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ON A PARALLEL DIAGONALLY IMPLICIT RKN METHOD FOR THE NUMERICAL INTEGRATION OF SECOND ORDER DIFFERENTIAL EQUATIONS POSSESSING OSCILLATORY SOLUTIONS

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ABSTRACT

This paper presents a new four stage fourth-order Parallel diagonally implicit Runge-Kutta-Nystrom (RKN) method for the numerical integration of second order initial value problems (IVPs) possessing oscillatory solutions. The stability analysis of the method was also investigated to show that the method can approximate oscillatory systems. Numerical example was presented to show the applicability of the method. The results obtained shows that the method compares favourably in terms of accuracy and convergence with existing methods in current literatures.

Keywords: Runge-Kutta-Nystrom, Oscillatory solutions, Stability, Initial value problem, Differential equations

INTRODUCTION

In applied sciences and engineering, oscillatory equations are commonly encountered. Numerical integration of oscillatory systems related to initial value problems of the of the form (1) is the focus of this research:

$$y''(t) = f(t, y), \quad y(t_0) = y_0, \; y'(t_0) = y'_0 \qquad (1)$$

$y \in\to \mathbb{R}^n, \; f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

Where the first order derivative does not appear explicitly. The solution to (1) can be obtained directly by the Runge-Kutta-Nystrom method (RKN) developed by E. J. Nystrom in 1925, without converting to a system of first order differential equations as in the case of the classical Runge-Kutta method. Numerous techniques for the numerical integration of equations from (1) have been proposed in the literature. Many studies have focused on the development and implementation of the explicit type of RKN method because they are not only easy to implement but also efficient.

However, since explicit Runge-Kutta methods have a limited region of absolute stability, they are not appropriate for the solution of (1). Implicit methods were developed in response to the volatility of explicit methods, these methods have been studied by Van der Houwen and Sommeijer (1989), Sharp et al. (1990), Ozawa (1999), Franco (2004), Imoni et al. (2006), Ismail (2009), Senu et al. (2010), Senu et al. (2011), Senu et al. (2012) and Moo et al. (2014), which shows that these methods are more efficient in terms of time and effectiveness than the explicit type in solving (1) and using the classical Runge-Kutta method by transforming the IVPs to a system of first order ODEs.

According to Wu (2012), stability analysis is crucial in numerical analysis of multidimensional adapted Runge–Kutta–Nyström (ARKN) methods for oscillatory systems. The test equation used in Franco (2006) and Wu and Wang (2010) to examine the stability of a multidimensional ARKN method does not fully capture stability; rather, there is a gap because of the assumptions made in the original works, which results in an incomplete stability region for an ARKN method.

A diagonally implicit Runge–Kutta–Nystrom (DIRKN) formula-pair of order 5(4) by Imoni and Ikhile (2014) shows that because of its diagonally implicit structure, it has a suitable region of stability and is computationally less expensive. Based on Simos (1998) technique, an explicit trigonometrically-fitted Runge–Kutta–Nystrom (ETFRKN) method is developed (Demba et al., 2016). The derived method demonstrates that the new method's global error is both less and more efficient than those of the other methods now in use. An optimization of the sixth-order explicit Runge-Kutta-Nystrom method with six stages derived by El-Mikkawy and Rahmo (2003) utilising the phase-fitted and amplification-fitted procedures with constant step-size was developed by Demba et al. (2021). The stability analysis is discussed, demonstrating the periodicity interval of the derived approach. Comparing the suggested scheme to other RKN codes that are currently in use with six stages and the same order, the numerical experiments show that it performs exceptionally well. Using the Verhulst logistic growth model, Lee et al. (2024) developed a five-stage, exponentially-fitted two-derivative Runge–Kutta–Nyström approach. In solving second-order ODEs with exponential solutions, the suggested method's exponentially-fitting strategy contributes to exceptional accuracy and efficiency by precisely simulating a few typical exponential functions. Numerical experiments are conducted for a suggested method using fitting methodology and other current methods in terms of maximum global error versus computation time.

In this paper, we develop a new four-stage fourth order parallel diagonally implicit Runge-Kutta-Nystrom (PDIRKN) method using constant step size to solve special second order differential equations having oscillatory solutions. The numerical example tested shows the effectiveness of the method in obtaining the solution to (1).

MATERIALS AND METHODS

The Runge-Kutta-Nystrom method is broadly divided to two types namely explicit ($a_{ij} = 0$ for $i \le j, \;\; i, j = 1,2, \dots, s$) and implicit elsewhere. The implicit type takes the form

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^{s} b_i k_i$$
$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} b'_i k_i \qquad (2)$$
$$k_i = f\left(t_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{s} a_{ij} k_j\right)$$

where:

h is the step size,

s is the number of stages,

k_i are the stage derivatives (intermediate stages),

c_i, a_{ij}, b_i and b'_i are the coefficients of the RKN method

The c_i satisfies the simplify assumptions

$$\frac{1}{2} {c_i}^2 = \sum_{j=1}^{s} a_{ij} \qquad i = 1, \dots, s \qquad (3)$$

$$b_i = b'_i (1 - c_i), \;\; i = 1, \dots, s \qquad (4)$$

The method's coefficients can be represented by the Butcher's tableau in matrix format as

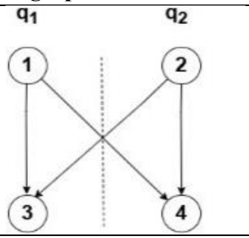
c	A
	b^T
	b'^T

where, $c = [c_1, \cdots, c_s]$,$b = [b_1, \dots, b_s]^T$, $b' = [b'_1, \dots, b'_s]^T$ and $A = [a_{ij}]$ with $c \in \mathbb{R}$, $b^T, b'^T \in \mathbb{R}^s$ and $A \in \mathbb{R}^{sxs}$

Construction of the Method

We construct a four-stage fourth order 2-parallel, 2-processors diagonally implicit RKN technique. The sparsity pattern and the diagraph of the method are shown in Table 1.

Table 1: Sparsity Pattern and Digraph of the Proposed Method

Sparsity Pattern	Digraph
$\begin{pmatrix} \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & 0 & \times \end{pmatrix}$	

From the table above, the symbol × denotes non-zero elements, q_1 *and* q_2 , denotes the number of processors

Order Conditions for RKN Methods

Algebraic conditions that an RKN method must satisfy are given in table 2 up to $p = 5$

Table 2: Order Conditions of RKN Methods

	$y \to b_i$		$y' \to b_i'$
1	h^2	$\frac{1}{2} = \sum_i b_i = 1$	h
2	h^3	$\frac{1}{6} = \sum_i b_i\, c_i = \frac{1}{2}$	h^2
3	h^4	$\frac{1}{12} = \sum_i b_i c_i^2 = \frac{1}{3}$	h^{3*}
4	h^{4*}	$\frac{1}{24} = \sum_i b_i\, a_{ij} c_j = \frac{1}{6}$	h^3
5	h^5	$\frac{1}{120} = \frac{1}{6}\sum_i b_i c_i^3 = \frac{1}{24}$	h^4
6	h^5	$\frac{1}{120} = \sum_{i,j} b_i a_{ij} c_j = \frac{1}{24}$	h^4
7	h^6	$\frac{1}{720} = \frac{1}{24}\sum_i b_i c_i^4 = \frac{1}{120}$	h^5
8	h^6	$\frac{1}{180} = \sum_{i,j} b_i c_i a_{ij} c_j = \frac{1}{30}$	h^5
9	h^6	$\frac{1}{720} = \frac{1}{2}\sum_{i,j} b_i c_i a_{ij} c_j^2 = \frac{1}{120}$	h^5

(see Hairer et al. (2008) and Imoni (2020))

For the proposed method, the order conditions are expressed in (5) – (13)

$order\ one$: $\sum_{i=1}^4 b_i' = b_1' + b_2' + b_3' + b_4' = 1$ (5)

$order\ two$: $\sum_{i=1}^4 b_i' c_i = b_1' c_1 + b_2' c_2 + b_3' c_3 + b_4' c_4 = \frac{1}{2}$ (6)

$order\ three$: $\sum_{i=1}^4 b_i' c_i^2 = b_1' c_1^2 + b_2' c_2^2 + b_3' c_3^2 + b_4' c_4^2 = \frac{1}{3}$ (7)

$order\ four$: $\sum_{i=1}^4 b_i' c_i^3 = b_1' c_1^3 + b_2' c_2^3 + b_3' c_3^3 + b_4' c_4^3 = \frac{1}{4}$ (8)

$\sum_{i,j=1}^4 b_i' a_{ij} c_j = b_1' a_{11} c_1 + b_2' a_{21} c_2 + b_2' a_{22} c_2 + b_3' a_{31} c_3 + b_3' a_{32} c_3 + b_3' a_{33} c_3 + b_4' a_{41} c_4 + b_4' a_{42} c_4 + b_4' a_{43} c_4 + b_4' a_{44} c_4 = \frac{1}{24}$ (9)

for y', and

$order\ two$: $\sum_{i=1}^4 b_i = b_1 + b_2 + b_3 + b_4 = \frac{1}{2}$ (10)

$order\ three$: $\sum_{i=1}^4 b_i c_i = b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{6}$ (11)

$order\ four$: $\sum_{i=1}^4 b_i c_i^2 = b_1 c_1^2 + b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{12}$ (12)

$order\ four$: $\sum_{i,j=1}^4 b_i a_{ij} c_j = b_1 a_{11} c_1 + b_2 a_{21} c_1 + b_2 a_{22} c_2 + + b_2 a_{22} c_2 + b_3 a_{31} c_1 + b_3 a_{32} c_2 + b_3 a_{33} c_3 + b_4 a_{41} c_1 + b_4 a_{42} c_2 + b_4 a_{43} c_3 + b_4 a_{44} c_4 = \frac{1}{24}$ (13)

for y

We have to satisfy nine equations, four for y and, five for y' and four compatibility conditions. Using the simplification assumption and since $a_{21} = a_{43} = 0$, we are left with eight equations in thirteen unknowns. There are six free parameters which are chosen to be c_1, c_2, c_3, c_4 *and* a_{32}. The Butcher's tableau for the proposed method is table 3.

Table 3: The Coefficients of the 4-stage New PDIRKN Method

c_1	a_{11}			
c_2	0	a_{22}		
c_3	a_{31}	a_{32}	a_{33}	
c_4	a_{41}	a_{42}	0	a_{44}
	b_1	b_2	b_3	b_3
	b'_1	b'_2	b'_3	b'_4

Equations (5) – (8) are solved to obtain the expression for b'_1, b'_2, b'_3 *and* b'_4 by using Maple 2021 as,

$$\left.\begin{aligned} b'_1 &= -\frac{12c_2c_3c_4 - 6c_2c_3 - 6c_2c_4 - 6c_3c_4 + 4c_2 + 4c_3 + 4c_4 - 3}{12(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)} \\ b'_2 &= \frac{12c_1c_3c_4 - 6c_1c_3 - 6c_1c_4 - 6c_3c_4 + 4c_1 + 4c_3 + 4c_4 - 3}{12(c_1 - c_2)(c_2 - c_3)(c_2 - c_4)} \\ b'_3 &= -\frac{12c_1c_2c_4 - 6c_1c_2 - 6c_1c_4 - 6c_2c_4 + 4c_1 + 4c_2 + 4c_4 - 3}{12(c_3 - c_1)(c_3 - c_2)(c_3 - c_4)} \\ b'_4 &= \frac{12c_1c_2c_3 - 6c_1c_2 - 6c_1c_3 - 6c_2c_3 + 4c_1 + 4c_2 + 4c_3 - 3}{12(c_1 - c_4)(c_4 - c_2)(c_4 - c_3)} \end{aligned}\right\} \quad (14)$$

Applying simplifying assumption (3) we have the following to obtain $a_{11}, a_{22}, a_{31}, and\ a_{41}$ expressed in (15) after simplification as:

$$a_{11} = \tfrac{1}{2}c_1^2,\ a_{22} = \tfrac{1}{2}c_1^2,\ a_{31} = \tfrac{1}{2}c_3^2 - 2a_{32} - \tfrac{1}{2}c_1^2,\ a_{41} = \tfrac{1}{2}c_4^2 - \tfrac{1}{2}c_1^2 - 2\,a_{42} \quad (15)$$

Also, from (9) and (13) we obtain expressions for a_{32} *and* a_{42} by using Maple 2021 as follows

$$a_{32} = \frac{A}{24b_3c_2} \quad (16)$$

$$a_{42} = \frac{B}{24b_4c_2} \quad (17)$$

where,

$A = 24a_{11}b_1c_1 + 24a_{21}b_2c_1 + 24a_{22}b_2c_2 + 24a_{31}b_3c_1 + 24a_{33}b_3c_3 + 24a_{41}b_4c_1 + 24a_{42}b_4c_2 + 24a_{43}b_4c_3 + 24a_{44}b_4c_4 - 1$

$B = 24a_{11}b_1c_1 + 24a_{21}b_2c_1 + 24a_{22}b_2c_2 + 24a_{31}b_3c_1 + 24a_{32}b_3c_2 + 24a_{33}b_3c_3 + 24a_{41}b_4c_1 + 24a_{43}b_4c_3 + 24a_{44}b_4c_4 - 1$

By applying the simplifying assumption (4) we obtain the expression for b_1, b_2, b_3 *and* b_4 in (18)

$$\left.\begin{aligned} b_1 &= -\frac{(1-c_1)(12c_2c_3c_4 - 6c_2c_3 - 6c_2c_4 - 6c_3c_4 + 4c_2 + 4c_3 + 4c_4 - 3)}{12(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)} \\ b_2 &= \frac{(1-c_2)(12c_1c_3c_4 - 6c_1c_3 - 6c_1c_4 - 6c_3c_4 + 4c_1 + 4c_3 + 4c_4 - 3)}{12(c_1 - c_2)(c_2 - c_3)(c_2 - c_4)} \\ b_3 &= -\frac{(1-c_3)(12c_1c_2c_4 - 6c_1c_2 - 6c_1c_4 - 6c_2c_4 + 4c_1 + 4c_2 + 4c_4 - 3)}{12(c_3 - c_1)(c_3 - c_2)(c_3 - c_4)} \\ b_4 &= \frac{(1-c_4)(12c_1c_2c_3 - 6c_1c_2 - 6c_1c_3 - 6c_2c_3 + 4c_1 + 4c_2 + 4c_3 - 3)}{12(c_1 - c_4)(c_4 - c_2)(c_4 - c_3)} \end{aligned}\right\} \quad (18)$$

The truncation error constant is given by

$$\left\|\tau^{(p+1)}\right\|_2 = \sqrt{\textstyle\sum_{i=1}^{p+1}\left(\tau_i^{(p+1)}\right)^2},\ \left\|\tau'^{(p+1)}\right\|_2 = \sqrt{\textstyle\sum_{i=1}^{p+1}\left(\tau_i'^{(p+1)}\right)^2} \quad (19)$$

(see Imoni (2020))

where, $\tau^{(p+1)}$ and $\tau'^{(p+1)}$ are the fifth order error equations associated with the method. Therefore, the truncation error constant for the proposed method is given by

$$\left\|\tau^{(5)}\right\|_2 = \sqrt{\textstyle\sum_{i=1}^{5}\left(\tau_i^{(5)}\right)^2},\ \left\|\tau'^{(5)}\right\|_2 = \sqrt{\textstyle\sum_{i=1}^{5}\left(\tau_i'^{(5)}\right)^2} \quad (20)$$

where,

$\tau_1^{(5)} = \sum_{i=1}^{4} b_i c_i^3 - \frac{1}{20},\ \tau_2^{(5)} = \sum_{i=1}^{4} b_i a_{ij} c_j - \frac{1}{120}$

$\tau_1'^{(5)} = \sum_{i=1}^{4} b_i c_i^4 - \frac{1}{5},\ \tau_2'^{(5)} = \sum_{i=1}^{4} b_i c_i a_{ij} c_j - \frac{1}{30},\ \tau_5'^{(5)} = \sum_{i=1}^{4} b_i c_i a_{ij} c_j^2 - \frac{1}{60}$

Substituting the expressions obtained in (14) and (15) into the truncation error constant (20) and minimizing using Python 3.0 subject to the bounds, $0 \le c_i \le 1, i = 1,2,3,4$. The 4-stage New DIRKN method obtained is expressed in the Butcher's tableau in table 4.

Table 4: The Coefficients of the New PDIRKN Method

$\frac{114}{137}$	$\frac{251}{725}$			
$\frac{282}{715}$	0	$\frac{251}{725}$		
$\frac{815}{961}$	$\frac{-332}{781}$	$\frac{435}{992}$	$\frac{251}{725}$	
$\frac{11}{232}$	$-\frac{106}{141}$	$\frac{316}{777}$	0	$\frac{251}{725}$
	$\frac{-14}{523}$	$\frac{158}{539}$	$\frac{29}{373}$	$\frac{147}{943}$
	$-\frac{81}{508}$	$\frac{91}{188}$	$\frac{457}{893}$	$\frac{9}{55}$

The coefficients of the Butcher's tableau are substituted into the fifth-order error formula (20) to obtain the error constants for y and y' respectively. $\left\|\tau^{(5)}\right\|_2 = 7.722466870818714 \times 10^{-2}$, $\left\|\tau'^{(5)}\right\|_2 = 4.425073327346878 \times 10^{-6}$

Stability Analysis

The stability characteristics of the derived method have been analyzed using the scalar harmonic oscillatory equation which is suitable for oscillatory systems.

$$y'' = -\omega^2 y,\quad y(0) = 1,\ y'(0) = i\omega,\quad \omega \in \mathbb{R} \quad (21)$$

The application of the RKN method (2) on (21) results to the following relations

$y_{n+1} = y_n + hy'_n + zb^T Y_n$

$hy'_{n+1} = hy'_n + zb'^T Y_n$

$Y_n = ey_n + chy'_n + zAy_n)$

Where $z = -\omega^2 h^2, Y_n = (ey_n + chy'_n)N^{-1},\ N = I + zA$

By eliminating the auxiliary vector Y_n , yields

$y_{n+1} = y_n(I + zb^T(I - zA)^{-1}e + hy'_n\, I + zb^T(I - zA)^{-1}c)$

$hy'_{n+1} = y_n zb'^T(I - zA)^{-1}e + hy'_n(1 + zb'^T(I - zA)^{-1}c)$

y_{n+1} *and* hy'_{n+1} can be written in a compact form as

$$\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} = H(z)\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \quad (22)$$

$$H(z) = \begin{bmatrix} 1 + zb^T(I - zA)^{-1}e & 1 + zb^T(I - zA)^{-1}c) \\ zb'^T(I - zA)^{-1}e & 1 + zb'^T(I - zA)^{-1}c \end{bmatrix} \quad (23)$$

And $A = \{a_{ij}\}_{i,j=1}^{s}$ $e = [1, \dots, 1]$, $b = [b_1, \dots, b_s]^T$, $b' = [b'_1, \dots, b'_s]^T$, $c = [c_1, \dots, c_s]$

The amplification matrix is the matrix $H(z)$ that is used to determine the method's stability (Imoni (2020)). The following functions, $s(z)$ and $p(z)$, are defined by Van der Houwen and Sommeijer (1989), where $s(z) = trace(H(z))$ and $p(z) = det(H(z))$,

The characteristic equation of the amplification matrix $R(z)$ is given by

$$\zeta^2 - s(z)\zeta + p(z) = 0 \quad (24)$$

Sharp et al. (1990) stated that the RKN method is R-stable if $\rho(R) \leq 1$ for all $z < 0$ and the eigen values on the unit disc are simple, provided that $\rho(R)$ represents the spectral radius of $R(z)$. This indicates that for any ω and h, the amplitude of the numerical solution to the test equation does not grow with time. The RKN approach is considered P-stable if $\rho(R) = 1$ for every $z < 0$, and RL-stable if the RKN approach is R-stable and $\rho(R) \to 0$ as $z \to \infty$. The interval of stability is defined as $(z_0, 0)$, $(z_0 < 0)$, on which $\rho(R) \leq 1$. (See Chawla and Sharma, 1981, Imoni and Ikhile, 2014 and Imoni, 2020). The interval of stability of the method obtained from the amplification matrix $H(z)$ defined in (23) is approximately $(-4.7, 0)$. The stability region of the method is given in figure 1

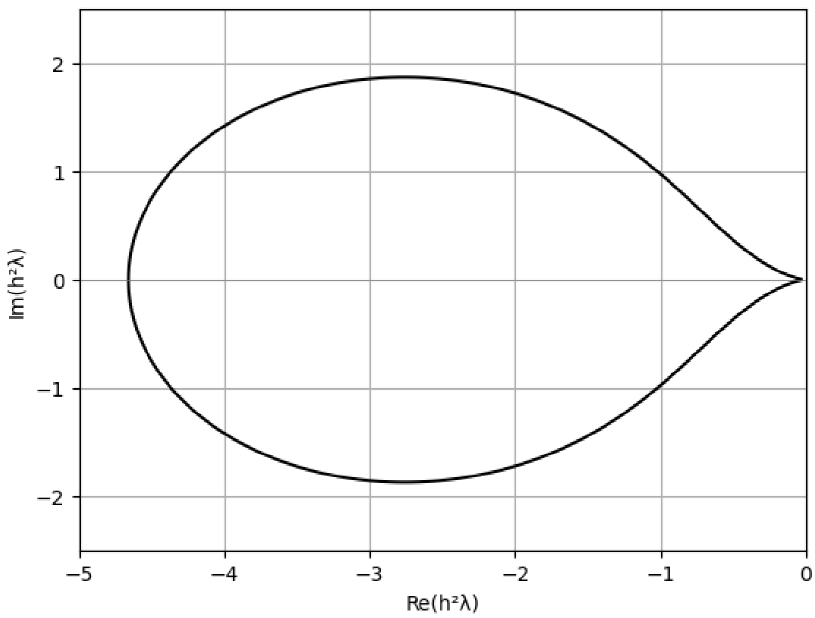


Figure 1: Stability region for the Fourth order New PDIRKN Method

The stability region of the derived method, which was obtained from the amplification matrix in (23), is shown in Figure 1. A significant portion of the negative real axis is covered by the plot, which is mostly located in the left half of the complex plane. This demonstrates the stability of the method with a stability interval of approximately (-4.7, 0).

RESULTS AND DISCUSSION

Numerical Example

The derived method will be applied to solve problems which appear in many papers on numerical methods for oscillatory problems. The following implicit RKN methods were selected for numerical comparison.

i. New DIRKN(4): the new method derived in this paper
ii. IRKN(4): four stage implicit RKN method derived by Ozawa (1999)
iii. DIRKN 3(4): three stage fourth order DIRKN method derived by Senu et al. (2011)
iv. DIRKN3(4,6): three stage fourth order DIRKN method derived by Sharp et al. (1990)
v. DIRKN3(4,4): DIRKN method derived by Senu et al. (2012)

Example

Consider the Duffing Equations

$y'' = -y - y^3 + \frac{1}{500} cos(1.01x)$, $y(0) = 0.20042678067$, $y'(0) = 0$

With exact solution

$y(x) = 0.200179477536\, cos(1.01x) + 0.00246946143\, cos(3.03x) + 0.304014 \times 10^{-6}\, cos(5.05x) + 0.374 \times 10^{-9}\, cos(7.07x)$

Source: Imoni and Ikhile (2014) and Li and Song (2006))

Notations

h: step size
Method: the method used
$\boldsymbol{y_n}$ **:** Numerical solution for each method
Error: $\|y_n - y(x_n)\|$ (absolute value of the numerical solution minus the exact solution)
FE: Function Evaluation
Time (s): Evaluation time in seconds

Table 5: Numerical Result for Example

h	Method	y_n	Exact	Error	FE	Time (s)
0.00125	New DIRKN(4)	0.178665	0.178963	0.000299	1599634	6.777
	IRKN(4)	0.178601	0.178963	0.000362	1437584	6.354
	DIRKN 3(4)	0.178601	0.178963	0.000362	1195224	4.028
	DIRKN3(4,6)	0.178601	0.178963	0.000362	1199450	4.071
	DIRKN3(4,4)	0.178601	0.178963	0.000362	1195584	5.204
0.0025	New DIRKN(4)	0.178668	0.178963	0.000295	799960	3.002
	IRKN(4)	0.178601	0.178963	0.000362	719698	2.444
	DIRKN 3(4)	0.178601	0.178963	0.000362	599406	2.984
	DIRKN3(4,6)	0.178601	0.178963	0.000362	599932	2.032
	DIRKN3(4,4)	0.178601	0.178963	0.000362	599442	1.983
0.0050	New DIRKN(4)	0.178682	0.178963	0.000281	399996	1.414
	IRKN(4)	0.178601	0.178963	0.000362	359964	1.243
	DIRKN 3(4)	0.178601	0.178963	0.000362	299926	1.267
	DIRKN3(4,6)	0.178601	0.178963	0.000362	299990	1.020
	DIRKN3(4,4)	0.178601	0.178963	0.000362	299930	0.972
0.0100	New DIRKN(4)	0.178739	0.178963	0.000225	199998	0.684
	IRKN(4)	0.178601	0.178963	0.000362	179996	0.622
	DIRKN 3(4)	0.178601	0.178963	0.000362	149990	0.506
	DIRKN3(4,6)	0.178601	0.178963	0.000362	149998	0.505
	DIRKN3(4,4)	0.178601	0.178963	0.000362	149992	0.498

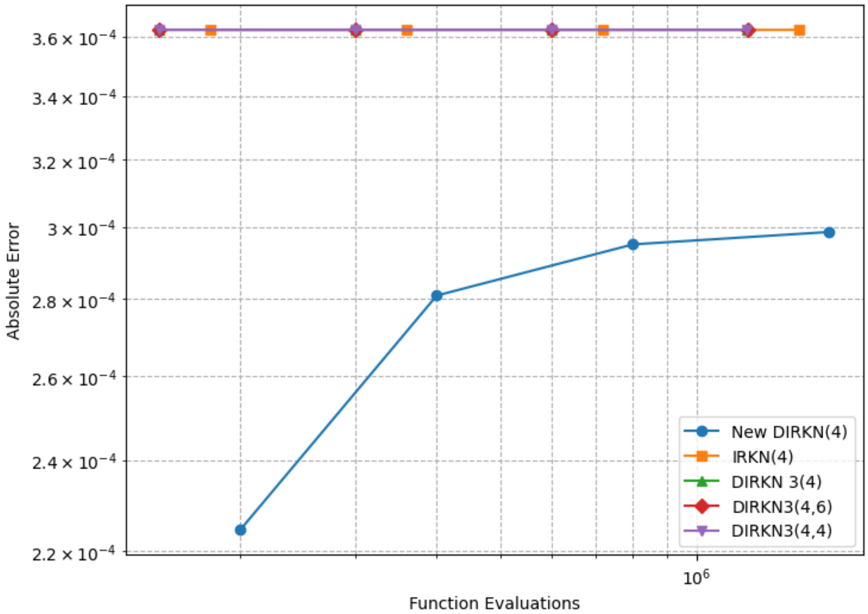


Figure 2: Efficiency Plot of Example

The New DIRKN(4) method's efficiency (function evaluation vs absolute error) plot are shown in figure 2 in comparison to the selected existing methods (IRKN(4), DIRKN 3(4), DIRKN 3(4,6) and DIRKN 3(4,4)). When compared to the other methods, that maintain the same error for all step sizes, as indicated by the straight lines, the New DIRKN(4) plot line, which is coloured blue, demonstrates least error about the exact solution across all step sizes tested, showing good accuracy of the method.

Discussion

From the results presented in Table 5 we have shown that the New DIRKN method derived in this paper shows the least error for all step sizes considered with respect to the exact solutions of the problems considered with more function evaluation being a four-stage method. This shows that New DIRKN method is more efficient and more accurate compared to the other order four methods it was compared with from the plot in Figure 2.

CONCLUSION

In this paper a four stage fourth-order Diagonally Implicit RKN method for the numerical integration of special second order IVPs of the form (1) possessing oscillatory solutions has been presented. The method has an appropriate region of stability. The derived method has also been applied to approximate second order IVPs possessing oscillatory solutions found in literatures to demonstrate its efficiency and accuracy. Results obtained compared favorably with the other methods derived by Ozawa (1999), Senu (2011), Sharp et al. (1990) and Senu (2012). However, it was observed that the new RKN method perform better in terms accuracy as shown in Table 5 and Figure2.

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