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A SEMI-ANALYTICAL SOLUTION TO THE TIME-FRACTIONAL ADVECTION–DIFFUSION EQUATION USING FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT
A potent tool for simulating models of physical, biological, and dynamic processes is well described by Fractional Partial Differential Equations (FPDE), which is due to their memory effect and non-local properties. The present study applies the Fractional Reduced Differential Transform Method (FRDTM), a semi-analytical technique, to solve Fractional Partial Differential Equations (FPDEs) relevant to advection-diffusion models. The study modifies an existing integer-order model by introducing fractional derivatives and analyzes the influence of these fractional parameters. The FRDTM is validated through comparison with the Fractional Variational Iteration Method (FVIM), thereby demonstrating its accuracy and computational efficiency. Results show that FRDTM performs well in finding an approximate solution for long-time simulations, especially at fractional order α < 1, and is preferable for high accuracy and low computational error, particularly for Advection-Diffusion problems.

Keywords: FPDEs, FRDTM, FVIM, Advection-Diffusion, Fractional order

INTRODUCTION

Fractional Partial Differential Equations (FPDEs) are generalizations of classical integral-order Partial Differential Equations, providing some degree of freedom in varying the rate of change of physical, biological, and dynamic processes (Meerschaert and Tadjeran, 2006). Models of Physical, biological, and dynamic processes are well described by fractional partial differential equations due to their non-local properties than the corresponding integer-order partial differential equations (Yadav et al., 2022), in which their recent data depends completely on the data of the past time. There are several fractional derivative operators used in defining the FPDEs, such as the Riemann-Liouville Fractional Derivative, the Caputo Fractional Derivative, the Gruunwald-Letnikov Fractional Derivative, and the Riesz Fractional Derivative (Li and Zeng, 2015). The Caputo Fractional Partial Derivatives allow for the natural interpretation of initial and boundary value problems. It is also useful in describing complex systems where the current state depends on the entire history of the system. The Caputo Fractional Partial derivative operator of order α of a function $u(x,t)$with respect to time t is defined as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\frac{\partial u(\tau,t)}{\partial \tau})}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad 0 < \alpha \le 1$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \le 1$ and $\Gamma(.)$ is the gamma function.

In recent years, the Caputo Fractional Partial Differential Equations (FPDEs) have been applied to model the dynamic behavior of so many physical phenomena, such as in Blood flow (Bansi et al., 2018), Biochemical reaction model (Akugl and Khoshnaw, 2019), Ebola epidemic model (Area et al., 2015), economic growth model (Muhamad et al., 2021), filtration-consolidation processes (Bohaienko and Bulavatsky, 2020), and many more.

Despite their potential, most Caputo fractional partial differential equations lack exact analytical solutions, requiring efficient numerical methods. Recent studies have explored various numerical schemes such as the Fractional Finite Element Method (Amattouch et al., (2023), the Fractional Differential Transform Method (Arikoglu and Ozkol, 2007), the Fractional Finite Difference Method (Guo et al., 2020), the Laplace Adomian Fractional Decomposition Method (Jafari et al., 2011), the Fractional Reduced Differential Transform Method (Keskin and Oturanc, 2010), the Natural Homotopy Perturbation Method (Maitama and Abdullahi, 2016), the Fractional Variational Iteration Method (VIM) (Singh and Kumar, 2017). Nevertheless, many of these approaches involve linearization, discretization, or perturbation techniques, which may reduce accuracy or increase computational cost.

The Fractional Reduced Differential Transform Method (FRDTM) has gained attention for its ability to handle nonlinearities and complex boundary conditions efficiently. The FRDTM is an extension of the Reduced Differential Transform Method modified by incorporating fractional derivatives (Keskin and Oturan, 2010). The solution is obtained as a convergent series and validated by comparing the result obtained with the results from existing methods such as the Fractional Variational Iteration Method (FVIM). FRDTM doesn't require dividing the problem domain into small intervals or linearizing the equations, and it provides solutions in the form of rapidly convergent series (Al-rabtah, 2021). FRDTM applies the fractional transform in the Caputo sense to handle initial value problems directly, and this method is implemented by first transforming the original fractional partial differential equation into an algebraic equation in the transformed domain using the fractional differential transform. Secondly, the transformed equation is then solved using algebraic manipulations, and finally, the inverse transformation is applied to obtain the solution in the original domain, which is easily programmable in symbolic software like Maple or Mathematica. FRDTM has been successfully applied to a wide range of models such as Biological population model (Srivastava et al, 2020), homogeneous case of gas dynamics equations (Keskin and Oturan, 2010), diffusion model (Kenea, 2018), oil pollution in the water (Patel et al., 2023) and mutualism model (Abdallah, M. A. and Ishag, 2023).

In this study, we aimed to create the fractional order partial differential equation equivalent of the non-steady state salt solute transport by replacing it with the Caputo fractional derivative. The FPDE will be solved using the Fractional Reduced Differential Transform Method (FRDTM). To show the reliability and accuracy of the FRDTM, the result obtained will be compared with that of the Fractional Variational Iteration Method (FVIM), and we will also analyze the effect of the various values of the fractional orderα.

MATERIALS AND METHODS

Considered the integer order partial differential equation of a non-steady states salt transport equation,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x},\ (x,t) \in (0,1)\times(0,T) \quad (1)$$

Subject to the initial and boundary conditions

$$u(x,0) = e^{-x}, \quad x \in [0,1] \quad (2)$$

$$u(0,t) = e^{(D+v)t}, \quad t \in [0,T] \quad (3)$$

$$u(1,t) = e^{-1+(D+v)t}, \quad t \in [0,T] \quad (4)$$

The exact solution is given as

$$u(x,t) = \left(e^{-x} + x - \frac{x}{e} - 1\right)e^{(D+v)t} - \left(x - \frac{x}{e} - 1\right)e^{(D+v)t} \quad (5)$$

where $D = 0.001$ is the diffusion coefficient and $v = 0.1$ is the flow velocity.

We replace the first-order time derivative $\frac{\partial u}{\partial t}$ with the Caputo fractional derivative of order α in equation (1), then, equation (1) becomes,

$$D_t^\alpha = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^2 u(x,t)}{\partial x^2} - v\frac{\partial u(x,t)}{\partial x}, \quad 0 < \alpha \le 1, (x,t) \in (0,1)\times(0,T) \quad (6)$$

$$u(x,0) = e^{-x}, \quad x \in [0,1] \quad (7)$$

$$u(0,t) = e^{(D+v)t}, \quad t \in [0,T] \quad (8)$$

$$u(1,t) = e^{-1+(D+v)t}, \quad t \in [0,T] \quad (9)$$

The Fractional Reduced Differential Transform Method (FRDTM) is used to solve equation (6).

The Basic Concept of Fractional Reduced Differential Transform Method (FRDTM)

Let $u(x,t)$ be a product of two variables, such that

$$u(x,t) = g(x)h(t) \quad (10)$$

Based on the properties of the fractional differential transform method, the function $u(x,t)$ is defined as;

$$u(x,t) = \sum_{i=0}^{\infty} g(i)x^i \times \sum_{j=0}^{\infty} h(j)t^j = \sum_{k=0}^{\infty} U_k(x)t^k \quad (11)$$

where U_k is the t-dimensional spectrum function of $u(x,t)$

Definition 1: According to (Kenea, 2018), If the function $u(x,t)$, is analytical and k-times continuously differentiable with α^{th} derivatives with respect to the time t and space x in the domain of interest, then let the Fractional Reduced Differential Transform (FRDT) of $u(x,t)$, be given as

$$FRDT[u(x,t)] = U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[(D_{a,t}^\alpha)^k(u(x,t)\right]_{t=t_0}$$

$$= \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}, \quad k = 0,1.2, \dots \quad (12)$$

where $\alpha > 0, t$-dimensional spectrum function $U_k(x)$, is the transformation function and $u(x,t)$ is the original function.

***Definition 2**:* The differential inverse fractional reduced transform of $U_k(x)$, denoted by $u(x,t)$ is given by;

$$FRDT^{-1}(U_k(x)) = u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \quad (13)$$

Combining equations (11) and (12), we have

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}(t-t_0)^{k\alpha} \quad (14)$$

Equation(14) shows clearly that the concept of the reduced differential transform is derived from the power series expansion.The fundamental operations of the fractional Reduced Differential Transform Method (FRDTM) (Moosav and Taghizadeh, 2020) are tabulated below.

Table 1: The Fundamental Operators of the FRDTM

S/N	Original function	Transformed Function
1	$u(x,t)$	$U_k(x) = \begin{cases} \frac{1}{(k/\alpha)!}\left[\frac{\partial^{\frac{k}{\alpha}}}{\partial t^{\frac{k}{\alpha}}}u(x,t)\right]_{t=t_0} & for\ k = 0,1,2,3,..(m\alpha-1)\ if\ k/\alpha\ \mathbb{Z}^+ \\ 0 & if\ k/\alpha \notin \mathbb{Z}^+ \end{cases}$
1	$u(x,t) = g(x,t) \pm h(x,t)$	$U_k(x) = G_k(x) \pm H_k(x)$
2	$u(x,t) = ag(x,t)$, where a is a constant	$U_k(x) = aG(x)$
3	$u(x,t) = \frac{\partial^n}{\partial x^n}u(x,t),$	$U_k(x) = \frac{\partial^n}{\partial x^n}U_k(x)$
4	$u(x,t) = \frac{\partial^{\alpha n}}{\partial t^{\alpha n}}u(x,t)$	$U_k(x) = \frac{\Gamma(\alpha(k+n)+1)}{\Gamma(k\alpha+1)}U_k(k+n),\ n = 1,2,3,\dots$
5	$u(x,t) = g(x,t)h(x,t)$	$U_k(x) = \sum_{m=0}^{k} G_k(m)H_k(k-m)$

The Implementation of Fractional Reduced Differential Transform Method (FRDTM) to Solve the Fractional Non-Steady State Salt Transport Equation

In this section, we apply the Fractional Reduced Differential Transform operator FRDT on both sides of equation (6) and (7 − 9), we have,

$$FRDT\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right] = FRDT\left[D\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x}\right], \quad 0 < \alpha \le 1, (x,t) \in (0,1)\times(0,T) \quad (15)$$

Applying the operators 3 and 4 in Table 1 to equation, equation (6) is transformed to

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = D\frac{\partial^2 U_k(x)}{\partial x^2} - v\frac{\partial U_k(x)}{\partial x} \quad (16)$$

or

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(D\frac{\partial^2 U_k(x))}{\partial x^2} - v\frac{\partial U_k(x)}{\partial x}\right) \quad (17)$$

The initial and the boundary condition in equation $(7-8)$can be transformed by applying operator 1 in Table 1 as follows:

$$FRDT[u(x,0)] = FRDT[e^{-x}] \quad (18)$$

$$U_0(x) = e^{-x} \quad (19)$$

$$FRDT[u(0,t)] = FRDT\left[e^{(D+v)t}\right], \quad t \in [0,T] \quad (20)$$

$$U_k(0,t) = e^{(D+v)t} \quad (21)$$

$$FRDT[u(1,t)] = FRDT\left[e^{-1+(D+v)t}\right], \quad t \in [0,T] \quad (22)$$

$$U_k(1,t) = e^{-1+(D+v)t} \quad (23)$$

Substituting equation (19) into equation (17), we obtain the following U_{k+1}values successively for $k = 0,1,2,3,4, \ldots$, we have,

$for\ k = 0$

$$U_{0+1}(x) = \frac{\Gamma(0\times\alpha+1)}{\Gamma(\alpha(0+1)+1)}\left(D\frac{\partial^2 U_0(x)}{\partial x^2} - v\frac{\partial U_0(x)}{\partial x}\right)$$

$$U_1(x) = \frac{\Gamma(1)}{\Gamma(\alpha+1)}\left(0.001\frac{\partial^2}{\partial x^2}(e^{-x}) - 0.1\frac{\partial}{\partial x}(e^{-x})\right)$$

$$U_1(x) = \frac{\Gamma(1)}{\Gamma(\alpha+1)}(0.001 \times e^{-x} + 0.1 \times e^{-x})$$

$$U_1(x) = \frac{0.101e^{-x}}{\Gamma(\alpha+1)}$$

$for\ k = 1$

$$U_{1+1}(x) = \frac{\Gamma(1\times\alpha+1)}{\Gamma(\alpha(1+1)+1)}\left(D\frac{\partial^2 U_1(x)}{\partial x^2} - v\frac{\partial U_1(x)}{\partial x}\right)$$

$$U_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(0.001\frac{\partial^2}{\partial x^2}\left(\frac{0.101e^{-x}}{\Gamma(\alpha+1)}\right) - 0.1\frac{\partial}{\partial x}\left(\frac{0.101e^{-x}}{\Gamma(\alpha+1)}\right)\right)$$

$$U_2(x) = \frac{1}{\Gamma(2\alpha+1)}(0.001 \times 0.101e^{-x} + 0.1 \times 0.101e^{-x})$$

$$U_2(x) = \frac{0.010201e^{-x}}{\Gamma(2\alpha+1)}$$

$for\ k = 2$

$$U_{2+1}(x) = \frac{\Gamma(2\times\alpha+1)}{\Gamma(\alpha(2+1)+1)}\left(D\frac{\partial^2 U_2(x)}{\partial x^2} - v\frac{\partial U_2(x)}{\partial x}\right)$$

$$U_3(x) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(0.001\frac{\partial^2}{\partial x^2}\left(\frac{0.010201e^{-x}}{\Gamma(2\alpha+1)}\right) - 0.1\frac{\partial}{\partial x}\left(\frac{0.010201e^{-x}}{\Gamma(2\alpha+1)}\right)\right)$$

$$U_3(x) = \frac{1}{\Gamma(3\alpha+1)}(0.001 \times 0.0102011e^{-x} + 0.1 \times 0.010201e^{-x})$$

$$U_3(x) = \frac{0.001030301e^{-x}}{\Gamma(3\alpha+1)}$$

$for\ k = 3$

$$U_{3+1}(x) = \frac{\Gamma(2\times\alpha+1)}{\Gamma(\alpha(2+1)+1)}\left(D\frac{\partial^2 U_3(x)}{\partial x^2} - v\frac{\partial U_3(x)}{\partial x}\right)$$

$$U_4(x) = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left(0.001\frac{\partial^2}{\partial x^2}\left(\frac{0.001030301e^{-x}}{\Gamma(3\alpha+1)}\right) - 0.1\frac{\partial}{\partial x}\left(\frac{0.001030301e^{-x}}{\Gamma(3\alpha+1)}\right)\right)$$

$$U_4(x) = \frac{1}{\Gamma(2\alpha+1)}(0.001 \times 0.001030301e^{-x} + 0.1 \times 0.001030301e^{-x})$$

$$U_4(x) = \frac{0.000104060401e^{-x}}{\Gamma(4\alpha+1)}$$

$$\vdots$$

Then, using the inverse transformation rule in equation(13), the approximate solution of $u(x,t)$ is obtained by

$$u(x,t) = \sum_{k=0}^{10} U_k(x)t^{k\alpha}, \quad k = 0,1,2,3,4 \ and\ \alpha = 0.5, 0.7\ and\ 1 \quad (24)$$

$$u(x,t) = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + U_4(x)t^{4\alpha} + \cdots$$

$$u(x,t) = e^{-x} + \frac{0.101e^{-x}}{\Gamma(\alpha+1)}t^{\alpha} + \frac{0.010201e^{-x}}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{0.001030301e^{-x}}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{0.000104060401e^{-x}}{\Gamma(4\alpha+1)}t^{4\alpha}$$

Using Maple21 software, we obtain the approximate solution, up to the tenth iteration approximation solution $u(x,t)$ ofthe Fractional Reduced Differential Transform Method.

Fractional Variational Iterational Method (FVIM) for Validation

To assess the accuracy of FRDTM, the Fractional Variational Iteration Method (FVIM) is applied. The FVIM is a semi-analytical technique based on correction functional and Lagrange multipliers (Wu and Lee, 2010; Odibat and Momani, 2006). In this study, it is used as a reference method. The FVIM correction functional for the time-fractional non-steady state salt transport equation is given as;

$$u_{k+1}(x,t) = u_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[D\frac{\partial^2 u(x,\tau)}{\partial x^2} - v\frac{\partial u(x,\tau)}{\partial x} - \frac{\partial^{\alpha} u(x,\tau)}{\partial \tau^{\alpha}}\right]d\tau \quad (25)$$

where $0 < \alpha \leq 1$ and u_{k+1} is the $(k+1)th$ approximation produced by the variational iteration correction functional. Refer to (Odibat and Momani, 2006) for details on the formulation and implementation of FVIM.

RESULTS AND DISCUSSION

To validate the accuracy of the Fractional Reduced Differential Transform Method (FRDTM), a series of numerical experiments were conducted by comparing its results with those obtained using the Fractional Variational Iteration Method (FVIM). The function $u(x,t)$was computed for various values of the fractional order α, while keeping the spatial variable fixed at $x = 0.3$ and varying the time variable $t[0,10]$. The three cases are presented for the fractional order$\alpha = 0.2, 0.5$ and 0.7 respectively. The numerical simulation was done using Maple21 software, and several interesting observations were made from the simulation results.

Table 2: The comparisons between the results obtained by the exact solution, FRDTM, and FVIM solutions $u(x,t)$for $\alpha = 1$ at$0 < t \leq 10$, $and\ 0 < x \leq 1$

x	t	Exact Solution	$FRDTM$	$FVIM$	Absolute error $u(x,t)$ for $\alpha = 1$ $\lvert Exact - FRDTM\rvert$	Absolute error $u(x,t)$ for $\alpha = 1$ $\lvert Exact - FVIM\rvert$
0	0	1	1	1	0	0
0.1	1.0	1.001001	1.001000499	1.182905468	1.00E-09	9.58E-02
0.2	2.0	1.002002	1.002002001	1.182905468	0	1.81E-01
0.3	3.0	1.003005	1.003004503	1.257767843	2.00E-09	2.55E-01
0.4	4.0	1.004008	1.004008011	1.3213456	0	1.73E-01
0.5	5.0	1.005013	1.005012521	1.373807108	0	3.69E-01
0.6	6.0	1.006018	1.006018036	1.415514729	0	4.09E-01
0.7	7.0	1.007025	1.007024554	1.446974591	4.00E-09	4.40E-01
0.8	8.0	1.008032	1.00803207	1.468795276	1.60E-08	4.61E-01
0.9	9.0	1.009041	1.009040575	1.481654086	4.80E-08	4.73E-01
1.0	10.0	1.01005	1.010050044	1.48626973	1.23E-07	4.76E-01

Table 2 presents the results of the approximate solution $u(x,t)$ obtained by the exact solutions, FRDTM and FVIM over a range of spatial ($x[0,1]$) and temporal values ($t[0,10]$) for fractional order of $\alpha = 1$. Additionally, the absolute errors are computed to assess the accuracy of the method. It was observed that the FRDTM solution matches the exact solution to high precision, with errors consistently in the range of 10^{-09}to10^{-07}indicating a high accuracy of the method for the integer-order case. While FVIM has higher errors, ranging from 9.58×10^{-01}at $t = 1.0$ to 4.74×10^{-01}at $t = 10.0$,FVIM's error increases with time, implying potential instability or accumulation of truncation error.

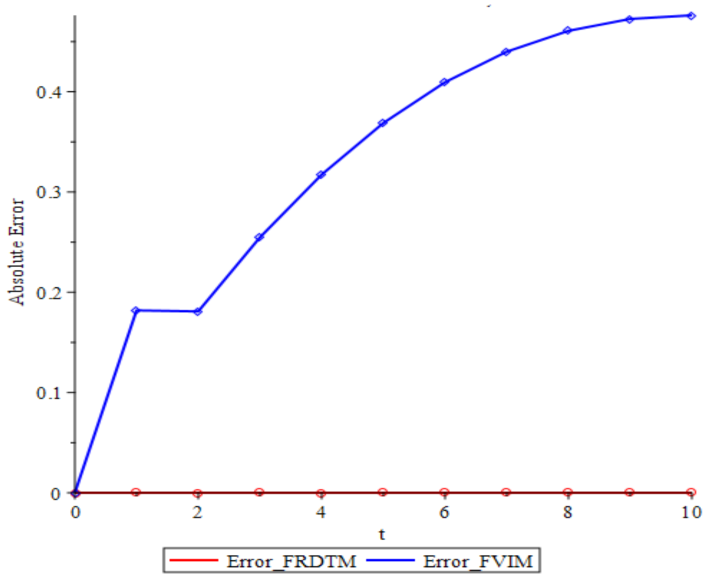

Figure 1: 2D surface plot of the absolute error of the FRDTM and FVIM solutions $u(x,t)$ for $\alpha = 1$ at $0 < t \leq 10,\ and\ 0 < x \leq 1$

It is seen from Figure 1 that the error of FRDTM is almost zero across time. This provides highly accurate solutions across all time steps and this shows that FRDTM is stable and reliable for long-time simulation. While FVIM is reasonably accurate at small time t, the error of FVIM increases significantly over time, thereby losing its precision as time increases, and this might not preserve solution accuracy well for large time t.

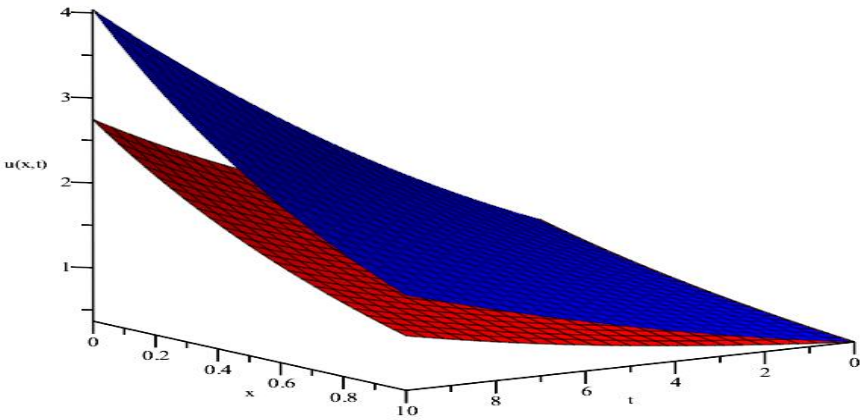

Figure 2: 3D surface plot of the EXACT, FRDTM and FVIM solutions $u(x,t)$for $\alpha = 1$ at $0 < t \leq 10,\ and\ 0 < x \leq 1$

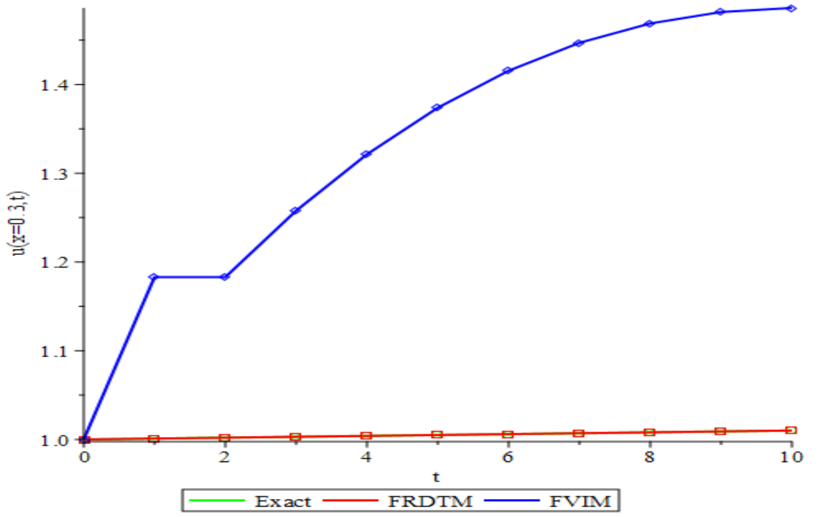

Figure 3: 2D surface plot of the EXACT, FRDTM and FVIM solutions $u(x,t)$ for $\alpha = 1$ at $0 < t \leq 10,\ and\ 0 < x \leq 1$.

Figures 2 and 3 show the 3D and 2D representation of the exact solution (green), the FRDTM (red), and the FVIM (blue). It is seen that the exact solutions show a smooth decay surface, with decreasing values as time increases, and the surface reflects the expected behavior of a dissipative advection-diffusion model. The FRDTM solution closely matches the exact solution; the curvature and slope of the surface match closely with the exact solution and no visible deviation is seen. While for the FVIM solution, it is observed that the surface is less steep and lower in magnitude than the other two solutions, which implies that FVIM underestimates the solution for higher $t$values.

Table 3: The Comparisons between the FRDTM and FVIM solutions $u(x,t)$for $\alpha = 0.2$ at $0 < t \leq 10\ \ and\ x = 0.3$

x	t	$FRDTM$)	$FVIM$
0.3	0	0.60653066	0.60653066
	1.0	0.671675597	0.673023306
	2.0	0.743817481	0.746741699
	3.0	0.823707824	0.824399945
	4.0	0.912178858	0.91277357
	5.0	1.23380268	1.224251577
	6.0	1.366320298	1.369114012
	7.0	1.513071081	1.511016512
	8.0	1.675583757	1.672322356
	9.0	1.855551243	1.853541851
	10.0	2.033991259	2.032486289

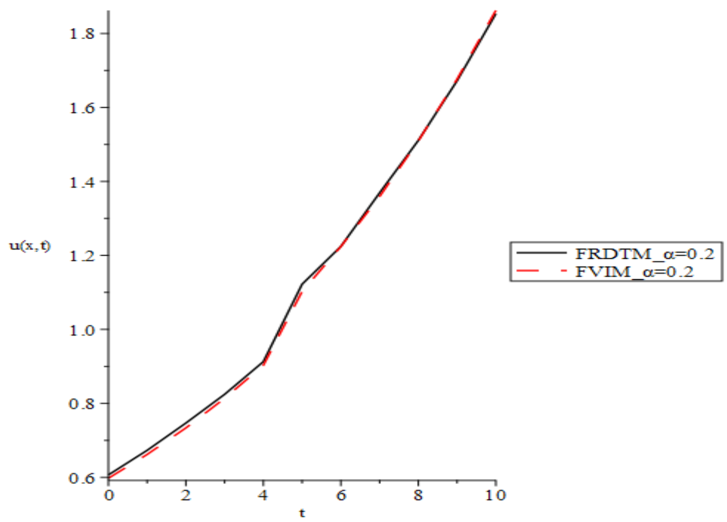

Figure 4: 2D surface plot of the comparisons between the FRDTM and FVIM solutions $u(x,t)$ for $\alpha = 0.2$ at $0 < t \leq 10\ \ and\ x = 0.3$.

Table 3andFigure 4 show the comparison between FRDM and FVM for$\alpha = 0.2$. It is observed that the values of the solutions $u(x,t)$ gradually increase as time progresses for both methods, this indicate a slow diffusion or transport process due to the low value of α. Also, both methods yield very similar results, with only slight differences appearing for higher$t$, especially after$t = 7$. Additional, the maximum absolute error remains minimal, suggesting that FRDM closely approximates the FVM results even for small fractional orders.

Table 4: The Comparisons between the FRDTM and FVIM solutions $u(x,t)$for $\alpha = 0.5$ at $0 < t \leq 10$ and $x = 0.3$

x	t	$FRDTM$	$FVIM$
0.5	0	0.740818221	0.729705947
	1.0	0.82435588	0.809617008
	2.0	0.906194684	0.90125395
	3.0	0.915826048	0.911093561
	4.0	1.006124227	1.005239696
	5.0	1.124172873	1.121398001
	6.0	1.240934708	1.248904416
	7.0	1.367837733	1.386688315
	8.0	1.507265969	1.534030278
	9.0	1.662470848	1.691230541
	10.0	1.854253675	1.860176024

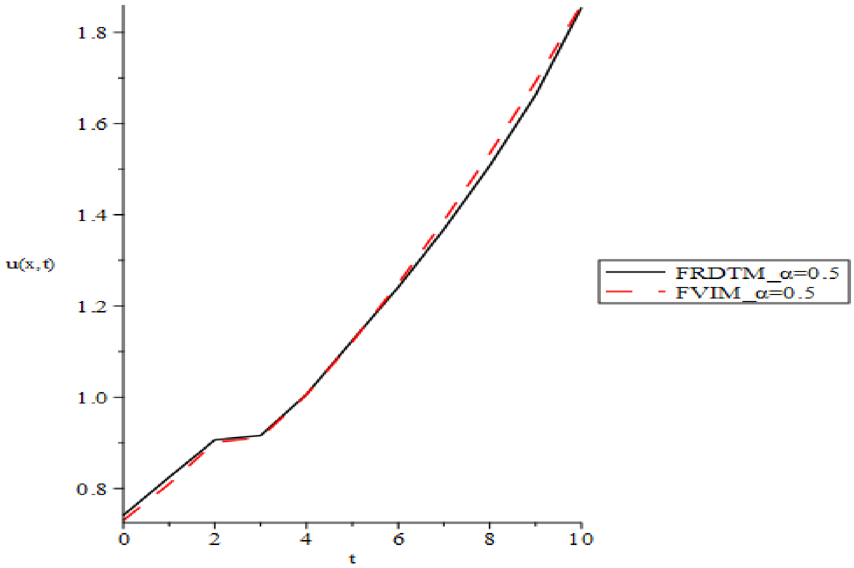

Figure 5: 2D surface plot of the comparisons between the FRDTM and FVIM solutions $u(x,t)$for $\alpha = 0.5$ at $0 < t \leq 10$ and $x = 0.3$.

Table 4 and Figure 5 present the solution values for $\alpha = 0.5$ at fixed point$x = 0.3$.Here, the solution$u(x,t)$values are higher than those of fractional order$\alpha = 0.2$, which implies that increasing the values of αleads to faster diffusion of the solution$u(x,t)$. The curves for both FRDTM and FVIM remain almost identical up to$t = 7$.After which, a slight deviations are observed. It is also observed that the FRDTM values are slightly underestimated compared to FVIM in the mid-time range, but gradually catch up by $t = 10$.

Table 5: Comparisons between the FRDTM and FVIM solutions $u(x,t)$ for $\alpha = 0.7$ at $0 < t \leq 10$ and $x = 0.3$.

x	t	$FRDTM$	$FVIM$
0.7	0	0.740818221	0.729705947
	1.0	0.825715735	0.811030723
	2.0	0.909474481	0.906374627
	3.0	1.010653892	1.013972327
	4.0	1.117571646	1.130036274
	5.0	1.229069021	1.250893426
	6.0	1.365138401	1.375601469
	7.0	1.4983637	1.50773999
	8.0	1.665757967	1.655225052
	9.0	1.857042444	1.827808
	10.0	2.047354309	2.0109898

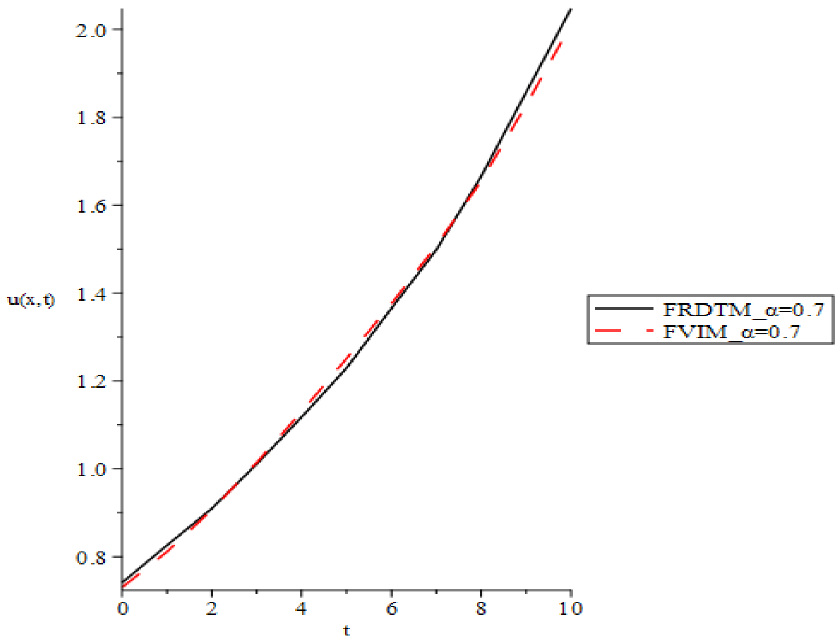

Figure 6: 2D surface plot of the Comparisons between the FRDTM and FVIM solutions $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ for $\boldsymbol{\alpha} = \mathbf{0.7}$ at $\mathbf{0} < t \leq 10$ *and* $x = 0.3$.

Table 5 and Figure 6 show the comparison for $\alpha = 0.7$ at fixed point$x = 0.3$.The solutions$u(x,t)$ is observed to show a sharp rise over time, stressing the impact of a higher fractional order on the dynamic behavior of the system. The results obtained by the FRDM and FVIM remain very close. However, the FRDM results tend to be slightly higher than FVIM in the later time steps. Also, it seems that the shape and slope of the curves in Figure 7 are nearly identical, thereby illustrating the consistency between the two methods.

CONCLUSION

The FRDTM successfully produced accurate solutions for the fractional non-steady state salt transport equation. Three cases with varying values of the fractional order α were analyzed. Results showed that lower values of α led to slower diffusion behavior, which is consistent with theoretical expectations. The 2D surface plots of the solutions at selected fixed spatial point sand varying temporal points demonstrate the convergence and reliability of FRDTM. The absolute error of FRDTM solutions was minimal compared to the FVIM solutions, highlighting the accuracy of FRDTM. The study shows there liability and stability of FRDTM for long-time simulation especially, at fractional order$\alpha < 1$and, also has the capability to accurately capture the complexity of higher-order fractional dynamics.

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