)$ and $\gamma_j(t)$ as:

$$u(x,t) \approx \sum_{i=0}^{n} \sum_{j=0}^{m} C_{ij}\varphi_i(x)\gamma_j(t) \quad (4)$$

The equation is enforced at selected collocation points resulting in a solvable algebraic system (Chen, 2017; Adamu, Aduroja & Kefas, 2023). This study extends the polynomial collocation method to solve 2D PVIDEs, offering high accuracy, stability, and applicability to complex kernel structures.

The use of a polynomial matrix based collocation method aligns with modern approaches that emphasize both computational simplicity and robustness. This method avoids the need for direct multiple integration by transforming the problem into an algebraic system. Our approach builds on Brunner's (2017) recommendation of spectral methods for smooth solutions and leverages Zhao and Zhao's (2021) observation that structured grid-based collocation offers a feasible trade-off between accuracy and speed in two-dimensional memory-dependent systems.

Preliminaries

Definition (Lipschitz Continuity)

A function $f : D \subset R^n \to R$ is said to satisfy a Lipschitz condition (or is Lipschitz continuous) on a domain D if there exists a constant $L > 0$ such that for all $x, y \in D$

$|f(x) - f(y)| \le L|x - y|$.

The constant L is called the Lipschitz constant (Burden & Faires, 2011; Evans, 2010).

Theorem (Lax-Richtmyer Equivalence Theorem)

For a well-posed initial value problem and a consistent difference scheme approximating it, stability is a necessary and sufficient condition for convergence (Lax & Richtmyer, 1956).

Theorem (Discrete Gronwall Inequality)

Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequence of non-negative real numbers. If

$$a_n \le b_n + \sum_{m=0}^{n-1} b_m c_m \, exp\left(\sum_{k=m+1}^{n-1} c_k\right), \text{for all } n \ge 0.$$

In particular, if $b_n = B \ge 0$ and $c_n \ge 0$ are constant, then

$a_n \le B\, exp(c_n)$, for all $n \ge 0$.

See (Hundsdorfer & Verwer, 2003; Thomas, 1995).

MATERIALS AND METHODS

To integrate partial derivative of the definite integral of equation (2) with respect to t from 0 to t, the principle of the Fundamental Theorem of Calculus is (Anton, Bivens & Davis, 2015) applied to the left hand side of equation (2) to have

$$\int_0^t \frac{\partial u(x,\xi)}{\partial \xi}\partial\xi = u(x,t) - u(x,0)$$

Now the complete integration of equation (2) is

$$u(x,t) = u(x,0) + \int_0^t g(x,\xi)d\xi + \int_0^t \left\{\int_0^x \int_0^\xi k(\xi,s,x,y)f(u(s,y))dyds\right\}d\xi \quad (5)$$

Since $s \in [0,\xi]$, $\xi \in [s,t]$, we can rewrite the triple integral as

$$\int_0^t \int_0^x \int_0^\xi k(\xi,s,x,y)f(u(s,y))dydsd\xi = \int_0^x \int_0^t \int_s^t k(\xi,s,x,y)d\xi.f(u(s,y))dsdy$$

Now equation (5) becomes

$$u(x,t) = u(x,0) + \int_0^t g(x,\xi)d\xi + \int_0^x \int_0^t \left\{\int_s^t k(\xi,s,x,y)d\xi\right\}.f(u(s,y))dsdy \quad (6)$$

which is the continuous time-integrated form of equation (2).

Now constructing a discretization method using a uniform hybrid grid of collocation points in both space and time, consider (6). Assume the spatial domain is $x \in [0,X]$, and time is $t \in [0,T]$.

Let $x_i = i.\Delta x$ where $\Delta x = \frac{X}{N_x - 1}$, $i = 0,1,2,\ldots,N_x - 1$, $t_n = n.\Delta t$ where $\Delta t = \frac{T}{N_t - 1}$, $n = 0,1,2,\ldots,N_t - 1$.

Let the collocation points be the Cartesian product of a subset of space-time grid points, i.e. choose N_c collocation point (x_i,t_n) where $i,n \in \{0,1,2,\ldots,N_x - 1\} \times \{0,1,2,\ldots,N_t - 1\}$. I_c can be defined based on desired resolution or application.

Consider these Discretized variables: $u_{i,n} \approx u(x_i,t_n)$ at each collocation point, $g_{i,n} = g(x_i,t_n)$, $f_{j,m} = f(u_{j,m})$, $K_{n,m,i,j} \approx \int_{s_m}^{t_n} k(\xi,s_m,x_i,y_j)d\xi$ - Computed numerically.

Therefore, the Discrete Time-Integration Equation is

$$u_{i,n} = u_{i,0} + \Delta t\sum_{m=0}^{n} g_{i,m} + \Delta x\Delta t\sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j}.f(u_{j,m}) \quad (7)$$

Now the matrix form at the collocation points is

$$\mathrm{u}_{i,n} = \mathrm{u}_{i,0} + \Delta t\sum_{m=0}^{n} \mathrm{g}_{i,m} + \Delta x\Delta t\sum_{j=0}^{i} \sum_{m=0}^{n} \mathrm{K}_{n,m,i,j}.f(\mathrm{u}_{j,m}) \quad (8)$$

where $\mathrm{u}_{i,n}$ is the solution at collocation points, $\mathrm{u}_{i,0}$ is the initial condition, $\mathrm{g}_{i,m}$ is the source term, and $\mathrm{K}_{n,m,i,j}$ is the kernel weights.

The variable x represents the spatial collocation points: This generates N_x points evenly distributed between 0 and 1. These points serve as the locations where the solution $u(x,t)$ is evaluated. The variable t represents the temporal collocation points: This generates N_t points evenly distributed between 0 and 1, indicating the times at which the solution is computed.

Convergence Analysis of the Discrete Time-Integrated Scheme

Assumptions:

Let $h := max\{\Delta x, \Delta t\}$. We make the following assumptions:

(A1) The functions $g(x,t)$, $k(\xi,s,x,y)$, and $u(x,t)$ are continuous and sufficiently smooth.

(A2) The nonlinearity $f : R \to R$ is Lipschitz continuous with constant $L > 0$.

(A3) The kernel integral is uniformly bounded:

$$\left|\int_s^t k(\xi,s,x,y)d\xi\right| \le M \quad \forall (x,y,s,t) \in [0,X]^2 \times [0,T]^2$$

Consistency of the Discrete Time-Integrated Scheme

Lemma 1 (Consistency). Let $u(x,t)$ be the exact solution to equation (2). Then the local truncation error $\tau_{i,n}$ defined by

$$\tau_{i,n} := u(x_i,t_n) - \left(u(x_i,0) + \Delta t\sum_{m=0}^{n} g(x_i,t_m) + \Delta x\Delta t\sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j}f(u(x_j,t_m))\right)$$

satisfies $|\tau_{i,n}| = O(h)$.

Proof: Define Local Truncation Error $\tau_{i,n}$ as

$$\tau_{i,n} = u(x_i,t_n) - \left(u(x_i,0) + \Delta t\sum_{m=0}^{n} g(x_i,t_m) + \Delta x\Delta t\sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j}f(u(x_j,t_m))\right).$$

Analysing each term of $\tau_{i,n}$

We approximate $\int_0^t g(x,\xi)d\xi$ using the composite trapezoidal or midpoint rule:

$\int_0^{t_n} g(x_i,\xi)d\xi ? \Delta t \sum_{m=0}^{n} g(x_i,t_m) + O(\Delta t^p)$

with $p = 2$ for trapezoidal rule, and the truncation error is $O(\Delta t^p)$.

The triple integral

$\int_0^x \int_0^{t_n} \left(\int_s^{t_n} k(\xi,s,x,y)d\xi\right) f(u(s,y))dsdy$

is approximated by

$\Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j} f(u(x_j,t_m)) + O(\Delta t^q + \Delta t^r)$.where q and r are the orders of the error terms.

The truncation error terms:

i. The integral $\int_0^x dy$ is approximated using quadrature rule that introduces error of order $O(\Delta t^q)$.
ii. Similarly, the integral $\int_0^t ds$ is approximated and introduces an error of order $O(\Delta t^r)$.
iii. The integral $\int_0^x k(\cdot)d\xi$ is also evaluated using quadrature of known accuracy order.

So total local truncation error from this terms is

$\tau_{i,n} = O(\Delta t^q + \Delta t^r)$

which satisfies $\lim_{\Delta x \Delta t \to 0} \tau_{i,n} = 0$.

Hence the proof.

Stability of the Discrete Time-Integrated Scheme

Lemma 2 (Stability). Let $u_{i,n}$ and $\tilde{u}_{i,n}$ be numerical solutions corresponding to different initial data with difference $\varepsilon_{i,n} := u_{i,n} - \tilde{u}_{i,n}$. Then:

$|\varepsilon_{i,n}| \le Ce^{\lambda t_n} \max_{j,m} |\varepsilon_{j,m}|$

for constants $C, \lambda > 0$ depending on L, M, T, and h .

Proof: We want to show the small changes in input (initial data or source data) lead to small changes in output. Let $u_{i,n}$ and $\tilde{u}_{i,n}$ be numerical solutions to the same problem with possibly different initial data or perturbed values.

The error is the difference between analytical solution and the approximate solution given as

$$\varepsilon_{i,n} = u_{i,n} - \tilde{u}_{i,n}$$

From (8), the difference gives

$\varepsilon_{i,n} = u_{i,0} - \tilde{u}_{i,0} + \Delta t \sum_{m=0}^{n} (g_{i,m} - \tilde{g}_{i,m}) + \Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j}[f(u_{j,m}) - f(\tilde{u}_{j,m})]$

Assume $u_{i,0} = \tilde{u}_{i,0}$ and $g_{i,m} = \tilde{g}_{i,m}$ i.e. only nonlinearity $f(u)$ is causing deviation. Then

$\varepsilon_{i,n} = \Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} K_{n,m,i,j}[f(u_{j,m}) - f(\tilde{u}_{j,m})]$.

Using Lipschitz property of f : Assume f is Lipschitz with constant $L > 0$, i.e.

$|f(u) - f(\tilde{u})| \le L|u - \tilde{u}|$.

Then

$|\varepsilon_{i,n}| = \Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} |K_{n,m,i,j}| \cdot L|\varepsilon_{j,m}|$.

Let

$M = max|K_{n,m,i,j}|$

Then

$|\varepsilon_{i,n}| = LM\Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} |\varepsilon_{j,m}|$.

Let $C = LM$, then

$|\varepsilon_{i,n}| = C\Delta x \Delta t \sum_{j=0}^{i} \sum_{m=0}^{n} |\varepsilon_{j,m}|$.

Applying Discrete Gronwall's Inequality, let

$E_n = \max_i |\varepsilon_{i,n}|$

then

$E_n \le C\Delta x \Delta t \sum_{m=0}^{n} \sum_{j=0}^{N_x} E_m = C\Delta x N_x \Delta t \sum_{m=0}^{n} E_m$

Let $C_1 \le C\Delta x N_x$

$E_n \le C_1 \Delta t \sum_{m=0}^{n} E_m$.

This is a discrete form of Gronwall's inequality. If $E_0 = 0$, it implies $E_n = 0$ (stability). If $E_0 \ne 0$ then

$E_n \le E_0 e^{C_1 t_n}$.

The error at time step n grows at most exponentially in time. Hence, the method is stable under Lipschitz nonlinearity.

Convergence of the Discrete Time-Integrated Scheme

Theorem (Convergence). Under assumptions (A1) - (A3), the discrete scheme (8) is consistent and stable. Therefore, the numerical solution $u_{i,n}$ converges to the exact solution $u(x_i, t_n)$ at the collocation points with first-order accuracy:

$\max_{i,n} |u(x_i,t_n) - u_{i,n}| \le Ch \to 0 \text{ as } h \to 0$

Proof: This follows from the standard Lax-Richtmyer Equivalence Theorem: consistency plus stability implies convergence.

RESULTS AND DISCUSSION

In this section, numerical examples are used to illustrate the new concept, efficiency, accuracy and simplicity of the new method. Let $u_N(t)$ and $u(t)$ be the approximate and numerical solution respectively, then $abs - e_N = |u_N(t) - u(t)|$ is the absolute error of N. All numerical solutions are presented in a figure. All computations in this section are performed using MATLAB. For all the examples, initial condition problems are considered.

Example 1 (Hussain *et al.*, 2016): Consider the linear partial integro differential two dimensional equation

$\frac{\partial u(x,t)}{\partial t} = x + \frac{tx(t-x)}{6}(tx+3) + \int_0^x \int_0^t (s - y)u(s,y)dyds, (x,t) \in [0,1] \times [0,1]$

with initial condition

$u(x,0) = 1, 0 \le x \le 1$

and exact solution

$u(x,t) = 1 + xt$.

Solution

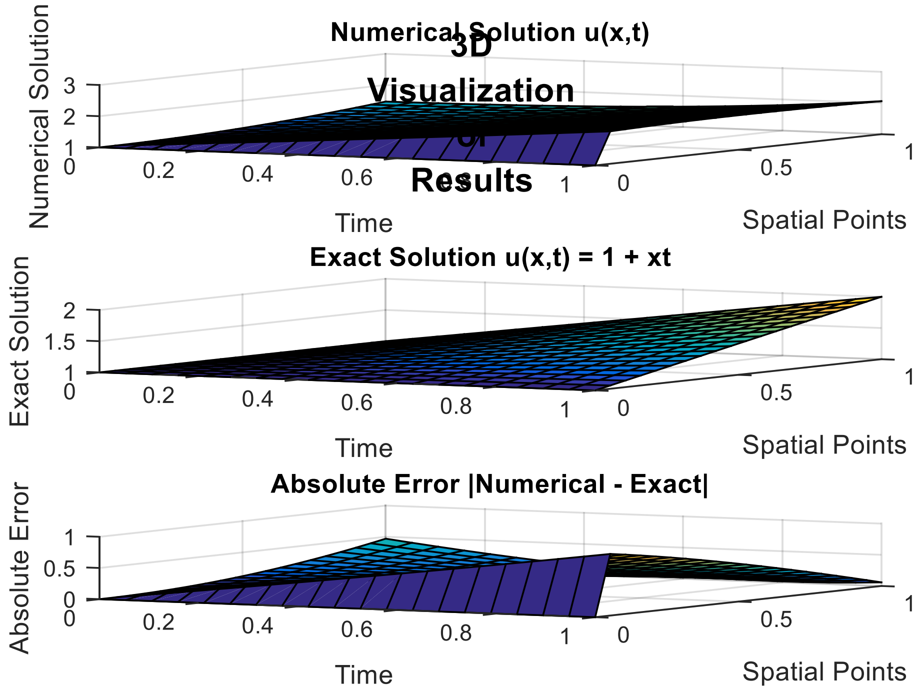

Figure 1: Results and Absolute Error

Table 1: Numerical Results for Example 1

Spatial point	Time point	Exact	Hussain et al., (2016)	Numerical Result	Error
0.00	0.00	1	1	1	0
0.00	0.50	1	1	1	0
0.00	1.00	1	1	1	0
0.25	0.00	1	1	1	0
0.25	0.50	1.125	1.125	1.125	0
0.25	1.00	1.25	1.25	1.25	0
0.50	0.00	1	1	1	0
0.50	0.50	1.25	1.25	1.25	0
0.50	1.00	1.5	1.5	1.5	0
0.75	0.00	1	1	1	0
0.75	0.50	1.375	1.375	1.375	0
0.75	1.00	1.75	1.75	1.75	0
1.00	0.00	1	1	1	0
1.00	0.50	1.5	1.5	1.5	0
1.00	1.00	2	2	2	0

Figure 1 presents the depict the 3D numerical results for the approximate and exact solutions for different values of x and t between 0 and 1, as well as the absolute error. Table 1 clearly show that this approach approximates to zero just like that of (Hussain et al., 2016).

Example 2 **(**Hussain *et al.*, 2016**)**: Consider the nonlinear partial two dimensional integro differential equation

$$\frac{\partial u(x,t)}{\partial t} = x - \frac{t^2x^2}{36}(4tx + 9) + \int_0^x \int_0^t [sy + u^2(s,y)]dyds, (x,t) \in [0,1] \times [0,1]$$

with initial condition

$u(x,0) = 1, 0 \leq x \leq 1$

and exact solution

$u(x,t) = xt.$

Solution

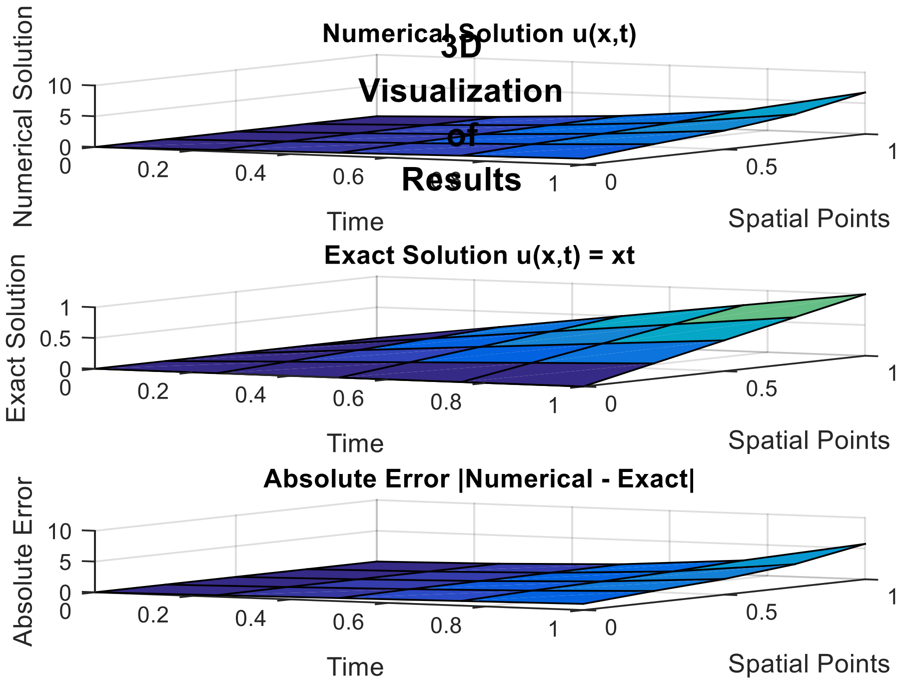

Figure 2: Results and Absolute Error

Table 2: Result of Example 2

Spatial point	Time point	Exact	Hussain et al., 2016	Numerical	Error
0.00	0.00	0	0	0	0
0.00	0.50	0	0	0	0
0.00	1.00	0	0	0	0
0.25	0.00	0	0	0	0
0.25	0.50	0.125	0.125	0.125	0
0.25	1.00	0.25	0.25	0.25	0
0.50	0.00	0	0	0	0
0.50	0.50	0.25	0.25	0.25	0
0.50	1.00	0.5	0.5	0.5	0
0.75	0.00	0	0	0	0
0.75	0.50	0.375	0.375	0.375	0
0.75	1.00	0.75	0.75	0.75	0
1.00	0.00	0	0	0	0
1.00	0.50	0.5	0.5	0.5	0
1.00	1.00	1	1	1	0

Figure 2 presents the 3D numerical view of the results for the approximate and exact solutions for different values of x and t between 0 and 1, and also the absolute error. Table 2 clearly show that this approach approximates to zero just like that of (Hussain et al., 2016).

CONCLUSION

In this research, a polynomial collocation method was developed for the numerical solution of two-dimensional Partial Volterra Integro-Differential Equations (PVIDEs). The method involved transforming the integro-differential problem into a continuous time-integrated formulation, which was then discretized using a structured hybrid grid of collocation points in both space and time. This discretization allowed the conversion of the continuous model into a system of algebraic equations, efficiently handled via MATLAB implementation.

The theoretical analysis confirmed the method's consistency, ensuring that the numerical formulation accurately reflects the original mathematical structure of the PVIDEs. The stability of the scheme was also proven under the assumption of a Lipschitz continuous nonlinearity and bounded kernel, thereby confirming that errors do not amplify as the mesh is refined. The convergence of the approximate solution to the exact solution was shown using the Lax-Richtmyer equivalence theorem and the discrete Grönwall inequality.

The results presented shows that, the method is reliable, offering accurate approximations as the grid becomes finer. The scheme is robust, making it suitable for simulating physical processes governed by memory effects, such as heat conduction and population dynamics. The convergence behaviour ensures that increasing spatial and temporal resolutions leads to improved numerical accuracy, which justifies its adoption in scientific and engineering applications.

Overall, the polynomial collocation method is simple to implement, computationally efficient, and highly accurate. It avoids restrictive transformations and preserves the physical behaviour of the original problem. Numerical experiments validated the performance of the method by showing excellent agreement between the exact and approximate solutions. This work adds a powerful and reliable numerical approach for solving Volterra-type PVIDEs, offering a promising tool for future modeling of real-world dynamical systems with memory.

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