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DEVELOPMENT OF A CONTINUOUS BACKWARD DIFFERENTIATION FORMULAE FOR SOLVING FIRST-ORDER AND SECOND-ORDER INITIAL VALUE PROBLEM

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ABSTRACT

Numerical methods for solving ordinary differential equations (ODEs) are essential in modeling dynamical systems across science and engineering. While specialized methods exist for first-order and second-order ODEs, developing a unified approach that efficiently handles both classes remain an active area of research. In this paper, we present a novel two-step hybrid block method based on the backward differentiation formula (BDF), capable of approximating solutions for both first- and second-order ODEs without requiring separate derivations. The method is constructed using interpolation and collocation techniques, and its numerical analysis confirms consistency, zero-stability, and convergence. Furthermore, stability analysis via the general linear method demonstrates that the scheme is A-stable, making it suitable for stiff systems. Numerical experiments including applications to the SIR epidemic model, Riccati differential equations, nonlinear stiff chemical systems, and second-order nonlinear ODEs—validate the method's accuracy and computational efficiency. Comparative results with existing methods in the literature highlight its superior performance in terms of error reduction and stability. This work contributes a versatile, high-precision tool for ODE solutions, bridging gaps in the adaptability of traditional BDF-based approaches.

Keywords: Backward Differentiation Formula, Block, Collocation, Ordinary Differential Equation, Stiff

INTRODUCTION

Ordinary differential equations (ODEs) are fundamental in modeling dynamic systems across engineering, physics, biology, and economics. Initial value problems (IVPs) involving first-order and second-order ODEs arise prominently in real-world applications, such as epidemic modeling (e.g., SIR systems), mechanical vibrations, chemical kinetics, and control theory. While analytical solutions are often intractable, numerical methods remain indispensable for approximating these systems efficiently and accurately. Among the diverse class of numerical integrators, Backward Differentiation Formulae (BDF) have emerged as a powerful tool for stiff and non-stiff ODEs due to their robust stability properties and high-order convergence.

Traditional BDF methods are widely implemented for first-order ODEs but require structural modifications or auxiliary techniques (e.g., reduction to first-order systems) when applied to second-order problems. This limitation motivates the development of unified BDF-based schemes capable of directly solving both first- and second-order IVPs without system transformations, thereby reducing computational overhead and preserving the inherent structure of the original problem. Recent advances in hybrid block methods and continuous formulations have further demonstrated the potential to enhance the adaptability and accuracy of BDF approaches.

In this work, we present a continuous BDF-type method derived through a hybrid interpolation-collocation framework, designed to approximate solutions for both first- and second-order IVPs within a single algorithmic structure. The proposed method leverages the advantages of BDFsuch as stiff stability and high-order convergence while extending its applicability to second-order ODEs without decoupling

MATERIALS AND METHODS

In this study, we intend to develop numerical schemes in the form of TBDF as follows:

$$Y(t) = \alpha_v(t)y_{n+v} + \sum_{i=0}^{2} \alpha_i(t)y_{n+i} + h\beta_2(t)y'_{n+2} + h^2\delta_2(t)y''_{n+2} \quad (1)$$

where h is the chosen step size and $\alpha_i(t)\ (i \in [0,2]), \beta_2(t), \delta_2(t)$ are continuous coefficients to be determined, while v is an off-step point.

Equation (1) is derived using interpolation and collocation technique of the trial function of the form:

$$Y(t) = \sum_{j=0}^{i+c-1} a_j t^j \quad (2)$$

where i is the number of interpolation points, c is the number of collocation points and a_j's are unknown coefficients of the power series function to be determined.

Equation (2) is interpolated at $\left(0, \frac{1}{4}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}\right)$ and collocating its first and second derivatives at t_{n+2}. These lead to a system of non-linear equations:

$$TA = y \quad (3)$$

and written explicitly as

$$\begin{pmatrix} 1 & t_n & (t_n)^2 & (t_n)^3 & (t_n)^4 & (t_n)^5 & (t_n)^6 & (t_n)^7 \\ 1 & \left(t_{n+1}^{\frac{1}{4}}\right) & \left(t_{n+1}^{\frac{1}{4}}\right)^2 & \left(t_{n+1}^{\frac{1}{4}}\right)^3 & \left(t_{n+1}^{\frac{1}{4}}\right)^4 & \left(t_{n+1}^{\frac{1}{4}}\right)^5 & \left(t_{n+1}^{\frac{1}{4}}\right)^6 & \left(t_{n+1}^{\frac{1}{4}}\right)^7 \\ 1 & \left(t_{n+1}^{\frac{3}{4}}\right) & \left(t_{n+1}^{\frac{3}{4}}\right)^2 & \left(t_{n+1}^{\frac{3}{4}}\right)^3 & \left(t_{n+1}^{\frac{3}{4}}\right)^4 & \left(t_{n+1}^{\frac{3}{4}}\right)^5 & \left(t_{n+1}^{\frac{3}{4}}\right)^6 & \left(t_{n+1}^{\frac{3}{4}}\right)^7 \\ 1 & \left(t_{n+1}^{\frac{1}{4}}\right) & \left(t_{n+1}^{\frac{1}{4}}\right)^2 & \left(t_{n+1}^{\frac{1}{4}}\right)^3 & \left(t_{n+1}^{\frac{1}{4}}\right)^4 & \left(t_{n+1}^{\frac{1}{4}}\right)^5 & \left(t_{n+1}^{\frac{1}{4}}\right)^6 & \left(t_{n+1}^{\frac{1}{4}}\right)^7 \\ 1 & \left(t_{n+1}^{\frac{5}{4}}\right) & \left(t_{n+1}^{\frac{5}{4}}\right)^2 & \left(t_{n+1}^{\frac{5}{4}}\right)^3 & \left(t_{n+1}^{\frac{5}{4}}\right)^4 & \left(t_{n+1}^{\frac{5}{4}}\right)^5 & \left(t_{n+1}^{\frac{5}{4}}\right)^6 & \left(t_{n+1}^{\frac{5}{4}}\right)^7 \\ 1 & \left(t_{n+1}^{\frac{3}{2}}\right) & \left(t_{n+1}^{\frac{3}{2}}\right)^2 & \left(t_{n+1}^{\frac{3}{2}}\right)^3 & \left(t_{n+1}^{\frac{3}{2}}\right)^4 & \left(t_{n+1}^{\frac{3}{2}}\right)^5 & \left(t_{n+1}^{\frac{3}{2}}\right)^6 & \left(t_{n+1}^{\frac{3}{2}}\right)^7 \\ 0 & 1 & 2(t_{n+2}) & 3(t_{n+2})^2 & 4(t_{n+2})^3 & 5(t_{n+2})^4 & 6(t_{n+2})^5 & 7(t_{n+2})^6 \\ 0 & 0 & 2 & 6(t_{n+2}) & 12(t_{n+2})^2 & 20(t_{n+2})^3 & 30(t_{n+2})^4 & 42(t_{n+2})^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+\frac{3}{2}} \\ y`_{n+2} \\ y``_{n+2} \end{pmatrix}$$

Solving (3) using the matrix inversion method via Maple 2015 program to acquire the values $\alpha_{j`s}$ which are then substituted into (2) to generate the continuous scheme of the present proposed method as:

$$Y(t) = \alpha_0(t)y_n + \alpha_{\frac{1}{4}}(t)y_{n+\frac{1}{4}} + \alpha_{\frac{3}{4}}((t)y_{n+\frac{3}{4}} + \alpha_1(t)y_{n+1} + \alpha_{\frac{5}{4}}((t)y_{n+\frac{5}{4}} + \alpha_{\frac{3}{2}}((t)y_{n+\frac{3}{2}} + h\beta_2(t)y`_{n+2} + h^2\delta_2(t)y``_{n+2} \tag{4}$$

where:

$$\alpha_0(t) = 1 - \frac{44738276\,t}{5125895\,h} + \frac{1289101042\,t^2}{46133055\,h^2} - \frac{2078610923\,t^3}{46133055\,h^3} + \frac{1870656598\,t^4}{46133055\,h^4} - \frac{190600024\,t^5}{9226611\,h^5} + \frac{51358496\,t^6}{9226611\,h^6} - \frac{28427648\,t^7}{46133055\,h^7}$$

$$\alpha_{\frac{1}{4}}(t) = \frac{15737088\,t}{1025179\,h} - \frac{371958912\,t^2}{5125895\,h^2} + \frac{2131219264\,t^3}{15377685\,h^3} - \frac{2106968896\,t^4}{15377685\,h^4} + \frac{379433856\,t^5}{5125895\,h^5} - \frac{319044608\,t^6}{15377685\,h^6} + \frac{36313088\,t^7}{15377685\,h^7}$$

$$\alpha_{\frac{3}{4}}(t) = -\frac{95895040\,t}{9226611\,h} - \frac{2126356736\,t^2}{9226611\,h^2} - \frac{5195873152\,t^3}{9226611\,h^3} + \frac{6011791232\,t^4}{9226611\,h^4} - \frac{3618341632\,t^5}{9226611\,h^5} + \frac{1095852032\,t^6}{9226611\,h^6} - \frac{132100096\,t^7}{46133055\,h^7}$$

$$\alpha_1(t) = \frac{52640340\,t}{1025179\,h} - \frac{406490202\,t^2}{1025179\,h^2} + \frac{3185715157\,t^3}{3075537\,h^3} - \frac{3907672090\,t^4}{3075537\,h^4} + \frac{821766744\,t^5}{1025179\,h^5} - \frac{774916448\,t^6}{3075537\,h^6} + \frac{96198272\,t^7}{3075537\,h^7}$$

$$\alpha_{\frac{4}{4}}(t) = -\frac{199367424\,t}{5125895\,h} - \frac{1578367872\,t^2}{5125895\,h^2} - \frac{4290192448\,t^3}{5125895\,h^3} + \frac{5498476608\,t^4}{5125895\,h^4} - \frac{722577024\,t^5}{1025179\,h^5} + \frac{235172864\,t^6}{1025179\,h^6} - \frac{150263808\,t^7}{5125895\,h^7}$$

$$\alpha_{\frac{5}{4}}(t) = \frac{37226176\,t}{3075537\,h} - \frac{4486506272\,t^2}{46133055\,h^2} + \frac{12490323568\,t^3}{46133055\,h^3} - \frac{16479914192\,t^4}{46133055\,h^4} + \frac{11166266176\,t^5}{46133055\,h^5} - \frac{3737950976\,t^6}{46133055\,h^6} + \frac{489389056\,t^7}{46133055\,h^7}$$

$$\beta_2(t) = -\frac{346905\,t}{1025179\,h} - \frac{2849994\,t^2}{1025179\,h} - \frac{8231951\,t^3}{1025179\,h^2} + \frac{11402230\,t^4}{1025179\,h^3} - \frac{8197560\,t^5}{1025179\,h^4} + \frac{2934176\,t^6}{1025179\,h^5} - \frac{409984\,t^7}{1025179\,h^6}$$

$$\delta_2(t) = -\frac{63360\,ht}{1025179\,h} - \frac{1047051\,t^2}{1025179\,h^2} + \frac{3053705\,t^3}{2050358\,h} - \frac{2143851\,t^4}{1025179\,h^2} + \frac{1570108\,t^5}{1025179\,h^3} - \frac{576336\,t^6}{1025179\,h^4} + \frac{83392\,t^7}{1025179\,h^5}$$

Evaluating (5) at c gives the discrete scheme

$$y_{n+2} = -\frac{25725}{1025179}y_n + \frac{115200}{1025179}y_{n+\frac{1}{4}} - \frac{1053696}{1025179}y_{n+\frac{3}{4}} + \frac{3087000}{1025179}y_{n+1} - \frac{4390400}{1025179}y_{n+\frac{5}{4}} + \frac{273630}{1025179}hy'_{n+2} - \frac{22050}{1025179}h^2y''_{n+2} \tag{5}$$

To obtain the sufficient schemes required for first order ODEs, we obtain the first derivative of (4) and evaluate at $t = t_{n+\frac{1}{4}}, t_{n+\frac{3}{4}}, t_n, t_{n+\frac{5}{4}}, t_{n+\frac{3}{2}}$ to obtain;

$$\frac{15377685}{33538883}hy'_{n+\frac{1}{4}} + \frac{6217425}{134155532}hy'_{n+2} - \frac{320175}{38330152}h^2y''_{n+2} \tag{6}$$

$$y_{n+\frac{3}{4}} = \frac{279463}{26607840}y_n - \frac{67089}{1108660}y_{n+\frac{1}{4}} + \frac{2890191}{1773856}y_{n+1} - \frac{860427}{1108660}y_{n+\frac{5}{4}} + \frac{654487}{3325980}y_{n+\frac{3}{2}} - \frac{1025179}{5543300}hy'_{n+\frac{1}{4}} - \frac{191043}{44346400}hy'_{n+2} + \frac{13239}{17738560}h^2y''_{n+2} \tag{7}$$

$$y_{n+1} = \frac{11184569}{65800425}y_n - \frac{1311424}{4386695}y_{n+\frac{1}{4}} + \frac{4794752}{7896051}y_{n+\frac{3}{4}} - \frac{16613952}{21933475}y_{n+\frac{5}{4}} - \frac{9306544}{39480255}y_{n+\frac{3}{2}} + \frac{1025179}{5543300}hy'_n + \frac{23127}{3509356}hy'_{n+2} - \frac{1056}{877339}h^2y''_{n+2} \tag{8}$$

$$y_{n+\frac{5}{4}} = -\frac{108517}{11685708}y_n + \frac{14495}{324603}y_{n+\frac{1}{4}} - \frac{1603300}{2921427}y_{n+\frac{3}{4}} + \frac{3155525}{1298412}y_{n+1} - \frac{2678530}{2921427}y_{n+\frac{3}{2}} + \frac{5125895}{8764281}hy'_{n+\frac{5}{4}} + \frac{370525}{35057124}hy'_{n+2} - \frac{39575}{23371416}h^2y''_{n+2} \tag{9}$$

$$y_{n+\frac{3}{2}} = \frac{1959905}{382090752}y_n - \frac{31471}{1326704}y_{n+\frac{1}{4}} + \frac{1459375}{5970168}y_{n+\frac{3}{4}} - \frac{34675225}{42454528}y_{n+1} + \frac{2110665}{1326704}y_{n+\frac{3}{2}} + \frac{5125895}{31840896}hy'_{n+\frac{3}{2}} + \frac{435675}{42454528}hy'_{n+2} + \frac{129225}{84909056}h^2y''_{n+2} \tag{10}$$

Similarly, we obtain the second derivative of (4), which evaluates to $t = t_{n+\frac{1}{4}}, t_{n+\frac{3}{4}}, t_n, t_{n+\frac{5}{4}}, t_{n+\frac{3}{2}}$, thereby generating the necessary number of schemes for solving second-order ODEs using the block method, as follows:

$$h^2y''_n = \frac{2578202084}{46133055}y_n - \frac{743917824}{5125895}y_{n+1} + \frac{4252713472}{9226611}y_{n+\frac{5}{4}} - \frac{812980404}{1025179}y_{n+1} + \frac{3156735744}{5125895}y_{n+\frac{5}{4}} - \frac{8973012544}{46133055}y_{n+\frac{3}{2}} + \frac{5699988}{1025179}hy'_{n+2} - \frac{1047051}{1025179}h^2y''_{n+2} \tag{11}$$

$$h^2y''_{n+\frac{1}{4}} = \frac{1188785003}{92266110}y_n - \frac{98443548}{5125895}y_{n+\frac{1}{4}} - \frac{39982600}{9226611}y_{n+\frac{3}{4}} + \frac{61612593}{2050358}y_{n+1} - \frac{152088636}{5125895}y_{n+\frac{5}{4}} + \frac{474026812}{46133055}y_{n+\frac{3}{2}} - \frac{661941}{2050358}hy'_{n+2} + \frac{247517}{4100716}h^2y''_{n+2} \tag{12}$$

$$h^2y''_{n+\frac{3}{4}} = -\frac{7072754}{9226611}y_n + \frac{4973772}{1025179}y_{n+\frac{1}{4}} + \frac{1025179}{9226611}y_{n+\frac{3}{4}} - \frac{25300830}{1025179}y_{n+1} + \frac{29148876}{1025179}y_{n+\frac{5}{4}} - \frac{78859328}{9226611}y_{n+\frac{3}{2}} + \frac{219870}{1025179}hy'_{n+2} - \frac{38961}{1025179}h^2y''_{n+2} \quad (13)$$

$$h^2y''_{n+1} = \frac{4226506}{46133055}y_n - \frac{2740608}{5125895}y_{n+\frac{1}{4}} + \frac{179493632}{9226611}y_{n+\frac{3}{4}} - \frac{39292242}{1025179}y_{n+1} + \frac{104447616}{5125895}y_{n+\frac{5}{4}} - \frac{48906848}{46133055}y_{n+\frac{3}{2}} - \frac{10206}{1025179}hy'_{n+2} + \frac{2396}{1025179}h^2y''_{n+2} \quad (14)$$

$$h^2y''_{n+\frac{5}{4}} = \frac{272597}{10251790}y_n - \frac{502764}{5125895}y_{n+\frac{1}{4}} - \frac{654952}{1025179}y_{n+\frac{3}{4}} + \frac{39590727}{2050358}y_{n+1} - \frac{188608716}{5125895}y_{n+\frac{5}{4}} + \frac{93273124}{5125895}y_{n+\frac{3}{2}} - \frac{217539}{2050358}hy'_{n+2} + \frac{64171}{4100716}h^2y''_{n+2} \quad (15)$$

$$h^2y''_{n+\frac{3}{2}} = -\frac{15361261}{46133055}y_n + \frac{7707072}{5125895}y_{n+\frac{1}{4}} - \frac{128988800}{9226611}y_{n+\frac{3}{4}} + \frac{41305089}{1025179}y_{n+1} - \frac{206243136}{5125895}y_{n+\frac{5}{4}} + \frac{588400832}{46133055}y_{n+\frac{3}{2}} + \frac{1006047}{1025179}hy'_{n+2} - \frac{282519}{2050358}h^2y''_{n+2} \quad (16)$$

It is important to note that equations (5) to (10) form the sufficient block method for solving first-order IVPs while the combined equations (5) to (16) is for second-order IVPs. And more so, all the discrete schemes are gotten from the same continuous method (4).

Convergence and Stability Analysis of the Method

The necessary and sufficient condition for the convergence of an LMM is consistency and zero stability (Lambert, 1973). In what follow, we shall discuss the consistency, zero stability and absolute stability of the method.

Consistency

We write the developed method as;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - h^n \sum_{j=0}^{k} \beta_j f_{n+j} = 0 \quad (17)$$

and the local truncation error defined as

$$\xi[y(t),h] = \sum_{j=0}^{k} \left[\omega_j y(t+jh) - h\sigma_j y'(t+jh) - h^2\delta_j y''(t+jh)\right] \quad (18)$$

We assume that $y(t)$ is sufficiently differentiable such that the linear operator ξdefined above can be expanded as a Taylor's series about the point t, then,

$$\xi[y(t),h] = c_0y(t) + c_1hy'(t) + c_2h^2y''(t) + \cdots + c_rh^rx^{(r)}(t) + \cdots \quad (19)$$

According Mohammed *et al.* (2022), (17) is said to be consistent if it has order of accuracy $p > 1$ for $c_0 = c_1 = c_2 = \cdots = c_{p+1} = 0$ and $c_{p+2} \neq 0$. The constant c_{p+2} is the error constant. Hence the block method in (5) to (16) has a uniform order 6 with error constants calculated as:

$$\begin{bmatrix} \frac{3675}{2099566592} & \frac{5979695}{1256002420736} & \frac{29214149443584}{4068799938560} & \frac{1173019}{1509303828480} \\ \frac{2447365}{5360841916416} & \frac{8744195}{29214149443584} & \frac{102431347}{146969661440} & \frac{274719821}{6046751784960} \\ \frac{1232971}{67186130944} & \frac{11798641}{4703029166080} & \frac{1125811}{529090781184} & \frac{796449}{36742415360} \end{bmatrix}^T$$

.

Since $p > 1$, then the method is consistent.

Zero Stability

We carry out the zero stability of the developed discrete schemes in (5) to (10) by considering the first characteristic polynomial defined as

$$\lambda(\rho) = \left|\rho P^{(1)} - P^{(0)}\right| \quad (20)$$

where

$$P^{(1)} = \begin{pmatrix} 1 & -\frac{3827500}{684467} & \frac{22427725}{2737868} & -\frac{3987495}{684467} & \frac{1196558}{684467} & 0 \\ \frac{67089}{1108660} & 1 & -\frac{2890191}{1773856} & \frac{860427}{1108660} & -\frac{654487}{3325980} & 0 \\ \frac{1311424}{4386695} & -\frac{4794752}{7896051} & 1 & -\frac{16613952}{21933475} & \frac{9306544}{39480255} & 0 \\ -\frac{14495}{324603} & \frac{1603300}{2921427} & -\frac{3155525}{1298412} & 1 & \frac{2678530}{2921427} & 0 \\ \frac{31471}{1326704} & -\frac{1459375}{5970168} & \frac{34675225}{42454528} & -\frac{2110665}{1326704} & 1 & 0 \\ -\frac{115200}{1025179} & \frac{1053696}{1025179} & -\frac{3087000}{1025179} & \frac{4390400}{1025179} & -\frac{3292800}{1025179} & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix}$$

$$P^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1308155}{2737868} \\ 0 & 0 & 0 & 0 & 0 & -\frac{279463}{26607840} \\ 0 & 0 & 0 & 0 & 0 & -\frac{11184569}{65800425} \\ 0 & 0 & 0 & 0 & 0 & \frac{108517}{11685708} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1959905}{382090752} \\ 0 & 0 & 0 & 0 & 0 & \frac{25725}{1025179} \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{3}{4}} \\ y_{n-1} \\ y_{n-\frac{5}{4}} \\ y_{n-\frac{3}{2}} \\ y_{n-2} \end{pmatrix}$$

Definition (Zero-stability): The block method (19) is said to be zero stable as $h \to 0$if the roots of the first characteristic polynomial $\lambda(\rho)$ satisfies $|\rho_j| \le 1, j = 1,2,3, ...$ and for those roots with$|\rho_j| = 1$, the multiplicity must not exceed 1, (Yakusak and Owolanke, 2018).
Applying equation (20), we have

$$\lambda(\rho) = \frac{27614603657861741663142025\text{0}}{383988424547843575851295737}\rho^5(\rho + 1) = 0 \qquad (21)$$
$$\rho = \{0,0,0,0,0,-1\}$$

Hence, it is safe to say that the method is zero stable since it satisfies $|\rho_j| \le 1$ and by extension, the convergence holds.

Regions of Absolute Stability of the Proposed Methods
Again, in the spirit of (Yakusak and Owolanke, 2018), a linear multi-step method is said to be A-stable if its region of absolute stability, contains the whole of the left-hand complex half-plane $R(hp) < 0$. It is important to investigate the performance of the proposed methods in the case of $h > 0$ fixed. The stability matrix is formulated using:

$$\lambda(z) = -(P^1 - zQ^{(1)}) - Z^2R^{(1)})^{-1}.(P^{(0)} - zQ^{(0)}) \qquad (22)$$

where

$$Q^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1025179}{52640340} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y'_{n-\frac{1}{4}} \\ y'_{n-\frac{3}{4}} \\ y'_{n-1} \\ y'_{n-\frac{5}{4}} \\ y'_{n-\frac{3}{2}} \\ y'_{n-2} \end{pmatrix}$$

$$R^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{320175}{38330152} \\ 0 & 0 & 0 & 0 & 0 & \frac{13239}{17738560} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1056}{877339} \\ 0 & 0 & 0 & 0 & 0 & -\frac{39575}{23371416} \\ 0 & 0 & 0 & 0 & 0 & \frac{129225}{84909056} \\ 0 & 0 & 0 & 0 & 0 & -\frac{22050}{1025179} \end{pmatrix} \begin{pmatrix} y^n_{n+\frac{1}{4}} \\ y^n_{n+\frac{1}{4}} \\ y^n_{n+1} \\ y^n_{n+\frac{1}{4}} \\ y^n_{n+\frac{1}{2}} \\ y^n_{n+2} \end{pmatrix}$$

$$Q^{(1)} = \begin{pmatrix} -\frac{15377685}{33538883} & 0 & 0 & 0 & 0 & \frac{6217425}{134155532} \\ 0 & -\frac{1025179}{5543300} & 0 & 0 & 0 & -\frac{191043}{44346400} \\ 0 & 0 & 0 & 0 & 0 & \frac{23127}{3509356} \\ 0 & 0 & \frac{5125895}{8764281} & 0 & 0 & \frac{370525}{35057124} \\ 0 & 0 & 0 & \frac{5125895}{31840896} & 0 & -\frac{435675}{42454528} \\ 0 & 0 & 0 & 0 & 0 & \frac{273630}{1025179} \end{pmatrix} \begin{pmatrix} y'_{n+\frac{1}{4}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1} \\ y'_{n+\frac{5}{4}} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \end{pmatrix}$$

The matrix $\chi(z)$ has eigenvalues$\{0,0,0,...,\rho_k\}$ and the dominant eigenvalue $\lambda_k: \mathbb{C} \to \mathbb{C}$is a rational function (called the stability function) with real coefficients given by

$$\rho_k(z) = \frac{(226800z^5 + 5210190z^4 + 46985073z^3 + 206730932z^2 + 428152500z + 326592000)}{32400z^6 - 101898z^5 - 70236z^4 - 6767247z^3 + 61670732z^2 - 225031500z + 326592000} \qquad (23)$$

Plotting $\rho_k(z)$ via Maple (2015) environment displays the stability region of new method in figure 1 which is A-stable.

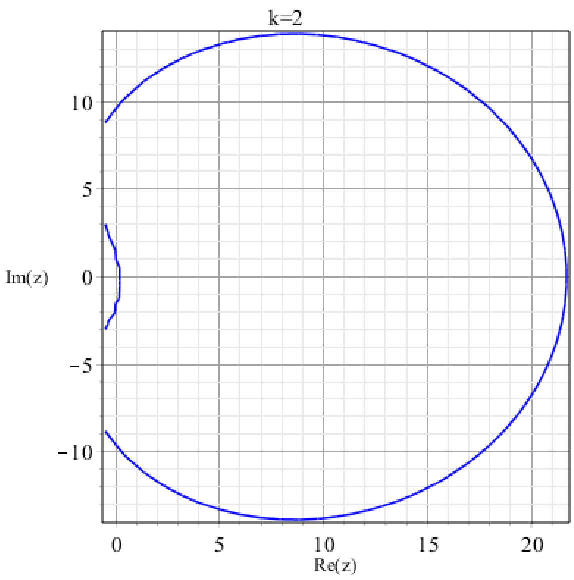

Figure 1: Stability Region of the Method

Numerical Experiments
In this section, the effectiveness of the new method is investigated through some numerical problems in first-order and second-order ODEs.

First-order problems
Problem 1
The SIR model is an epidemiological model that counts the notional number of people in a confined population that have an infectious disease over time. This class of models gets its name from the fact that it consists of linked equations that relate the number of susceptible people *S(t)*, the number of infected people *I(t)*, and the number of recovered people *R(t)*. This is an excellent and straightforward model for many infectious diseases, including measles, mumps, and rubella. It is given by the three coupled equations shown below:

$$\frac{dS}{dt} = \mu(1 - S) - \beta IS \quad (24)$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS \quad (25)$$

$$\frac{dR}{dt} = \mu R + \gamma I \quad (26)$$

where μ, β and γ are positive parameters. Define *x* to be:

$y = S + I + R$

and adding (24), (25) and (26), the following evolution equation for *y* is obtained.

$$y` = \mu(1 - y) \quad (27)$$

Kuboye and Adeyefa (2021) solved this problem with the following parameters:

$\mu = \frac{1}{2}, y(0) = \frac{1}{2}, h = 0.1$

Exact solution: $y(t) = 1 - \mu e^{-i\mu t}$

Problem 2
We consider the Riccati differential equation solved in Kashkaria and Syamb (2019)

$y` = 1 + 2y - y^2; y(0) = 0, 0 \leq t \leq 10$

Exact solution: $y(t) = 1 + \sqrt{2}\ tanh\left(\sqrt{2t} + \frac{1}{2} log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$

Problem 3
We consider the nonlinear system of stiff chemical problem

$y`_1 = \lambda y_1 + y_{2,}^2 y_1(0) = \frac{1}{\lambda+2}$

$y`_2 = -y_2(0) = 1$

The exact solution is given as

$Y_1(t) = -\frac{\exp(-2t)}{\lambda+2}, Y_2(t) = \exp(-t)\, where\ \lambda = 10000$

RESULTS AND DISCUSSION
Table 1: Comparative Analysis of Errors for Problem 1 (*h* = 0.1)

t	Exact solution	Error in Kashkari and Syam (2019)	Error in Abolarin *et al.* (2020)	Error in Kuboye and Adeyefa (2021)	Error in New Method
0.1	0.524385287749643	1.998×10^{-15}	9.104×10^{-15}	3.846×10^{-13}	1.9200×10^{-16}
0.2	0.547581290982020	3.886×10^{-15}	7.105×10^{-15}	7.319×10^{-13}	2.371×10^{-16}
0.3	0.569646011787471	5.440×10^{-15}	8.882×10^{-15}	1.044×10^{-12}	3.990×10^{-16}
0.4	0.590634623461009	6.994×10^{-15}	2.121×10^{-14}	1.324×10^{-12}	4.287×10^{-16}
0.5	0.610599608464298	8.216×10^{-15}	1.368×10^{-13}	1.575×10^{-12}	5.651×10^{-16}
0.6	0.629590889659141	9.548×10^{-15}	7.983×10^{-13}	1.797×10^{-12}	5.821×10^{-16}
0.7	0.647655955140644	1.055×10^{-14}	3.699×10^{-12}	1.995×10^{-12}	6.960×10^{-16}
0.8	0.681185924189114	1.132×10^{-14}	-	2.168×10^{-12}	7.023×10^{-16}
0.9	0.681185924189114	1.221×10^{-14}	-	2.320×10^{-12}	7.968×10^{-16}
1.0	0.696734670143684	-	-	2.452×10^{-12}	7.944×10^{-16}

Table 2: Comparative Analysis of Errors for Problem 2

t	**Exact solution**	**Error in Kashkari and Syam (2019) (*h* = 0.05)**	**Error in Abolarin *et al.* (2020) (*h* = 0.1)**	**Error in New Method (*h* = 0.1)**
1.0	1.68949839159439	1.418×10^{-11}	9.104×10^{-15}	1.480×10^{-09}
2.0	2.35777165329150	7.234×10^{-13}	7.105×10^{-15}	3.477×10^{-11}
3.0	2.41081368593662	1.163×10^{-13}	8.882×10^{-15}	7.959×10^{-13}
4.0	2.41401238260570	2.132×10^{-14}	2.121×10^{-14}	4.365×10^{-13}
5.0	2.41420167069693	2.664×10^{-15}	1.368×10^{-13}	5.735×10^{-14}
6.0	2.41421285950392	4.441×10^{-16}	7.983×10^{-13}	5.265×10^{-15}
7.0	2.41421352082949	4.441×10^{-16}	3.699×10^{-12}	4.220×10^{-16}
8.0	2.41421355991764	4.441×10^{-16}	-	3.150×10^{-17}
9.0	2.41421356222798	4.441×10^{-16}	-	2.249×10^{-18}
10	2.41421356236453	4.441×10^{-16}	-	1.558×10^{-19}

Table 3: Comparative Analysis of Errors for problem 3

t	y_i	**Error in Akinfenwa *et al.* (2017) ($h = 0.01$)**	**Error in Khalsaraei *et al.* (2020) ($h = 0.0001$)**	**Error in Mohammed *et al.* (2022) ($h = 0.1$)**	**Error in New Method (*h* = 0.1)**
3	(y_1)	2.030×10^{-19}	1.779×10^{-20}	3.790×10^{-22}	4.882×10^{-24}
	(y_2)	1.440×10^{-14}	2.079×10^{-20}	2.998×10^{-18}	4.249×10^{-17}
5	(y_1)	1.200×10^{-20}	2.493×10^{-19}	1.60×10^{-21}	1.488×10^{-25}
	(y_2)	3.210×10^{-15}	4.664×10^{-13}	6.740×10^{-19}	3.013×10^{-18}
	(y_1)	1.110×10^{-20}	5.743×10^{-20}	7.120×10^{-20}	1.349×10^{-29}
10	(y_2)	4.380×10^{-17}	6.346×10^{-12}	9.080×10^{-21}	4.995×10^{-20}

Second Order Problems
Three problems from second order ordinary differential equations are considered. These include a stiff linear equation and nonlinear problems.

Problem 4
Consider the nonlinear problem: $y`` + (y`)^2 + y^2 = 1 - sint$
with the following initial conditions
$y(0) = 0, y`(0) = 1 \; and \; h = 0.1.$
Exact Solution: $Y(t) = sint$

Problem 5
Consider the nonlinear problem:
$y`` = t(y`)^2$
with the following initial conditions
$y(0) = 1, y`(0) = 0.5 \; and \; h = 0.1$
Exact Solution: $Y(t) = \text{arctanh}\left(\frac{1}{2}t\right) + 1$

Problem 6
Consider the Stiff linear problem:
$y`` + 1001y` + 1000y = 0$
with the following initial conditions
$y(0) = 1, y`(0) = -1 \; and \; h = 0.1$
Exact solution: $Y(t) = \exp(-t)$

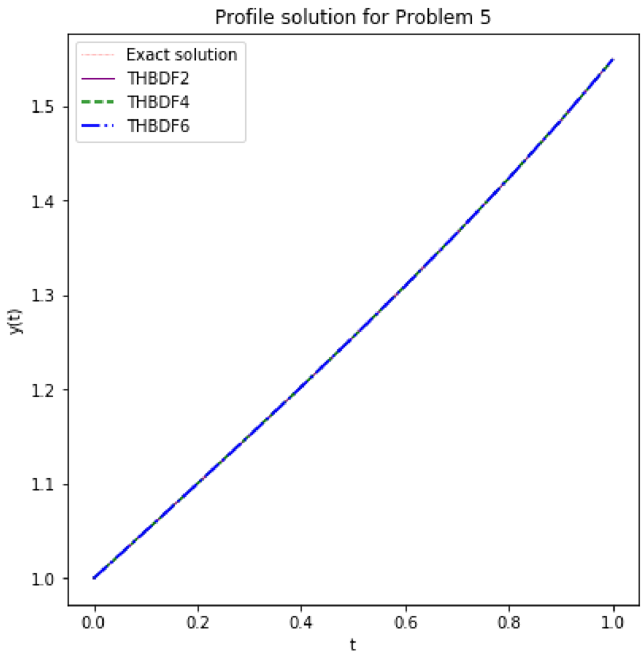

Figure 2: Profile solution for Problem 5

Table 4: Comparative Analysis of Errors for Problem 4

t	Exact solution	Error in Guler *et al.* (2019)	Error in Ogunlaran and Kehinde (2022)	Error in New Method
0.1	0.099833416646828152307	3.490×10^{-08}	9.300×10^{-10}	1.739×10^{-15}
0.2	0.19866933079506121546	1.160×10^{-07}	1.990×10^{-09}	7.136×10^{-15}
0.3	0.29552020666133957511	-	-	4.113×10^{-14}
0.4	0.38941834230865049167	1.130×10^{-07}	3.180×10^{-09}	9.785×10^{-14}
0.5	0.47942553860420300027	4.610×10^{-07}	3.230×10^{-09}	2.262×10^{-13}
0.6	0.56464247339503535720	7.800×10^{-07}	3.510×10^{-09}	3.780×10^{-13}
0.7	0.64421768723769105367	1.400×10^{-06}	3.740×10^{-09}	6.377×10^{-13}
0.8	0.71735609089952276163	4.1600×10^{-06}	3.530×10^{-09}	9.145×10^{-13}
0.9	0.78332690962748338846	1.400×10^{-05}	3.030×10^{-09}	1.327×10^{-12}
1.0	0.84147098480789650665	4.100×10^{-05}	2.750×10^{-09}	1.747×10^{-12}

Table 5: Comparative Analysis of Errors for Problem 5

t	Exact solution	Error in Abdelrahim & Omar (2016)	Error in Kuboye & Omar (2015)	Error in New Method
0.1	1.0500417292784912682	1.310×10^{-16}	1.446×10^{-14}	9.607×10^{-16}
0.2	1.1003353477310755806	3.975×10^{-14}	3.779×10^{-13}	1.095×10^{-15}
0.3	1.1511404359364668053	1.021×10^{-14}	3.428×10^{-11}	7.123×10^{-15}
0.4	1.2027325540540821910	3.304×10^{-13}	6.987×10^{-08}	9.470×10^{-15}
0.5	1.2554128118829953416	-	-	3.455×10^{-14}
0.6	1.3095196042031117155	-	-	4.668×10^{-14}
0.7	1.3654437542713961691	-	-	1.528×10^{-13}
0.8	1.4236489301936018069	-	-	2.065×10^{-13}
0.9	1.4847002785940517416	-	-	7.451×10^{-13}
1.0	1.5493061443340548457	-1.293×10^{-12}	2.017×10^{-07}	1.014×10^{-12}

Table 6: Absolute Error $\left(Y(t)-y(t)\right)$ in the proposed methods for Problem 6

t	Exact solution	Error in Adeniyi & Adeyefa (2013)	Error in Ajileye *et al.* (2017)	Error in New Method
0.1	0.90483741803595957316	2.360×10^{-10}	1.206×10^{-06}	2.674×10^{-15}
0.2	0.81873075307798185867	4.780×10^{-10}	1.584×10^{-05}	9.633×10^{-14}
0.3	0.74081822068171786607	5.817×10^{-10}	2.299×10^{-05}	1.030×10^{-13}
0.4	0.67032004603563930074	7.356×10^{-10}	1.210×10^{-05}	3.526×10^{-14}
0.5	0.60653065971263342360	8.126×10^{-10}	2.996×10^{-05}	5.268×10^{-14}
0.6	0.54881163609402643263	8.941×10^{-09}	4.678×10^{-05}	2.389×10^{-13}
0.7	0.49658530379140951470	9.914×10^{-09}	2.742×10^{-05}	2.624×10^{-13}
0.8	0.44932896411722159143	1.017×10^{-09}	1.384×10^{-04}	2.613×10^{-13}
0.9	0.40656965974059911188	1.041×10^{-08}	2.224×10^{-04}	3.169×10^{-13}
1.0	0.36787944117144232160	1.071×10^{-08}	2.810×10^{-04}	6.777×10^{-13}

Discussion

Tables 1–3 present a comparative analysis of errors for each method at different step sizes (h) for first-order problems. The results demonstrate that the newly developed method achieves the smallest error values, confirming its superior accuracy and efficiency. Notably, the errors produced by the proposed method are significantly lower than those of existing methods (Akinfenwa et al., 2017; Khalsaraei et al., 2020; Kuboye et al.), highlighting its improved convergence properties. The consistently larger error values in the alternative methods indicate comparatively lower accuracy and computational efficiency. Thus, the findings clearly establish that the new method outperforms existing approaches in terms of precision, efficiency, and convergence rate.

Similarly, Tables 4 and 5 display error comparisons for second-order problems, where the new method again exhibits the smallest error values, reinforcing its robustness and reliability. Additionally, Table 6 presents the absolute errors of the proposed method, further confirming its enhanced performance over competing techniques.

Based on these results, it can be concluded that the newly developed method is the most accurate and exhibits the fastest convergence among the tested approaches

CONCLUSION

In this paper, a continuous formulation of a multipurpose numerical method that has the ability to provide numerical solutions to both first and second-order ODEs is sought. In the course of achieving this milestone, this research focuses on the derivation of a 2-step block hybrid BDF. It is important to emphasis that the method formulated is capable of solving first-order ODEs and second-order without necessarily deriving separate methods for different orders of ODEs. Analysis of the numerical properties established that the method is consistent and zero-stable which confirm its convergence. Also the stability analysis yielded regions of absolute stability that is A-stable which is peculiar to BDF methods. Some numerical experiments considered in this work include an application problem in SIR model, the Riccati differential equation, nonlinear system of stiff chemical problem and other nonlinear second order differential equations. Finally, comparative analysis shows that the derived method has better effectiveness than some methods found in the literature.

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