FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645-2944
Vol. 9 No. 6, June 2025, pp 10 - 18
DOI: https://doi.org/10.33003/fjs-2025-0906-3703

FRACTIONAL DERIVATIVES APPROACH ON INTEGRO-DIFFERENTIAL EQUATIONS USING EXPONENTIAL-FITTED COLLOCATION

Uwaheren Ohigweren Airenoni[1], Abubakar Usman Jos[1], Oduola Samson Oluyemi[2] and Owoeye Christiana Aina[1]

[1] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2] Department of Mathematics, FCT College Education, Zuba, Abuja.
*Corresponding authors' email: uwaheren.ao@unilorin.edu.ng

Abstract

In this article, the numerical solution of fractional order integro-differential equations using Exponential-Fitted Collocation (EFC) approach is discussed. The normal integro-differential equations with n^{th}-integer ordered derivatives were considered and solved at some neighborhood $\lceil\alpha\rceil^-$ and $\lceil\alpha\rceil^+$ where $\lceil\alpha\rceil^- \leq n \leq \lceil\alpha\rceil^+$ indicate some carefully chosen fractional order values to the left and right of n. The solutions are compared to the exact solution which was given at n values. Here, we desire to find the value of α that will produce a faster and better convergent to the exact solution given at the value of n. A trial solution in shifted Chebyshev polynomials basis function was assumed and substituted into the slightly perturbed problem considered. After collocation, $(N+1)$ algebraic system of equations was obtained. The values of unknown constants coefficients from the system of equations were substituted into the assumed trial solution to get the required approximate solution. Three (3) illustrative examples are solved to verify the reliability, simplicity and accuracy of the collocation method considered.

Keywords: Fractional order Integro-differential Equations and Exponentially Fitted Collocation method Solution.

INTRODUCTION

According to Al-Zuhairi et al (2024), integro-differential, differential and fractional integro-differential equations alike are important in describing physical, natural, and biological phenomena as well as in engineering and biological technologies. To solve these equations, numerous numerical methods have been advanced for the solution of which includes Adomian Decomposition Method (ADM), Finite Difference Method (FDM), Collocation Method (CM) to mention a few.
Ajileye et al., (2024) and Ajileye et al., (2023) studied the solution of Volterra integro-differential equations using collocation approach and standard collocation points method respectively to convert their equations to set of linear system of equations. The results obtained in both investigations showed that the methods are good and are capable of solving the class of problems accurately. More recently, Olotu et al., (2025) carried out a comprehensive analysis study of the Differential Transform Method (DTM) for solving ordinary differential equations (ODEs) of various orders. In the study, the authors focused the method's ability to handle both linear and nonlinear ODEs without linearization, discretization, or perturbation and obtained remarkably expressive results that are comparable the to existing literature. Falade and Taiwo (2023) proposed an exponentially fitted collocation algorithm (EFCA) for the solutions of n^{th}-order Fredholm type integro-differential equations. The proposed numerical algorithm was experimented on some examples and the results were compared with the exact solutions and some existing methods and were found to be accurate and reliable.
Aduroja et al., (2023) used collocation approximation method to solve some classes of Volterra integro-differential equations with polynomial basis functions and obtained the required algebraic equation after some transformations. The obtained system of equations was solved by Gaussian elimination method and the constants obtained were substituted into the trial solution and this yielded accurate solutions.
Uwaheren et al., (2022) used on Akbari-Ganji's method to solve Volterra type of integro differential difference equations and the approximate re-

sults obtained compared to the exact solution was found to be good and they converged rapidly.

Alshbool et al., (2022) advanced two techniques of Bernstein operational fractional polynomials and Bernstein operational matrices of differentiation metheds to solve fractional integro-differential equations (FIDEs) and the schemes were generated based on the idea of the conventional operational matrices. After collocation, the approximate solutions obtained showed that the proposed methods were good and converged to the exact solution.

Uwaheren et al., (2021) applied Legendre Galerkin method for solving fractional integro-differential equations of Fredholm type. Using the equation of the problem the authors were able to reduce the errors of the approximate solution without the use of another method of linearization on the non-linear part of the problem. Oyedepo et al., (2021) worked on the modified homotopy perturbation technique on fractional integro-differential difference equations and and had good results that converged to the exact solution.

Owolanke et al., (2019) applied exponentially fitted collocation method to solve singular multi-order fractional integro-differential equations. In the work, Canonical polynomials were constructed and used as basis functions for the solution of slightly perturbed singular multi-order integr-differential equation and the experiment produced very good results as compared with the exact solutions.

Falade (2019) Solved integro differential equations using exponential fitted collocation approximate technique. In the work, 1^{st}, 2^{nd}, 3^{rd}, and 5^{th}–orders linear Volterra and Fredholm integro-differential equations were solved using the proposed technique. The authors concluded that the proposed technique was successfully used and the results compared favorably to the exact solution.

Owolanke et al., (2017) developed and implemented a new two-step hybrid method for the solution of general second order ordinary differential equation. The research used eight order two-step Taylor series algorithm and the obtained equation were collocated at all grid and off-grid points to get the required approximate solution.

Amer, Saleh, Mohamed, and Abdelrhman (2013) worked on solution of linear and nonlinear boundary value problems fourth-order fractional integro-differential equations using Variation iteration method (VIM) and Adomian decomposition method (ADM). The authors converted the integro-differential equations to infinite series of convergent equations by two methods and Compared the results obtained with the exact solutions. The results showed that the proposed methods are accurate and efficiency.

Some relevant terms used in this work are briefly defined below

Integro-differential Equation: An integro-differential equation is an equation that involves both integrals and derivatives of a function. The general order of linear integro-differential equation is expressed as

$$u^{(n)}(x)+\int_{x_0}^{x_n} f(x,u(t))dt = g(x,u(x)), \quad u(x_0)=u_0 \tag{1}$$

Fractional Integro-differential Equation: A Fractional Integro-differential equation is similar to the conventional integro-differential equation. The general fractional order integro-differential equation (Weibeer, 2005) is expressed as:

$$\begin{aligned} D^{\alpha}u(x) &+ \int_{x_0}^{x_n} k(x,t)f(x,u(t))dt \\ &= g(x,u(x)), u(x_0) \\ &= u_0 \end{aligned} \tag{2}$$

where $u(x)$ is the unknown function, $D^{\alpha}(x)$ is the derivative, $k(x,t)$ is the kernel of the problem and x_0 and x_n are the limits of the integral. The fractional derivative D^{α} of a function $f(t)=t^n$ with an arbitrary parameter α is given by

$$D^{\alpha}(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha} \tag{3}$$

Higher Order Fractional Integro-differential Equations: The general Nth order Fractional integro-differential equations is of the form

$$\begin{aligned} & D^{\alpha}y(x) + D^{\alpha-1}y(x) + \cdots + y(x) \\ & \quad + \lambda \int_{x_0}^{x_n} k(x,t)y(t)dt = f(x) \qquad x \\ & \quad \in [a,b] \end{aligned} \tag{4}$$

together with the initial conditions

$$\begin{aligned} y(a) = \alpha_0, \quad & y^{i}(a) \\ &= \alpha_1, y^{ii}(a) \\ &= \alpha_2, \cdots, \quad y^{n-1}(a) \\ &= \alpha_{n-1} \end{aligned} \tag{5}$$

Where $k(x,t)$ is a kernel functions, $f(x)$ is a given function and $y(x)$ is the unknown functions to be determined.

Collocation Method: Collocation method is a numerical technique which involves selecting a finite-dimensional space of potential solutions (often polynomials of a specific degree) and a set of points in the domain, known as collocation points. The trial solution is then chosen such that it satisfies the given equation at these collocation points. It simply means evaluating a problem at equally spaced interval in the domain of consideration.

Basic Chebyshev Polynomials: The Chebyshev polynomials of the first kind and of degree k are defined on the interval $[-1,1]$ as: $T_k(x) =$

$\cos(k\cos^{-1}(x))$ The recurrence relation is given as:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, 3, \cdots \quad (6)$$

The shifted Chebyshev polynomials of degree n on the closed interval $[0,1]$ is defined as: $T_n^*(x) = T_n(2x-1)$ The recurrence formula on the closed form interval $[0,1]$ is:

$$T_{n+1}^*(x) = 2(2x-1)T_n^*(x) - T_{n-1}^*(x), \quad n > 1 \quad (7)$$

Also, a few terms are listed thus:

$$\begin{aligned}
T_0^*(x) &= 1 \\
T_1^*(x) &= 2x - 1 \\
T_2^*(x) &= 8x^2 - 8x + 1 \\
T_3^*(x) &= 32x^3 - 48x^2 + 18x - 1 \\
T_4^*(x) &= 128x^4 - 256x^3 + 100x^2 - 32x + 1 \\
T_5^*(x) &= 512x^5 - 128x^4 + 1120x^3 - 400x^2 + 50x - 1.etc
\end{aligned} \quad (8)$$

Basic Legendre Polynomials: The well-known Legendre polynomials P(s) is defined in the interval [-1,1] by the Rodrigues' formula:

$$P_n(s) = \frac{1}{2^n n!}\frac{d^n}{ds^n}[(s^2-1)^n] \quad (9)$$

where, $P_0(s) = 1$ and $P_1(s) = s$ for $n = 0, 1$. It will be observed that an n^{th} derivative of the formula has to be carried out before a polynomial of degree n is obtained at all times. That process is however time consuming and so a recurrence formula to obtain the polynomial for $n > 2$ is given as:

$$P_{n+1}(s) = \frac{(2n+1)}{n+1}sP_n(s) - \frac{n}{n+1}P_{n-1}(s) \quad (10)$$

To transform (2.12) to shifted Legendre polynomial in the interval [0,1], set $s = 2t - 1$ where $t \in [0,1]$, then $P_n(s) = L_n(2t-1)$ and $L_{n+1}(t)$ will be defined as:

$$\begin{aligned}
L_{n+1}(t) &= \frac{(2n+1)(2t-1)}{n+1}L_n(t) \\
&- \frac{n}{n+1}L_{n-1}(t), \quad n \\
&= 1, 2, ...,
\end{aligned} \quad (11)$$

A few numbers of the Legendre polynomials are given as follows:

$$\begin{aligned}
L_0(t) &= 1 \\
L_1(t) &= (2t-1) \\
L_2(t) &= (6t^2 - 6t + 1) \\
L_3(t) &= (20t^3 - 30t^2 + 12t - 1) \\
L_4(t) &= (70t^4 - 140t^3 + 90t^2 - 20t + 1)
\end{aligned} \quad (12)$$

METHODOLOGY

Exponential-fitted collocation method

We considered the general class of Fredholm Integro-differential equation given in equations (4) and (5):

To find a single approximation solution to the equations, we assumed a trial solution of the form

$$\begin{aligned}
u_N(x) &= \sum_{i=0}^{N} a_i T_i^*(x) \\
&= a_0 + a_1(2x-1) + a_2(8x^2 - 8x + 1) \\
&+ a_3(32x^3 - 48x^2 + 18x - 1) \\
&+ a_4(128x^4 - 256x^3 + 100x^2 - 32x + 1) \\
&+ a_5(512x^5 - 128x^4 + 1120x^3 - 400x^2 \\
&+ 50x - 1) + \cdots
\end{aligned} \quad (13)$$

where a_i are the unknown constants to be determined and $T_i^*(x)$ are shifted Chebyshev polynomial An exponentially fitted approximate term proposed by Falade (2019) is modified here to be,

$$\begin{aligned}
H_N(x) &= \sum_{i=0}^{n}\left(a_i L^*(x) + \tau_i e^x\right) \\
&= \sum_{i=0}^{\lceil\alpha\rceil}\left(a_i L_{N-n-i}(x) + \tau_i e^i\right) \\
&\text{a} \le x \le b
\end{aligned} \quad (14)$$

where L^* is the perturbation polynomial in terms of the shifted Legendre polynomials, $\lceil\alpha\rceil$ is called the ceiling α, nearest upper or lower integer to α and τ_i are free tau parameters.

Substituting equation (11) into equation (4), gives

$$\begin{aligned}
&D^\alpha\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\
&+ D^{\alpha-1}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\
&+ \cdots + \left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\
&+ \lambda\int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right)dt \\
&= g(x)
\end{aligned} \quad (15)$$

We slightly perturb equation (8) by adding $H_N(x)$ defined in equation (12) and we have

$$\begin{aligned} D^{\alpha}&\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\ &+ D^{\alpha-1}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\ &+ \cdots + \left(\sum_{i=0}^{N} a_i T_i^*(x)\right) \\ &+ \lambda \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right)dt \\ &+ \sum_{i\ =0}^{\lceil\alpha\rceil}\left(a_i L_{N-n-i}(x) + \tau_i e^i\right) = g(x) \end{aligned} \tag{16}$$

where n is the order of the problem and N is the degree of the assumed approximant

We collocation equation (14) at an equi-distance points x_k: $x_k = a + \frac{(b-a)k}{N}$,
for $k = 0, 1, 2, \cdots N$ to get

$$\begin{aligned} D^{\alpha}&\left(\sum_{i=0}^{N} a_i T_i^*(x_k)\right) \\ &+ D^{\alpha-1}\left(\sum_{i=0}^{N} a_i T_i^*(x_k)\right) \\ &+ \cdots + \left(\sum_{i=0}^{N} a_i T_i^*(x_k)\right) \\ &+ \lambda \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right)dt \\ &+ \sum_{i\ =0}^{\lceil\alpha\rceil}\left(a_i L_{N-n-i}(x_k) + \tau_i e^i\right) = g(x_k) \end{aligned} \tag{17}$$

Equation (15) give rise to $(N + 1)$ algebraic system of equations at $k = 0, 1, 2, \cdots N$ with $(N + 1)$ unknown constants.

Additional n equations are obtained from the initial conditions given in equation (5) so that altogether we have $(N + n + 1)$ equations.

However, we can achieve the $(N + n + 1)$ system of equations if equation (15) is collocated straight forward at $x_k = a + \frac{(b-a)k}{(N+n)}$; $k = 0, 1, 2, \cdots (N + n)$

Thus, unique values of the constants $a_0, a_1, a_2, \cdots, a_N, \tau_0, \tau_1, \tau_2, \cdots, \tau_n$ are obtained by solving system of equations. The values are substituted back into equation (6) to obtain a single polynomial approximation which is the required exponential-fitted approximate solution.

Numerical Examples

In this section, we solve four examples following the proposed methodology

Example 1

Consider the seven-order Fredholm integro-differential equations

$$y^{(7)}(x) - \int_0^1 x^2 y(t)dt = -8e^x + x^2 + e^x(1-x) \tag{18}$$

subject to the initial conditions

$$\begin{aligned} y(0) = 1,\quad & y^{i}(0) \\ &= 0, y^{ii}(0) \\ &= -1,\quad y^{iii}(0) \\ &= 0,\quad y^{iv}(0) \\ &= -1,\quad y^{v}(0) \\ &= 0, \\ & y^{vi}(0) \\ &= 1 \end{aligned} \tag{19}$$

The exact solution is $y(x) = cosx$

To solve equation (16) for $N = 8$, $n = 7$ and at $\alpha = \frac{15}{2}$, the assumed approximate solution, equation (11) together with exponential-fitted perturbation solution, equation (12) are substituted into it. After some necessary simplifications in accordance with the proposed method algorithm, we obtained the values of the constant coefficients:

$$\begin{aligned} a_0 &= 0.6634996325 \quad a_1 \\ &= -0.4780893692 \quad a_2 \\ &= -0.1622098711 \quad a_3 \\ &= -0.0220799461 \\ a_4 & \\ &= -0.00191577123 \quad a_5 \\ &= -0.0001225927 \quad a_6 \\ &= -0.0000062252 \quad a_7 \\ &= -2.6206 \times 10^{-7} \\ a_8 & \\ &= -9.4413 \times 10^{-9} \qquad \tau_1 \\ &= -2.8835 \times 10^{-8} \quad \tau_2 \\ &= 3.2820 \times 10^{-8} \quad \tau_3 \\ &= -1.0352 \times 10^{-7} \\ \tau_4 & \\ &= 1.3676 \times 10^{-7} \quad \tau_5 \\ &= -2.2008 \times 10^{-7} \quad \tau_6 \\ &= 9.0765 \times 10^{-7} \quad \tau_7 \\ &= 1.4729 \times 10^{-5} \end{aligned}$$

Substituting the values of $a_i \quad (0 \le i \le 7)$ obtained into equation (11) and after simplification gives

$$\begin{aligned} y_8(x) &= -0.000003400739338\,x - 0.4001700368\,x^2 \\ &- 0.3333900124\,x^3 - 0.1250141698\,x^4 - \\ &0.03333616728\,x^5 - 0.006944916766\,x^6 \\ &- 0.0011905436\,x^7 - 0.0001736275128\,x^8 \end{aligned} \quad (20)$$

Also, solving equation (16) for $N = 8$, $n = 7$ at $\alpha = \frac{15}{2}$, and following the same process, we obtained the values the constants as follows:

$$\begin{aligned} a_0 &= 0.8639999325 \quad a_1 \\ &= -0.6480951112 \quad a_2 \\ &= -1.1644008711 \quad a_3 \\ &= -0.02205454261 \\ a_4 & \\ &= -0.040194219 \quad a_5 \\ &= -3.0601225928 \quad a_6 \\ &= -0.0000062302 \quad a_7 \\ &= -2.31625 \times 10^{-6} \\ a_8 & \\ &= -8.4543 \times 10^{-7} \quad \tau_1 \\ &= -2.8586 \times 10^{-8} \quad \tau_2 \\ &= 2.2428 \times 10^{-8} \quad \tau_3 \\ &= -1.14203 \times 10^{-6} \\ \tau_4 & \\ &= 1.7536 \times 10^{-7} \quad \tau_5 \\ &= -2.2520 \times 10^{-7} \quad \tau_6 \\ &= 4.0697 \times 10^{-7} \quad \tau_7 \\ &= 3.47317 \times 10^{-5} \end{aligned}$$

Substituting the values of a_i into equation (11) and after simplification gives

$$\begin{aligned} y_8(x) &= -0.03400739338\,x - 0.5001700368\,x^2 \\ &- 0.3333900124\,x^3 \\ &- 0.1250141698\,x^4 - 0.033336 \\ &16728\,x^5 - 0.006944916766\,x^6 \\ &- 0.0011905436\,x^7 - 0.00017362751\,x^8 \end{aligned} \quad (21)$$

Equations (19) and (20) are the required approximate solutions for example 1

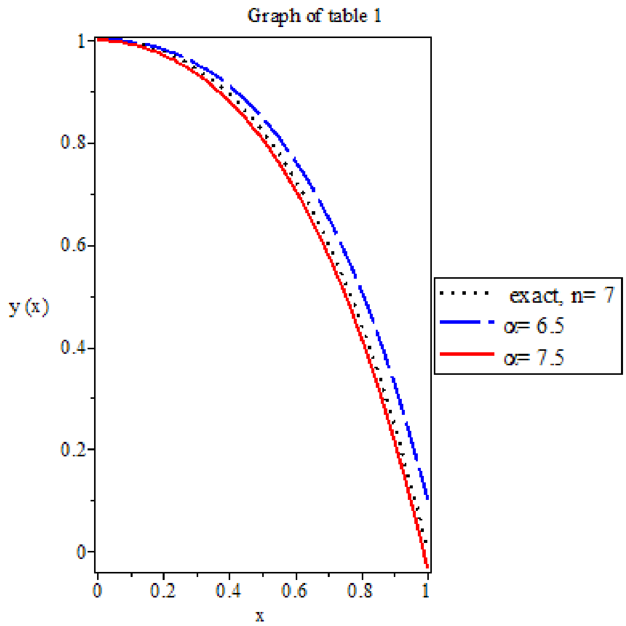

Figure 1: Graph representation of example 1

Example 2

Consider the eight-order Fredholm integro-differential equations

$$y^8(x) - y(x) - \int_0^1 y(t)dt = -8e^x + x^2 \quad (22)$$

subject to the initial conditions

$$\begin{aligned} y(0) &= 0, \quad y^i(0) \\ &= -1, \quad y^{ii}(0) \\ &= 0, \quad y^{iii}(0) \\ &= -1, \quad y^{iv}(0) \\ &= 0, \quad y^v(0) \\ &= 1, \\ y^{vi}(0) & \\ &= 0, \quad y^{vii}(0) \\ &= -1 \end{aligned} \quad (23)$$

The exact solution is $y(x) = sin(x)$

Solving equation (21) for $N = 12$, $n = 8$, at $\alpha = \frac{15}{2}$ and $\alpha = \frac{17}{2}$, using the assumed approximate solution, equation (11) together with exponential-fitted perturbation equation (12) and following the same procedure, we obtained respectively the required approximate solutions as follows:

$$\begin{aligned} y_{12}(x) &= 0.9999889\,x - 0.00000007\,x^2 \\ &- 0.1666666601\,x^3 - 0.000000001798\,x^4 \\ &- 0.00724983328\,x^5 \\ &- 6.94491 \times 10^{-11}\,x^6 \\ &- 0.0001499989\,x^7 - 0.000000017362\,x^8 \\ &- 0.0000002116\,x^9 - 0.00000254876\,x^{10} \\ &- 0.0000000250522\,x^{11} \\ &- 0.000000001642\,x^{12} + 0.0009996599 \end{aligned} \quad (24)$$

Table 1: Error of Results for Example 1

x	Exact	$\alpha = \frac{13}{2}$ EFC	Error	$\alpha = \frac{15}{2}$ EFC	Error
0.0	1.000000	1.000000	0.0000	1.000000	0.0000
0.1	0.994653	0.994653	4.90e-10	0.994653	5.00e-10
0.2	0.977122	0.977122	4.10e-10	0.977121	2.32e-10
0.3	0.944901	0.944901	1.30e-10	0.944901	3.04e-09
0.4	0.895094	0.895094	1.20e-09	0.895095	3.50e-09
0.5	0.824360	0.824360	3.50e-08	0.824360	3.10e-09
0.6	0.728847	0.728847	3.70e-07	0.728847	2.22e-09
0.7	0.604125	0.604125	3.82e-07	0.604125	1.21e-08
0.8	0.445108	0.445106	4.20e-06	0.445108	3.19e-08
0.9	0.245960	0.245961	4.10e-06	0.245960	1.60e-07
1.0	0.999847	0.999846	1.64e-06	0.999847	1.36e-07

Table 2: Error of Results for Example 2

x	Exact	$\alpha = \frac{15}{2}$ EFC	Error	$\alpha = \frac{17}{2}$ EFC	Error
0.0	0.000000	0.000000	1.3397e-10	0.001340	1.3397e-10
0.1	0.099833	0.099833	1.3742e-10	0.099833	1.3742e-10
0.2	0.198669	0.198669	1.4078e-09	0.198669	1.4078e-10
0.3	0.295520	0.295520	1.4175e-09	0.295520	1.4175e-10
0.4	0.389418	0.389418	1.3430e-09	0.389418	1.3430e-09
0.5	0.479426	0.479426	1.0682e-08	0.479426	1.0682e-08
0.6	0.564642	0.564642	4.0220e-08	0.564642	4.0220e-08
0.7	0.644218	0.644218	9.3852e-08	0.644218	9.3852e-08
0.8	0.717356	0.717356	3.3490e-07	0.717356	3.3490e-06
0.9	0.783327	0.783317	7.3537e-07	0.783329	6.1367e-06
1.0	0.841471	0.841398	1.3625e-06	0.841471	1.3425e-05

$$
\begin{aligned}
y_{12}(x) = {} & 0.99988888\,x - 0.0000990007\,x^2 \\
& - 0.16663369833\,x^3 \\
& - 0.0000000016\,x^4 - 0.0025983328\,x^5 \\
& - 6.94491 \times 10^{-11}\,x^6 - 0.1000428993\,x^7 \\
& - 0.0000000136\,x^8 - 0.0000002692\,x^9 \\
& - 0.000000025476\,x^{10} \\
& - 0.00000024505211110002\,x^{11} \\
& - 0.00801642\,x^{12} + 0.000999999
\end{aligned} \tag{25}
$$

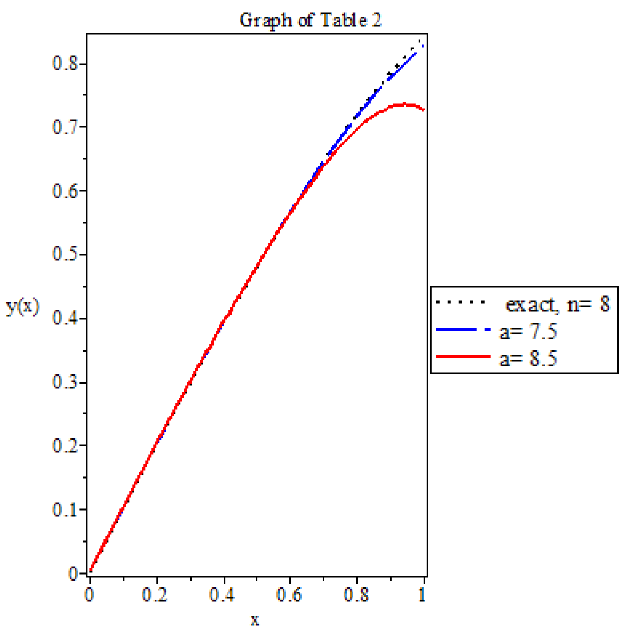

Figure 2: Graph representation of example 2

Example 3

Consider the tenth-order Fredholm integro-differential equations

$$y^{10}(x) - \int_0^{\frac{\pi}{2}} (x^3 + t^2 sinx) y^{vii}(t) dt = cosx - \left(1 - \frac{\pi^2}{4}\right) sinx + 2x^3 + (\pi - 4) \quad (26)$$

subject to the initial conditions

$$y(0) = -1, \quad y^i(0) = 1, \quad y^{ii}(0) = 1, \quad y^{iii}(0) = -1, \quad y^{iv}(0) = -1, \quad y^v(0) = 1, \quad y^{vi}(0) = 1, \quad y^{vii}(0) = -1 \quad y^{viii}(0) = -1 \quad y^{ix}(0) = 1 \quad (27)$$

The exact solution is $y(x) = sinx - cosx$
Solving equation (25) for $N = 12$, $n = 10$, at $\alpha = \frac{19}{2}$ and $\alpha = \frac{21}{2}$, using the assumed approximate solution, equation (11) together with exponential-fitted perturbation equation (12) and following the same procedure, we obtained respectively the required approximate solutions as follows:

$$\begin{aligned} y_{12}(x) = {} & 0.999988999\,x - 0.50000007003898\,x^2 \\ & - 0.1666690012564\,x^3 \\ & - 0.041000000012698\,x^4 \\ & + 0.00833528\,x^5 + 0.001494491\,x^6 \\ & - 0.00019998999\,x^7 - 0.00024000173\,x^8 \\ & + 0.00091191100222\,x^9 \\ & - 0.0000000254876\,x^{10} \\ & - 0.0000000111102\,x^{11} \\ & - 0.000000001642\,x^{12} - 1.0009996599 \end{aligned} \quad (28)$$

$$\begin{aligned} y_{12}(x) = {} & 0.9888998998\,x - 0.490099000700368\,x^2 \\ & - 0.1667336983900\,x^3 \\ & - 0.04200000001698\,x^4 \\ & + 0.0083328\,x^5 + 0.0013694491\,x^6 \\ & - 0.0001428999936\,x^7 \\ & - 0.0000250173627\,x^8 + 0.0909090222\,x^9 \\ & - 0.0000000254876\,x^{10} \\ & - 0.090000000021111\,x^{11} \\ & - 0.00801642\,x^{12} - 1.00090099999 \end{aligned} \quad (29)$$

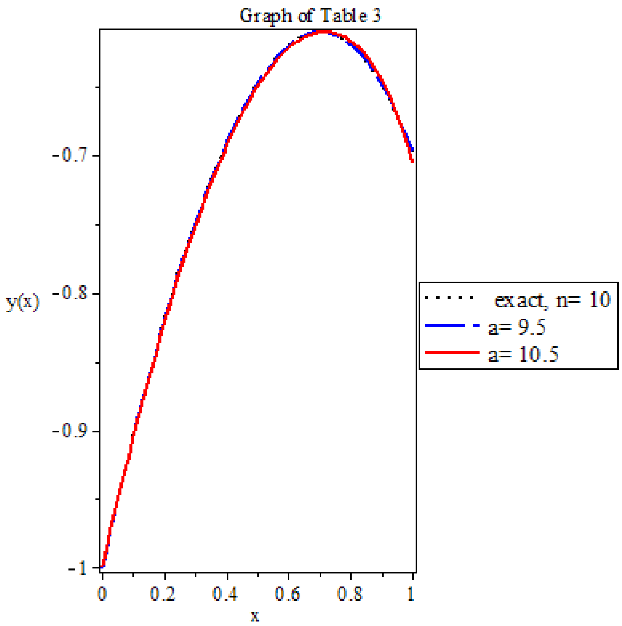

Figure 3: Graph representation of example 3

Example 4
Consider the third-order Fredholm integro-differential equations

$$y^{(3)}(x) = sinx + x - \int_0^{\frac{\pi}{2}} (xt) y'(t) dt \quad (30)$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1 \quad (31)$$

The exact solution is $y(x) = cosx$
Solving equation (29) for $N = 6$, $n = 3$, at $\alpha = \frac{5}{2}$ and $\alpha = \frac{7}{2}$, using the assumed approximate solution, equation (11) together with exponential-fitted perturbation equation (12) and following the procedure of proposed method, we obtained respectively the required approximate solutions as follows:

$$\begin{aligned} y_6(x) = {} & 0.0001200005\,x + 0.50000007368\,x^2 \\ & - 0.000000166669\,x^3 + 0.041007500016\,x^4 - 0.00833728\,x^5 - 0.0013777725\,x^6 \\ & + 1.0009996599 \end{aligned} \quad (32)$$

$$\begin{aligned} y_6(x) = {} & 0.000112166\,x + 0.4990700368\,x^2 \\ & - 0.0000166733698\,x^3 \\ & + 0.0410000001698\,x^4 - 0.0083329\,x^5 - 0.00138888\,x^6 + 1.0000099999 \end{aligned} \quad (33)$$

Table 3: Error of Results for Example 3

x	*Exact*	$\alpha = \frac{19}{2}$ EFC	Error	$\alpha = \frac{21}{2}$ EFC	Error
0.0	-1.000000	-1.000000	0.0000	-1.000000	0.0000
0.1	-0.895171	-0.895170	2.00e-10	-0.895170	2.00e-10
0.2	-0.781397	-0.781397	2.12e-10	-0.781397	1.45e-10
0.3	-0.659816	-0.659816	2.00e-10	-0.6598162	2.00e-10
0.4	-0.531642	-0.531642	1.40e-09	-0.531642	1.40e-09
0.5	-0.398157	-0.398157	1.07e-08	-0.398157	1.07e-08
0.6	-0.260693	-0.260693	5.64e-08	-0.260693	5.64e-08
0.7	-0.120624	-0.120624	2.60e-07	-0.120624	2.69e-08
0.8	0.020649	0.020648	7.57e-07	0.0206486	7.57e-07
0.9	0.161717	0.161714	2.19e-06	0.161714	2.19e-06
1.0	0.301168	0.301163	5.66e-06	0.3011630	5.66e-06

Table 4: Error of Results for Example 4

x	*Exact*	$\alpha = \frac{5}{2}$ EFC	Error	$\alpha = \frac{7}{2}$ EFC	Error
0.0	1.000000	1.000099	9.996e-05	1.000010	1.000e-05
0.1	1.005004	1.006016	1.010e-05	1.005015	1.120e-05
0.2	1.020068	1.020108	1.017e-05	1.020059	1.076e-05
0.3	1.045342	1.045346	1.005e-05	1.045270	7.080e-05
0.4	1.081069	1.082006	9.364e-04	1.080864	2.054e-04
0.5	1.127596	1.128341	7.440e-04	1.127113	4.827e-04
0.6	1.185355	1.185673	3.182e-04	1.184342	1.012e-03
0.7	1.254868	1.254366	5.019e-04	1.252912	1.955e-03
0.8	1.336738	1.334799	1.939e-03	1.333202	3.535e-03
0.9	1.431644	1.427357	4.287e-03	1.425597	6.046e-03
1.0	1.540333	1.532412	7.921e-03	1.530468	9.864e-03

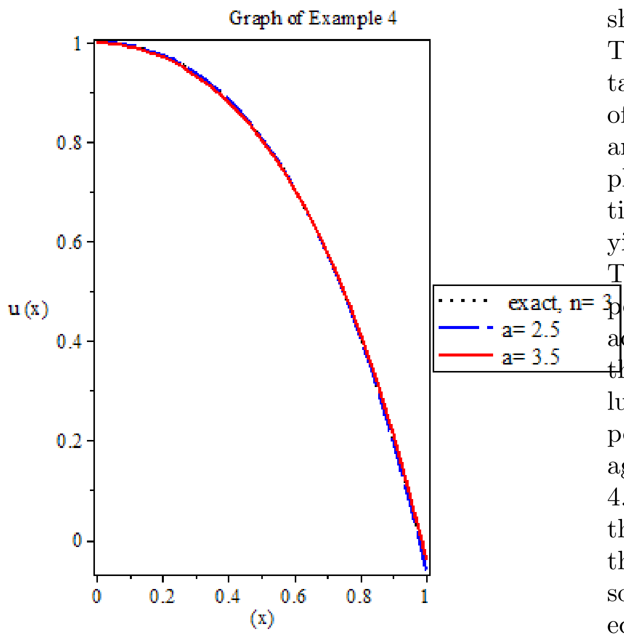

Figure 4: Graph representation of example 4

CONCLUSION

Exponential-fitted collocation methods have been discussed and used successfully to solve higher order fractional integro-differential equations using shifted Chebyshev polynomial as basis function. Three examples were solved and the results obtained are tabulated in tables 1-3. From the tables of results, it is observed that the results obtained are close to the exact solution for the three examples considered. It means that fractional differentiation at $\lceil\alpha\rceil^-$ and $\lceil\alpha\rceil^+$ where $\lceil\alpha\rceil^- \leq n \leq \lceil\alpha\rceil^+$ yield good result close to the exact solution at n. The basis functions applied and the one used to perturb performed very well in terms of accuracy achieved. However, from the graphs, it can be seen that the results obtained agreed with the exact solution more closely at lower values than the upper values on the x-axis. Still, the results obtained agreed more closely throughout in problems 3 and 4. This maybe due to the transcendental nature of the problems and the exact solution. It is concluded that the proposed method is an effective tool for solving higher order fractional integro-differential equations and the derivatives of the class of problems considered yield same results whether fractional or integer orders.

REFERENCES

Ajileye, G., Adiku, L., Auta, J. T., Aduroja, O. O., and Oyedepo, T. (2024). linear and

nonlinear Fredholm integro- differential equations: An application of collocation method. *Journal of Fractional Calculus and Applications,* 15(2) 6, 1-10
http://jfca.journals.ekb.eg/

Ajileye G., and Amoo, A. S. (2023). Numerical solution to Volterra integro-differential equations using collocation approximation. *Mathematics and Computational Sciences,* 4(1), 2023, 1-8
DOI:10.30511/mcs.2023.1978083.1099

Aduroja, O. O., Adiku, L., and Gbolade, O. (2023). Collocation approximation method for the numerical solution of the Volterra integro-differential equations. *Fuoye Journal of Management, Innovation and Entrepreneurship,* 2(1) 2023, 240-245.
https://fuoye-jmie.com ISSN:2814-2578.

Al-Zuhairi W. K, Al-Humedi H. O. (2024): Reproducing Kernel Hilbert Space Method for Solving Nonlinear Integro-Differential Equations. Journal of Basrah Resources (sciences) 50 (2) 182 (2024) DOI: https://doi.org/10.56714/bjrs.50.2.15

Alshbool, M.H.T., Mutaz Mohammad, Osman Isik and Ishak Hashim c (2022) Fractional Bernstein operational matrices for solving integro-differential equations involved by Caputo fractional derivative. Results in Applied Mathematics journal. www.elsevier.com/locate/results-in-applied-mathematics

Amer, S.M. Saleh, M. H. Mohamed, M. A. and Abdelrhman, N. S. (2013): Variational Iteration Method and Adomian Decomposition Method for Fourth-Order Fractional Integro-Differential Equations. International Journal of Computer Applications (0975 – 8887) 80 (6), 2013.

Falade, K. I. (2019). Solving integro differential equations using exponential fitted collocation approximate technique. *Middle East Journal of Science.*, 5**(1)**: 73-85.
https://doi.org/10.23884/mejs.2019.5.1.08

Falade K. I. and Taiwo O. A. (2023): Application of Exponentially Fitted Collocation Algorithm for Solving Nth-Order Fredholm Integro-differential Equations. Journal of Science and Technology 15 (1) (2023) 29-42
http://penerbit.uthm.edu.my/ojs/index.php/jst

Olotu O. T, Salaudeen K. A, Folaranmi R. O., Omole E. O. and Odetunde O (2025): An in-deth study of the application of the differential transform method for effectively solving various-order ordinary differential equations. FUDMA Journal of Sciences (FJS). (9) 4, 2025, 347 - 356. DOI: https://doi.org/10.33003/fjs-2025-0904-3491

Owolanke, A. O., Uwaheren, O. A., and Obarhua F. O. (2017): An Eight Order Two-Step Taylor Series Algorithm for the Numerical Solutions of Initial Value Problems of Second Order Ordinary Differential Equations. Open Access Library Journal, 4: e3486. https://doi.org/10.4236/oalib.1103486

Owolanke, A. O., Uwaheren, O. A. and Taiwo, O. A. (2019): Exponentially fitted collocation method for solving singular multi-order fractional integro differential equations. ASUU Journal of Research and Development. 6 (1 & 2) 2019. 10-23
https://ejournals.asuu.org.ng/archives.*php*? *page* = 4&*type* = 3

Oyedepo, T., Uwaheren, O. A., Okperhie, E. P., and Peter, O. J., (2021). Solution of fractional integro-differential equation using modified homotopy perturbation technique and constructed orthogonal polynomials as basis functions. *Journal of Science Technology and Education,* 7(3), 157-164.
http://www.atbuftejoste.net/index.php/joste/issue/view/36
https://www.atbuftejoste.com

Weilbeer, M. (2005): Efficient Numerical Methods for Fractional Differential Equations and their Analytical Background.

Uwaheren, O. A., Adebisi, A. F., Ishola C.Y., Rajji, M. T., Yekeem, A. O, and Peter, O. J. (2022). Numerical Solution of Volterra Integro-differential Equations by Akbari-Ganji's Method. *BAREKENG: Journal of Mathematics and its Application,* 16 (3) 1123-1130.
https://ojs3.unpatti.ac.id/index.php/barekeng/issue/view/72
https://doi.org/10.30598/barekengvol16iss3pp1123-1130

Uwaheren, O. A., Adebisi, A. F., Olotu, O. T., Etuk, M. O., and Peter, O. J. (2021). Legendre Galerkin Method for Solving Fractional Integro-Differential Equations of Fredholm Type. *Aligarh Bulletin of Mathematics, Aligarh Muslim University, Aligarh, India,* 40(1) 15-27.
http://tabmaths.com/issues/Tabmath-Vol40-Number-1-2021/5-Uwaheren.pdf