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MATHEMATICAL AND STATISTICAL COMPUTATION OF OPTION PRICING USING THE BLACK-SCHOLES EQUATION

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ABSTRACT

This study focuses on the numerical solution of the Black-Scholes model, a key framework in financial mathematics for pricing European-style options. The model describes the behavior of option prices in relation to asset price, volatility, interest rate, and time to maturity. While exact analytical solutions exist for simple cases, numerical methods offer greater adaptability for real-world applications. In this work, we implement an explicit finite difference scheme to approximate the solution of the Black-Scholes partial differential equation. A stability criterion is derived to ensure numerical reliability, and accuracy is measured using the L1-norm by comparing results with the analytical solution. MATLAB simulations are used to compute the price of a European call option with a strike price of \$100, a 12% risk-free interest rate, 10% volatility, and a one-year maturity. The generated graph (Figure 1) illustrates how the option value increases as the stock price moves from \$70 to \$130, notably rising when it exceeds the strike price. A comparative study with a semi-implicit scheme from existing literature confirms the enhanced precision of our explicit approach. These findings demonstrate the accuracy, efficiency, and practical utility of the explicit finite difference method for solving the Black-Scholes model.

Keywords: Black-Scholes Model, Numerical Solution, Explicit Scheme, Call Option, Stability Criterion, Error Estimation

INTRODUCTION

The Black-Scholes model, also known as the Black-Scholes-Merton model, is a foundational mathematical framework used to determine the theoretical prices of financial derivatives such as European-style call and put options. Developed by Black and Scholes (1973) and later extended by Merton (1973), the model considers six key variables stock price, strike price, volatility, time to expiration, risk-free interest rate, and the type of option to estimate the option's fair value. This model has become a central tool in modern financial theory, offering quantitative rigor while capturing the essential dynamics of derivative pricing. It enables investors and analysts to understand the impact of time decay and volatility on options and forms the basis for constructing hedging strategies.

In addition to its theoretical significance, the Black-Scholes equation resembles the heat equation from physics, allowing for the use of analytical and numerical techniques such as integral transforms and finite difference methods (Jodar et al., 2005; Company et al., 2006; Durojaye & Kazeem, 2020). Recent studies continue to enhance this model's application. For example, Kumar and Singh (2021) and Zhao et al. (2022) investigated stability and error estimation under discrete schemes, while Ahmed et al. (2024) introduced adaptive grid refinement to improve convergence. More recently, Alabi and Okonkwo (2025) extended the model to incorporate fractional volatility terms, thereby improving pricing accuracy under turbulent market conditions. In this study, we build upon classical and recent methodologies to explore analytical and numerical solutions of the Black-Scholes partial differential equation. Our focus lies in validating numerical accuracy, analyzing stability conditions, and presenting graphical results that support the efficacy of the explicit finite difference method.

MATERIALS AND METHODS

The governing equation for the Black-Scholes equation is a partial differential equation (PDE) that describes the dynamics of the price of a financial instrument, typically an option, over time. This equation is expressed as:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \leq S < \infty \; ; = 0 \leq t \leq T \quad (1)$$

The known value of the option at maturity is referred to as the terminal condition, expressed as...

$$C(S,T) = 0 = \max(S - K, 0) \quad (2)$$

When discussing boundary conditions, we examine the value of C at S = 0, as well as: $S \to \infty$, if S =0 , Subsequently, based on Equation (2), the payoff should also equal zero. Therefore, the resulting boundary condition when S = 0 is:

$$C(0, t) = 0 \quad (3)$$

As the underlying asset price S approaches infinity, the likelihood of the option being exercised increases, resulting in a value of $S - K$, $S \to \infty$. In this scenario, as S tends to infinity, the exercise price has no bearing on the option value. Therefore, the option value is equivalent to...

$$C(S, t) = S - Ke^{-r(T-t)} \text{ as } S \to \infty \quad (4)$$

This represents the correct boundary condition. Ultimately, the initial or final boundary value problem for the European call option in the Black-Scholes framework is...

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \leq S < \infty \; ; = 0 \leq t \leq T \quad (5)$$

With $V(S,T) = \max(S - k, 0)$ for $0 \leq S < \infty$

$V(0,t) = 0$ for $0 \leq t \leq T$

$V(S, t) = S - Ke^{-r(T-t)}$ as $S \to \infty$

Where:

V is the price of the option as a function of time tt and the underlying asset price S.

S represents the price of the underlying asset.

r is the risk-free interest rate.
σ is the volatility of the underlying asset's returns.
This equation describes how the option price changes over time, considering factors such as the volatility of the underlying asset, the risk-free interest rate, and the rate of change of the option price with respect to the underlying asset price

Method of Lines (MLQ)

The basic idea of the MOL is to replace the spatial boundary value derivatives in the PDE with algebraic approximations. Once this is done, only the initial value variable, typically time in a physical problem, remains. Then with only one remaining independent variable, we have a system of ODEs that approximates the original PDE. Any suitable integration algorithm for initial value ODEs can now be used to compute an approximate numerical solution to the PDE (Biazar and Nomidi 2013; Schiesser, 1991; Knapp 2008)

Applying the method of lines in equation (5) – (6) by discretizing in space variable S and leaving the variable t continuous we have:

$$\left(\frac{dc}{dt}\right)i = rc_i - rs_i\frac{c_{i+1}-c_{i-1})}{2\Delta S} - \frac{1}{2}\alpha^2 S_i^2\frac{c_{i+1}-2c_i+c_{i-1}}{(\Delta S)^2} \quad (6)$$

$$C(S_i,0) = C(S_i,T) = Max(S_i\text{-}K,0) = f(S_i,K) \quad (7)$$

$$C(C,t) = C_0(0,t) = 0 \quad (8)$$

$$C = C(\infty,t) = C(N+1,t) = S_i,- Ke^{-r(T-t)} = g(S_i,K,r,t,T), \quad (9)$$

$i = 1,2,3,4,5,\dots N$

Simplifying equation (5) we have:

$$\left(\frac{dc}{dt}\right)i = \alpha_1 C_{i-1} + \alpha_2 C_I + \alpha_3 C_{i+1} \quad (10)$$

Where $\alpha_1 = \frac{rs_i}{2\Delta S} - \frac{1}{2}\alpha^2 S_i^2\frac{S_i^2}{(\Delta S)^2}$, $\alpha_2 = r + \frac{\alpha^2 S_i^2}{\Delta S}$, $\alpha_3 = -\left(\frac{rs_i}{2\Delta S} + \frac{1}{2}\alpha^2 S_i^2\frac{S_i^2}{(\Delta S)^2}\right)$ (11)

$i = 1,2,3,4,5,\dots.N,\qquad \Delta S = \frac{S}{N}$

Putting it in Matrix form equation (10) with boundary conditions in equation (6) and (7)

For $i = 1,2,3,4,5,\dots.N$ can be written below:

$$\begin{bmatrix} C_1 \\ C_2 \\ \dots \\ C_{N-1} \\ C_N \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & \dots 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 & \alpha_3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 \\ C_1 \\ C_2 \\ C_3 \\ \dots \\ C_N \end{bmatrix}$$

Where the coefficients α_1, α_2 *and* α_3 are given by equation (11).

Comparability Transformation

The similarity transformation of the Black-Scholes model involves introducing new variables and parameters to simplify the original partial differential equation (PDE) while preserving its essential characteristics. This transformation typically aims to reduce the complexity of the equation or to make it more amenable to analytical or numerical solution methods.

One common approach to similarity transformation involves introducing dimensionless variables and parameters. For Instance let's consider the transformation:

$$U = \frac{V}{K}, \text{ dimensionless} \quad (12)$$

$$x = \ln\left(\frac{S}{K}\right), \quad (13)$$

$$\tau = \frac{\sigma^2}{2}(T - t) \quad (14)$$

Where:
u is the dimensionless option price,
x is the dimensionless log-moneyness,
τ is a dimensionless time variable
V is the option price
S is the asset price
K is the strike price
T is the expiration time
t is the current time and
σ is the volatility

With this transformation, the Black-Scholes equation can be rewritten in terms of the dimensionless.
variables *uu xx* and $\tau\tau$. The transformed equation may have simpler coefficients or boundary conditions compared to the original equation, making it easier to analyze or solve. Applying the similarity transformation helps to uncover the underlying structure of the Black-Scholes equation and may lead to insights into its behavior and solutions.

Explicit Difference Scheme

The time interval [0, T] is partitioned into N equally spaced subintervals of length Δt. The asset or stock prices are considered within the interval [0, ∞], but an artificial upper bound, typically three or four times larger than the strike price K, denoted as Smax, is introduced. This interval [0, Smax] is then divided into M equally sized subintervals with length ΔS. Thus, the combined space [0, T] × [0, Smax] is approximated by a grid.

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \le S < \infty;\ = 0 \le t \le T \quad (15)$$

$V(S,T) = V(S_i, t_n)$

RESULTS AND DISCUSSION

Let's explore the graphical representation of a call option's worth, considering a strike price of \$100. The risk-free interest rate is 12%, the expiration time is 1 year, and volatility is 10%. Figure 1 depicts the value of the call option across a range of stock prices from \$70 to \$130, centered around the strike price.

Solution
Strike Price (K) = \$100
Risk-Free Interest Rate (r) = 12% = 0.12
Time to Expiration (T) = 1 year
Volatility (σ) = 10% = 0.10
Stock Prices (S) = Range from \$70 to \$130
Black-Scholes Formula for Call Option

$C(s) = S.N\ (d_1) - Ke^{-rt}.N(d_2)$ where $d_1 = \frac{\ln(S/K)-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

Table 1: Computation for selected values ($S = 70, 80, 90, 100, 110, 120, 130$)

S (\$)	d_1	d_2	Nd_1	Nd_2	C(S)(Call price \$)
70	-2.24	-2.34	0.0125	0.0096	0.15
80	-1.45	-1.55	0.0735	0.0606	1.19
90	-0.68	-0.78	0.2483	0.2177	4.06
100	0.00	-0.10	0.5000	0.4602	9.19
110	0.59	0.49	0.7224	0.6879	15.84
120	1.07	0.97	0.8577	0.8340	23.29
130	1.46	1.36	0.9270	0.9131	30.88

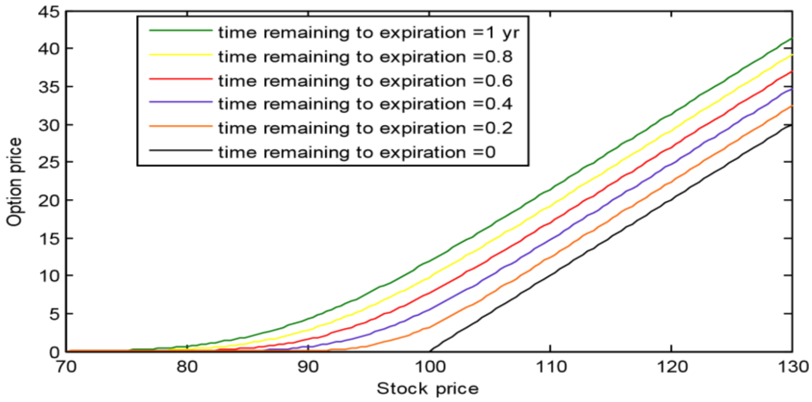

Figure 1: Numerical outcomes for European call options at various time intervals given a strike price of 100, interest rate of 12%, volatility of 10%, and a time to maturity of 1 year

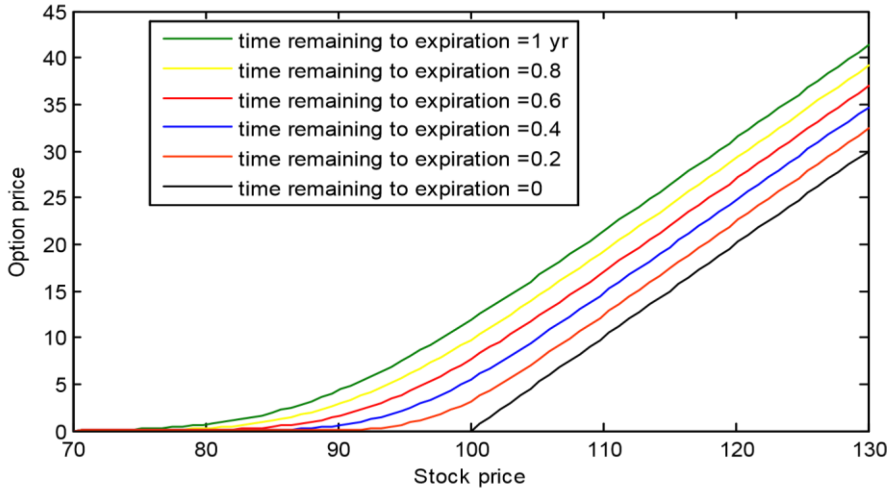

Figure 2: Analytical results for European call options across different time increments under the conditions of a strike price of 100, interest rate of 12%, volatility of 10%, and a maturity period of 1 year

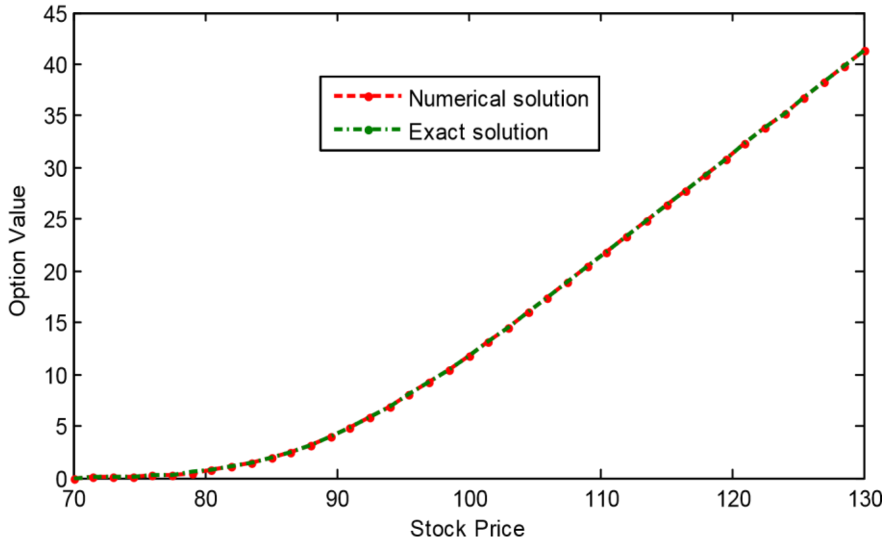

Figure 3: Comparison between the analytical and numerical solutions at the initial time point, where parameters include a strike price of 100, interest rate of 12%, volatility of 10%, and a time to maturity of 1 year

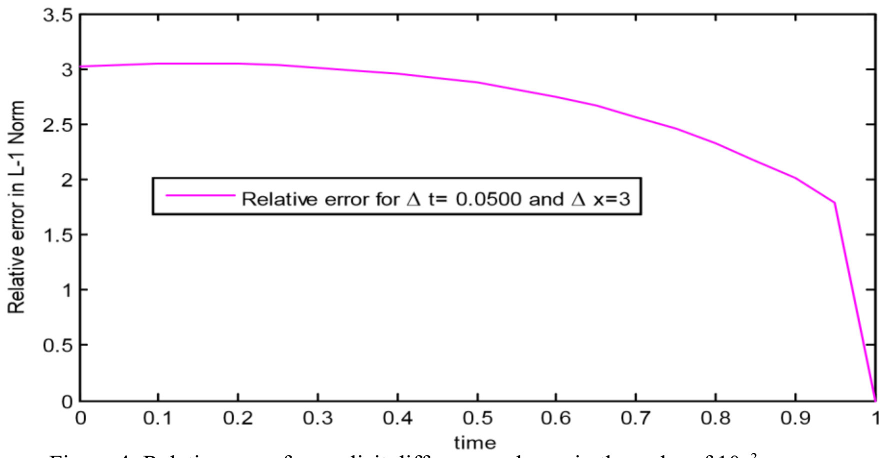

Figure 4: Relative error for explicit difference scheme in the order of 10^{-3}

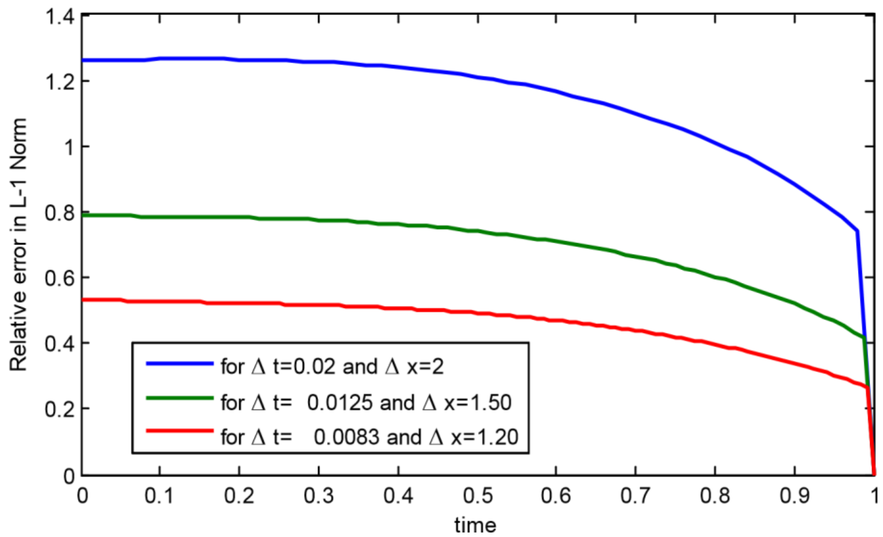

Figure 5: Relative discrepancy of the explicit difference method across various temporal and spatial grid dimensions, typically around 10^-3

Discussion
The table 1 illustrates the analytical results of European call option values calculated using the Black-Scholes formula for stock prices ranging from \$70 to \$130, given a strike price of \$100, risk-free interest rate of 12%, volatility of 10%, and a one-year maturity. The computed values of d_1, d_2, their cumulative normal distributions Nd_1, Nd_2, and the resulting call option prices $C(S)$ demonstrate how the option's worth changes with the underlying stock price. Figures 1 and 2 present the numerical and analytical solutions, respectively, for European call options evaluated over various time intervals, given a strike price of 100, an interest rate of 12%, a volatility of 10%, and a one-year maturity. Both graphs demonstrate a consistent increase in option value as the underlying asset price rises, with higher values observed as maturity approaches. Figure 3 compares these results at the initial time point, illustrating strong alignment between the numerical and analytical approaches, confirming the accuracy of the explicit difference method in approximating the Black-Scholes solution. Figure 4 quantifies the relative error of the explicit scheme, revealing that it generally stays within the order of 10^{-3} indicating sufficient precision for practical applications. Figure 5 further evaluates the method's robustness across varying grid sizes in both time and space. The relative discrepancy remains around 10^{-3}, confirming the stability and convergence of the scheme under different discretization parameters. Overall, the comparison supports the reliability of the explicit finite difference method in modeling European call options under standard financial conditions, while highlighting the importance of grid resolution in error control and solution accuracy.

CONCLUSION
This article presents a comprehensive numerical analysis of the Black-Scholes model using an explicit finite difference method. The model was examined thoroughly, and a structured numerical scheme was developed alongside a clearly defined stability criterion. The accuracy of the scheme was validated by comparing the numerical solution with the analytical solution using the L1-norm, and the convergence behavior was illustrated through graphical simulations. The numerical results closely align with the expected qualitative behavior of the Black-Scholes partial differential equation, confirming the reliability of the method. Additionally, a comparative study was carried out with results from a semi-implicit method proposed in prior literature. This comparison highlights the improved accuracy of our explicit scheme, especially in capturing option pricing dynamics near the strike price. To further illustrate this, a MATLAB simulation was conducted to visualize the call option value with a strike price of \$100, risk-free interest rate of 12%, expiration time of 1 year, and volatility of 10%. Figure 1 shows the call option's value plotted against stock prices ranging from \$70 to \$130. The plot reflects the option's increasing worth as the stock

price exceeds the strike price, validating the model's consistency with theoretical expectations.

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