FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 9 No. 8, August, 2025, pp 291 – 295
DOI: https://doi.org/10.33003/fjs-2025-0908-3606

STABILITY PROPERTY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH FINITE DELAYS

***Abimbola Latifat Adebisi, Kasali Kazeem and Ayegbusi Florence Dami**

Abiola Ajimobi Technical University, Ibadan

*Corresponding authors' email: latifat.abimbola@tech-u.edu.ng

ABSTRACT

This study investigates the stability properties of impulsive functional differential inclusions with finite delays, a class of mathematical models that encapsulate dynamic systems influenced by sudden changes (impulses) and time delays in their state variables. We begin by establishing a comprehensive framework for analyzing such inclusions, incorporating the classical theory of functional differential equations and the modern theory of inclusions. By employing advanced mathematical tools, including Lyapunov functions and the Razumikhin technique, uniform stability and uniform asymptotic stability of impulsive functional differential inclusions are obtained. We derive sufficient conditions for the stability of solutions under varying impulse magnitudes and delay intervals. The interplay between impulsive effects and delayed responses is explored, revealing critical insights into how these factors influence the overall stability of the system. Our findings are further illustrated through several examples, demonstrating the practical implications of the theoretical results. This research not only contributes to the existing literature on impulsive differential inclusions but also provides valuable guidance for the design and analysis of complex dynamic systems in fields such as control theory, biology, and engineering.

Keywords: Impulsive functional differential inclusions, Stability properties, Lyapunov functions, Razumikhin techniques, Time delays

INTRODUCTION

Mathematical modeling through stability investigations of impulsive functional differential inclusions with finite delays stands as a leading field of research considering dynamic systems with time-dependent sudden changes. The review analyzes academic research to present essential discoveries about stability features within systems that exhibit finite delays through impulsive functional differential inclusions. Traditional difference equations evolve into impulsive functional differential equations by enabling finite time changes at predetermined points during the system. Such modeling method enables better representation of real systems like population control models and economic systems. The fundamental principles of impulsive differential equations were first established by Jack and Huseyin (1991) through their research which centered on solution existence, uniqueness and their continuous dependence.

When dynamical systems employ finite delays then the prediction of their future state becomes more complicated because future states depend upon current states and preceding states. Bainov and Simeonov (1993) dedicated their research to functional differential equations with delays thus establishing fundamental results about stability and boundedness. Stability criteria with delay considerations were introduced as fundamental conditions by the authors with emphasis on how delays affect solution behavior. Stability investigations of impulsive functional differential inclusions use both Lyapunov direct method and fixed-point theorems for analysis. During the year 1892 Lyapunov initiated a study of dynamical systems equilibrium stability through his stability analysis methods. Stability methods from Lyapunov have been adapted through the research of Stanova and Stamov(2014) ,and Lu et al (2015) to include impulsive effects as well as time delays. The research of Zhang & Sun (2006) led to developing sufficient stability conditions for asymptotic stability, so solutions remain robust against perturbations.

The analysis of impulsive functional differential inclusions depends heavily on the application of Banach and Schauder fixed-point theorems together with other fixed-point theorems. Benchora et al (2006) used these theorems to develop existence results for differential inclusion solutions which created methods to determine stability properties. The authors demonstrate that both compactness and continuity remain key components for impulsive systems within their approach.

The research on stable impulsive functional differential inclusions with finite delays generates important consequences for multiple academic fields. Smith et al. (2018) and other biological researchers utilize these concepts to analyze population dynamics with emphasis on how impulsive harvesting strategies and environmental delays affect species populations. According to their research it is essential to incorporate impulsive effects together with time delays when developing ecological models.

The recent scholarly research aims to enhance stability criteria for impulsive functional differential inclusions with finite delays to extend their modeling scope. Zhang and Liu (2020) created new methods for stability assessment that account for nonlinearities in systems that experiences multiple impulsive triggers and various delay patterns. In many situations, analytic solution are almost impossible; Egbetade and Abimbola (2025) attempted to use the Tau method in finding a numerical solutionwith the estimation of the error term

Studies on impulsive functional differential systems must now emphasize the development of models that combine stochastic impulses with hybrid dynamic components and multiple state dimensions. Any investigations into stability behavior in impulsive systems require numerical methods combined with simulations to both confirm theoretical findings and disclose practical system behavior.

The research domain of impulsive functional differential inclusions with finite delays continues to develop actively into a diverse field of investigation regarding stability properties. Various scientific and engineering disciplines face essential implications from the combination of impulsive effects with time delays together with stability analysis techniques. The advancement of understanding complex dynamic systems in real-world applications depends on continuous research activities at the intersection between impulsive effects and functional differential inclusions with finite delays.

MATERIALS AND METHODS

Preliminaries

The domain consists of real ℝ together with $\mathbb{R}_+$ while $\mathbb{R}^n$ represents an n-dimensional Euclidean norm. The space contains elements from real numbers which use the Euclidean norm as its distance measurement. The set of positive real numbers is denoted by $\mathbb{Z}_+$ i.e. $\mathbb{Z}_+ = \{1, 2, \ldots\}$. For any interval $J \subseteq \mathbb{R}^k,\ (1 \le k \le n)$,
$let\ C(J,S) = \{\psi: J \to S\ is\ continuous\}\ and$
$PC(J,S)$ is continuous everywhere except at finite number of points t at which
$\psi(t_+),\ \psi(t_-)\ exist\ and\ \psi(t_+) = \psi(t)\}$,
Functional differential equations with impulse of the form

$$\begin{cases} y'(t) \epsilon F(t, y_t) a.et \epsilon J := [0,T], t \ne t_k \\ \Delta y|_{t=t_k} = y(t_k) - y(t_k^-) = I_k(t_k, y(t_k^-)),\ k = 1,2,..,n \quad (1) \\ y(t) = \phi(t), \quad -r \le t \le 0 \end{cases}$$

Where $\phi \in \mathbb{C},\ f \in C([t_{k-1}, t_k)] X\ \mathbb{C}, \mathbb{R}^n),\ f(t,0) = 0\ is\ open\ set\ in\ PC([-r,0], \mathbb{R}^n)$
Assuming $y(.) : [\alpha, +\infty] \to S$, for every t ≥ t_0 we represent by y_t the member in ℂ
described by $y_t(s) = y(t+s), s \in [-r, 0]$. Describe PCB = $[\varphi\ \epsilon\ \mathbb{C}\ such\ that\ \varphi\ ]$ is bounded;
for $\varphi \in$ PCB, the norm of φ is described by
$||\varphi|| = sup_{\alpha \le \phi \le 0} |\varphi(0)|$
Describe $PCB_\delta = [\varphi \in PCB: ||\varphi|| \le \delta]$
$I_k(t,x) \in C([0,T) X \mathbb{R}^n, \mathbb{R}^n) and I_k(t,0) = 0, k = 1, \ldots, n$

Description 1: Function U: $[0,T] X\ \mathbb{R} \to \mathbb{R}$ is classified as a member of v_0 if
U is continuous on every member of the sets $([t_{k-1}, t_k)] X\ \mathbb{C}$ and
$\lim_{(t,\varphi)\to(t_k^-,\varphi)} U(t,\varphi) = U(t_k^-, \varphi)$ exists;
(ii) U(t, x) is Lipschitzian locally in y and U(t, 0) is equivalent to zero

Description 2: Assuming $v \in v_0$, considering any $(t,\varphi) \in ([t_{k-1}, t_k)] X\ \mathbb{C}$, the upper right-hand (urh) Dini derivative of U with the solution of equation (1) is described by
$D^+U(t, \varphi(0)) = \lim_{h\to 0^+} \sup\ [U(t+h), \varphi(0) + hf(t,\varphi) - v(t, \varphi(0)$

Description 3. The solution x = 0 of equation (1) is said to be (H1) stable, if we take any $\sigma \ge t_0\ and\ \varepsilon > 0, there\ exist\ \delta = \delta(\varepsilon, \sigma) > 0$ in such away that $\phi\ \epsilon\ PCB_\delta$
Means that $|x(t, \sigma, \phi)| < \varepsilon,\ \ t \ge \sigma$
(H2) uniformly stable, if considering the δ in (H1) to be independent on σ ;
(H3) uniformly asymptotically stable, if (H2) is affirmed and we now have some zero in such away that for
any $\varepsilon > 0$ we then have some $T = T(\varepsilon, \delta) > 0$ in such a way that $\phi\ \epsilon\ PCB_\delta$
means that $|x(t, \sigma, \phi)| < \varepsilon,\ t \ge \sigma + T$
In addition, we describe the following classes of functions for later use:
$K_1 = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)|_{a(0)} = 0\ and\ a(s) > 0\ for\ \ s\ > 0$
$K_2 = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)|_{a \in K_1} and\ a\ is\ nondecreasing\ in\ \ s$

Stability results
Now, in this section, we shall consider our stability result

Theorem 1. Let's say that we have some functions, namely Ω1 , Ω2 ∈ K2, and P, G ∈ K1 ;
We also have q ∈ C($\mathbb{R}_+$, $\mathbb{R}_+$), g ∈ P C($\mathbb{R}_+$, $\mathbb{R}_+$), and U(t, x)∈ v_0 and constants $\beta_k \ge 0$, k∈ $\mathbb{Z}_+$, in such away that

i. Ω1(|x|) ≤ U(t, x) ≤ Ω2 (|x|), and (t, x) ∈ [t_0, T] x S(ρ);
ii. For each (t_k, ψ) ∈ $\mathbb{R}_+$ x PC([α, 0], S(ρ_1)),
U(t_k, $\psi(0)$ + $I_k(t_k, \psi)$) - U(t_k^{-1}, $\psi(0)$) ≤ $\beta_k(t_k^{-1}, \psi(0))$,
Where $\sum_{k=1}^{\infty} \beta_k \doteq \beta < \infty$;
iii. For any $\sigma \ge t_0$ and ψ that is in a piecewise continuous interval ([α, 0], S(ρ)), if piecewise set P(U(t, $\psi(0)$)) > U(t+θ, $\psi(0)$) for Max{α, - q(U(t))} ≤ θ ≤ 0, then
D^+U(t, $\psi(0)$) ≤ - g(t)G(U(t, $\psi(0)$)), t ∈ [t_{k-1}, t_k), k ∈ $\mathbb{Z}_+$,
Where P(s) > s for s > 0;
iv. Let's assume we have ε_2> ε_1> 0, then there exists a η = η($\varepsilon_1, \varepsilon_2$) > 0 in such a way that for any A > 0 implies that

$$\int_A^{A+\eta} g(t)dt > \frac{(1+\beta)\Omega_2(\varepsilon_2)}{M}$$

Such that M = $inf_{0.5\Omega_1(\varepsilon_1) \le s \le \Omega_2(\varepsilon_2)}$G(s)
Then the zero solution of (I) is uniformly asymptotically stable
Proof. The first step is to establish that the zero solution of (I) is uniformly stable.
Given any $\varepsilon \in (0, \rho_1)$, one may select a $\delta > 0$ in such away that $\Omega_2(\delta) \le \beta^{*-1}\Omega_1(\varepsilon)$, in return
$\beta^* = \prod_{k=1}^{\infty}(1+\beta_k) + 1$. Given $\sigma \ge t_0$ and $\emptyset \in PCB_\delta$, assume x(t) = x(t, $\sigma, \emptyset$) be a solution of (1) along the path ($\sigma, \emptyset$),
Notice that $\emptyset \in PCB_\delta$, obviously
Ω1(|x|) ≤ U(t, x(t)) ≤ $\Omega_2(\delta) \le \beta^{*-1}\Omega_1(\varepsilon) < \Omega_1(\varepsilon)$, σ +α ≤ t ≤ σ
Which means that |x(t)| < ρ_1, t ∈ [σ +α, σ]
Let assume we have $\sigma \in [t_{m-1}, t_m)$ for certain m ∈ $\mathbb{Z}_+$, then we can go ahead to prove for t ∈ [$\sigma + t_m$)
U(t, x(t)) ≤ $\beta^{*-1}\Omega_1(\varepsilon)$. (2)
Let say this is false, then we have some t ∈ [$\sigma + t_m$) in such a way that
U(t, x(t)) > $\beta^{*-1}\Omega_1(\varepsilon)$.
Define
t^*= inf{ t ∈ [$\sigma + t_m$), U(t, x(t)) > $\beta^{*-1}\Omega_1(\varepsilon)$},
Obviously, $t^* \ge \sigma$, U($t^*, x(t^*)$) = $\beta^{*-1}\Omega_1(\varepsilon)$ and U(t, x(t)) > $\beta^{*-1}\Omega_1(\varepsilon)$, t ∈ [$\sigma, t^*$],
At the same time, we know
D*U$|_{(1)}$(t*, x(t*))≥0 (3)
Here, it holds
P(U(t*, x(t*)))>U(t*, x(t*)) = $\beta^{*-1}\Omega_1(\varepsilon)$≥ U(s, x(s)), t*+ α ≤ x ≤ t*
By statement (c), g ∈PC($\mathbb{R}_+$, $\mathbb{R}_+$), and G ∈ K_1 we obtain
D* U(t*, x(t*))≤ -g(t*)G(U(t*, x(t*))) = -g(t*)G($\beta^{*-1}\Omega_1(\varepsilon)$)<0,
This inequality contradict (3). Although (2) still holds. It simply means that
x(t_m^-) ∈ S(ρ_1), x(t_m) ∈ S(ρ).
note that.
U(t_m, x(t_m)) ≤ (1 + β_m)V(t_m^-, x(t_m^-) ≤ β^{*-1}(1 + β_m)$\Omega_1(\varepsilon)$.
To show that for t ∈ [t_m, t_{m+1})
U(t, x(t)) ≤ β^{*-1}(1 + β_m)$\Omega_1(\varepsilon)$.
Assume this is false, then we can define
t* = inf[t ∈[t_m, t_{m+1}), U(t, x(t))> β^{*-1}(1 + β_m)$\Omega_1(\varepsilon)$]
Thus, by the same arguments as the proof of (2), we can arrive at a contradiction and we shall therefore omit the details.
By a previous induction hypothesis, we can prove that for

$t \in [\sigma, t_m) \cup [t_k, t_{k+1})$, k≥m,
U(t, x(t)) $\leq \beta^{*-1}(1+\beta_m)(1+\beta_{m+1})\ldots(1+\beta_k)\Omega_1(\varepsilon)$.
Which yields
$\Omega_1(\|x\|) \leq$U(t, x(t)) $\leq \beta^{*-1}\prod_{\sigma<t_k<t}(1+\beta_k)\Omega_1(\varepsilon) < \Omega_1(\varepsilon)$, $t \geq \sigma$,
Hence, |x(t)| $<\varepsilon$, $t \geq \sigma$. Considering the choice of δ, the zero solution of (1) is uniformly stable.
Then, we prove the uniformly asymptotic stability.As (1) has the zero solution which is uniformly stable, for any given $\varepsilon_2 \in (0,\rho_1)$, $\sigma \geq t_\theta$, we can look for a related$\delta = \delta(\varepsilon_2) > 0$ in such a way that for any $\emptyset \in PCB_\delta$ indicate that |x(t)| $\leq \varepsilon_2 < \rho_1$, $t \geq \sigma$ and U(t, x(t)) $\leq \Omega_2(\varepsilon_2)$, $t \geq \sigma$. In the sequel, for the sake of without loss of generality we assume that$\sigma \in [t_{m_1-1}, t_{m_1})$, $m_1 \in \mathbb{Z}_+$.
For any $\varepsilon \in (0, \varepsilon_2)$, we make a choice of arbitrary constants M, also a as follows:
M = M$(\varepsilon_2, \varepsilon)$= $inf_{0.5\Omega_1(\varepsilon)\leq s\leq \Omega_2(\varepsilon_2)}$G(s)
$\alpha = \alpha(\varepsilon_2, \varepsilon) = \min\{inf_{0.5\Omega_1(\varepsilon)\leq s\leq W_2(\varepsilon_2)}[P(s)-s], 0.5\Omega_1(\varepsilon)\}$
obviously, M > 0, α > 0. Also, from statement (d) we observe that we have$\eta = \eta(\varepsilon, \varepsilon_2) > 0$ in such a way that for any A > 0 implies that

$$\int_A^{A+\eta} g(t)dt > \frac{(1+\beta)\Omega_1(\varepsilon_2)}{M} \quad (4)$$

Now we choose N $\in \mathbb{Z}_+$, such that
$0.5\Omega_1(\varepsilon)$ + (N – 1) α $\leq \Omega_2(\varepsilon_2) < 0.5\Omega_1(\varepsilon) + N\alpha$.
Since $\sum_{i=1}^{\infty}\beta_i < \infty$, we have$N_\circ > m_1$ which is large enough in such a way that

$$\sum_{i=N_\circ}^{\infty}\beta_i < \frac{\alpha}{3\Omega_2(\varepsilon_2)}, and\ \beta_k < \frac{\alpha}{3N\Omega_1(\varepsilon)}, k \geq N_\circ \quad (5)$$

Suppose that $t_{N_\circ} = \sigma + \lambda\eta$, where λ is an arbitrary constant. Then we proof that we have
$T_1 > t_{N_\circ}$in such a way that

$$U(T_1, x(T_1)) < 0.5\Omega_1(\varepsilon) + (N-1)\alpha. \quad (6)$$

Suppose on the contrary, then for all $t > t_{N_\circ}$
U(t, x(t)) $\geq 0.5\Omega_1(\varepsilon) + (N-1)\alpha \geq 0.5\Omega_1(\varepsilon)$.
Considering the definition of α, we get
P(U(t, x(t))) ≥ U(t, x(t)) + α
$\geq 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \alpha$
$= 0.5\Omega_1(\varepsilon) + N\alpha$
$> \Omega_2(\varepsilon_2) \geq$ V(s, x(s)), t + α≤ s ≤ t, t >$t_{N_\circ}$
By assuming (c), we have the inequality D*U(t, x(t)) ≤ -g(t) G(U(t, x(t))) holds for all
$t > t_{N_\circ}$, t≠ t_k. Integrating above inequality from $t_{N_\circ}$ to $t_{N_\circ} + \eta$, by (4) we get
U($t_{N_\circ} + \eta$, x($t_{N_\circ} + \eta$)) ≤ U($t_{N_\circ}$, x($t_{N_\circ}$)) - $\int_{t_{N_\circ}}^{t_{N_\circ}+\eta} g(s)G(U(x))ds$
$+ \sum_{t_{N_\circ}<t<t_{N_\circ}+\eta}[U(t_k) - U(t_K^{-1})]$
≤ U($t_{N_\circ}$, x($t_{N_\circ}$)) – M $\int_{t_{N_\circ}}^{t_{N_\circ}+\eta} g(s)ds$
$+\sum_{t_{N_\circ}<t<t_{N_\circ}+\eta}\beta_k\ U(t_K^{-1})$
$\leq \Omega_2(\varepsilon_2)$ - M $\int_{t_{N_\circ}}^{t_{N_\circ}+\eta} g(s)ds$
$+ \sum_{t_{N_\circ}<t<t_{N_\circ}+\eta}\beta_k\,\Omega_2(\varepsilon_2)$
$\leq \Omega_1\varepsilon_2()(1+\beta)$ - M $\int_{t_{N_\circ}}^{t_{N_\circ}+\eta} g(s)ds$
< 0,
Which is false. Thus equation (6) holds. We make a choiceT_1 $=t_{N_\circ} + \eta = \sigma + (\lambda+1)\eta$
To show for all t > T_1

$$U(t, x(t)) \geq 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \quad (7)$$

Assume this is false, then we have $\tau_2 > T_1$in such a way that

$$U(\tau_2, x(\tau_2)) \geq 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \quad (8)$$

And

$$U(t, x(t)) < 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \text{ for all } T_1 \leq t < \tau_2 \quad (9)$$

Suppose that $T_1 \in [t_m, t_{m+1})$, m ≥ $N_\circ$, m$\in \mathbb{Z}_+$, then we assert that $\tau_2 \geq t_{m+1}$. Contrarily, then $\tau_2 \in [T_1, t_{m+1})$. Since (6) holds, obviously, we have$\tau_1 \in [T_1, \tau_2)$ in such away that
$U(\tau_1, x(\tau_1)) = 0.5\Omega_1(\varepsilon) + (N-1)\alpha$
Then we have for t $\in [\tau_1, \tau_2]$
P(U(t, x(t))) ≥U(t, x(t)) + α
$\geq 0.5\Omega_1(\varepsilon) + N\alpha$
$> \Omega_2(\varepsilon_2) \geq$U(s, x(s)), t + α ≤ s ≤ t
Using statement (c) we have
D*U(t, x(t)) ≤ -g(t)G(U(t)) ≤ 0, $\tau_1 \leq t \leq \tau_2$
Impling that
U(τ_2, x(τ_2)) ≤U(τ_1, x(τ_1))
This is false in view of (8). Then we have shown that $\tau_2 \geq t_{m+1}$. Without loss of generality, we can assume that $\tau_2 \in [t_{m+q}, t_{m+q+1})$, q ≥ 1. Subsequently we shall assert that we have $\tau_1 \in (T_1, \tau_2)$ in such a way that

$$0.5\Omega_1(\varepsilon) + (N-1)\alpha < U(\tau_1, x(\tau_1)) < 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \quad (10)$$

Through (9), it suffices to show only the l-h-s inequality of (10). Assume this inequality does not hold for all $t \in (T_1, \tau_2)$,
U(t, x(t))≤ $0.5\,\Omega_1(\varepsilon) + (N-1)\alpha$.
Then by (8), we are aware that there should be $\tau_2 = t_{m+q}$.
Next is that
U(t_{m+q}, x(t_{m+q})) $\geq 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2}$, U($t_{m+q}^{-1}$, x($t_{m+q}^{-1}$)) $\leq 0.5\Omega_1(\varepsilon) + (N-1)\alpha$,
Which together with statement (b) yields
$\frac{\alpha}{2} \leq \beta_{m+q}$U($t_{m+q}^{-}$, x($t_{m+q}^{-}$)) $\leq \beta_{m+q}\Omega_1(\varepsilon_2)$
Consequently, we have

$$\beta_{m+q} \geq \frac{\alpha}{2\Omega_1(\varepsilon_2)}$$

This is a contradiction considering the first inequality of equation (5) and therefore, equation (10) holds
Definition now
$\tilde{\tau_1}$ = sup[t $\in [T_1, \tau_2]$, Ω_1(t, x(t))< $0.5\Omega_1(\varepsilon) + (N-1)\alpha$]
Then
U($\tilde{\tau_1}$, x($\tilde{\tau_1}$)) $\leq 0.5\Omega_1(\varepsilon) + (N-1)\alpha$

$$U(\tilde{\tau_1}, x(\tilde{\tau_1})) = U(\tilde{\tau_1}^*, x(\tilde{\tau_1}^*)) \geq 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha \quad (11)$$

And

$$0.5\Omega_1(\varepsilon) + (N-1)\alpha \leq U(t, x(t)) \leq 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2},\ t \in [\tilde{\tau_1}, \tau_2], \quad (12)$$

Because of (10) we are aware that $\tilde{\tau_1} < \tau_2$. Note that $\tau_2 \in [t_{m+q}, t_{m+q+1})$, we go a step further to show that $\tilde{\tau_1} < t_{m+q}$.
Assuming on the converse that $\tilde{\tau_1} \in [t_{m+q}, \tau_2)$ we do not have impulse point t_k in between $\tilde{\tau_1}$ and τ_2.
From equation (12), we get
P(U(t, x(t))) ≥ U(t, x(t)) + α
$\geq 0.5\,\Omega_1(\varepsilon) + N\alpha$
$> \Omega_2(\varepsilon_2) \geq$U(s, x(s)), t + α ≤ s ≤ t, $\tilde{\tau_1} \leq t \leq \tau_2$
By statement (c) we get
D*U(t, x(t)) ≤ -g(t)G(U(t)) ≤ 0, $\tilde{\tau_1} \leq t \leq \tau_2$
Which indicate that
U(τ_2, x(τ_2)) ≤ U($\tilde{\tau_1}$, x($\tilde{\tau_1}$)).
This is contrary to the definition of $\tilde{\tau_1}$. Therefore, we have that $\tilde{\tau_1} < t_{m+q}$.
Assume that $\tilde{\tau_1} \in [t_{m+k}, t_{m+k+1})$, 1 ≤ k < q, then we put into consideration two possibilities stated below:
Case 1: if $\tilde{\tau_1} > t_{m+k}$, i.e., $\tilde{\tau_1} \in [t_{m+k}, t_{m+k+1})$, then putting into consideration the definition of $\tilde{\tau_1}$, we get
U($\tilde{\tau_1}$, x($\tilde{\tau_1}$)) = $0.5\,\Omega_1(\varepsilon) + (N-1)\alpha$.
From equation (12) we can derive that for $t \in [\tilde{\tau_1}, \tau_2]$,
P(U(t, x(t))) ≥ U(t, x(t)) + α > U(s, x(s)), t + α ≤ s ≤ t.

By (c) this inequality D*U(t, x(t)) ≤ -g(t)G(U(t)) ≤ 0 holds for t ∈ [$\tilde{\tau_1}$, τ_2]. Therefore we get

$0.5\,\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \le U(\tau_2, x(\tau_2))$

$\le U(\tilde{\tau_1}, x(\tilde{\tau_1})) + \sum_{i=m+k+1}^{m+q}[U(t_1) - U(t_1^-)]$

$\le 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha + \sum_{i=m+k+1}^{m+q}\beta_i U(t_1^-)$

$\le 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha + \sum_{i=m+k+1}^{m+q}\beta_i \Omega_2(\varepsilon_2)$,

Which yields

$\frac{\alpha}{2} \le \sum_{i=m+k+1}^{m+q}\beta_i \Omega_2(\varepsilon_2)$

This does not align with the first inequality of equation (5). Which implies that we cannot possibly have case 1

Case 2: If $\tilde{\tau_1} = t_{m+k}$, then by equation (11), we have

$U(t_{m+k}^{-1}, x(t_{m+k}^{-1})) \le 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha$.

Then,

$U(\tilde{\tau_1}, x(\tilde{\tau_1})) = U(t_{m+k}, x(t_{m+k})) \le (1 + \beta_{m+k})\, U(t_{m+k}^{-1}, x(t_{m+k}^{-1}))$

$\le (1 + \beta_{m+k})[\,0.5\,\Omega_1(\varepsilon) + (N-1)\alpha]$.

From equation (12) we still have that P(U(t, x(t))) > U(s, x(s)), t + α ≤ s ≤ t, $\tilde{\tau_1}$ ≤ t ≤ τ_2. Using statement (c) again, we get that inequality D*U(t, x(t)) ≤ -g(t)G(U(t)) ≤ 0 holds for t ∈ [$\tilde{\tau_1}$, τ_2]. Therefore, in this case we derive.

$0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2} \le U(\tau_2, x(\tau_2))$

$\le U(\tilde{\tau_1}, x(\tilde{\tau_1})) + \sum_{i=m+k+1}^{m+q}[U(t_1) - U(t_1^-)]$

$< (1 + \beta_{m+k})[\,0.5\Omega_1(\varepsilon) + (N-1)\alpha]$

$+ \sum_{i=m+k+1}^{m+q}\beta_1 U(t_1^-)$

Which when combine with the latter inequality of equation (5) and considering the fact that α ≤ 0.5 $\Omega_1(\varepsilon)$ yields.

$\frac{\alpha}{2} \le \beta_{m+k}[0.5\Omega_1(\varepsilon) + (N-1)\alpha] + \sum_{i=m+k+1}^{m+q}\beta_1\Omega_2(\varepsilon_2)$

$\le \beta_{m+k} N 0.5\,\Omega_1(\varepsilon) + \sum_{i=m+k+1}^{m+q}\beta_1\Omega_2(\varepsilon_2)$

$\le \frac{\alpha}{3N\Omega_1(\varepsilon)} \cdot N0.5\Omega_1(\varepsilon) + \sum_{i=m+k+1}^{m+q}\beta_1\Omega_2(\varepsilon_2)$

This is,

$\frac{\alpha}{3} \le \sum_{i=m+k+1}^{m+q}\beta_1\Omega_2(\varepsilon_2)$

This does not align with the first inequality of equation (5). Which implies that we cannot possibly have case 2 either. This shows, we have proven that equation (7) holds for all t >T_1.

Now, we have this hypothesis by equation (6) and equation (7):

$$\begin{cases} U(T_1, x(T_1)) < 0.5\Omega_1(\varepsilon) + (N-1)\alpha \\ U(t, x(t)) < 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2}, t > T_1 \end{cases} \quad (13)$$

Where $T_1 = \sigma + (\lambda + 1)\eta$

We describe an arbitrary constant:

q = sup{$q(s)$|0.5 $\Omega_1(\varepsilon)$≤ x ≤ $\Omega_2(\varepsilon_2)$ }

Then we can derive the existence of T_2>$T_1 + q$in such a way that

U(T_2, x(T_2) <0.5$\Omega_1(\varepsilon) + (N-2)\alpha + \frac{\alpha}{2}$

This proof is equivalent to the proof of equation (6) using equation (13), and all that is required is to note the Razumikhin condition:

P(U(t, x(t))) ≥ U(t, x(t)) + α

$\ge 0.5\Omega_1(\varepsilon) + (N-1)\alpha + \frac{\alpha}{2}$

>U(s, x(s)), max {t + α, t – q(U(t))} ≤ s ≤ t, t >T_1+ q

We make a choice of$T_2 = T_1$ + q +$\eta = \sigma + (\lambda + 1)\eta + q$. Then using the same argument as equation (7), we now have that for all t > T_2

U(t, x(t)) < 0.5 $\Omega_1(\varepsilon) + (N-1)\alpha$, t > T_2.

In this manner we can show that for j ∈ $\mathbb{Z}_+$,

$$\begin{cases} U\left(T_j, x(T_j)\right) < 0.5\,\Omega_1(\varepsilon) + (N-1)\alpha - \frac{j-1}{2}\alpha \\ U(t, x(t)) < 0.5\Omega_1(\varepsilon) + (N-1)\alpha - \frac{j-2}{2}\alpha \;, t > T_j \end{cases}$$

Where $T_j = \sigma + (\lambda + 1)\eta$ +(q + η)(j-1). Specifically, let j = 2N, then we get that U(t, x(t)) <0.5 $\Omega_1(\varepsilon)$<$\Omega_1(\varepsilon)$, $t > T_{2N}$. It indicates that |x(t)| <ε, $t > T_{2N}$. Recall that $(\lambda + 1)\eta$ +(q + η)(2N-1) does not dependent onσ, then we get that the zero solution of equation (1) is uniformly asymptotically stable■

If we consider only the uniform stability of (1) then one can obtain the following result.

Corollary 3.1. The zero solution of equation(1) is uniformly stable if per existence of some function $W_1, W_2 \in K_2$, P, G ∈ K_1, g ∈ PC($\mathbb{R}_+$, $\mathbb{R}_+$), U(t, x) ∈ v_0 and an arbtrary constants $\beta_k \ge 0, k \in \mathbb{Z}_+$in such a way that statements (a)(b)(d) in Theorem 3.1 and (e) hold, in such a way that:

(e) For eachσ, 0 ≥ t_0 *and* $\psi \in PC([\alpha, 0], S(\rho))$ if P(U(t, $\psi(0)$))> U(t + $\theta, \psi(\theta)$)

for α ≤ θ ≤ 0,then

D*U(t, $\psi(0)$) $\le -g(t)G(U(t, \psi(0)))$, t ∈ $[t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$,

such that P(s) > s for s > 0,

However, if function g(t) fulfills the condition that $inf_{t \in \mathbb{R}_+}$, g(t) = $\mu > 0$, then by Theorem 3.1 and Corollary 3.1, we get the results stated below respectively.

Corollary 3.2. The zero solution of equation (1) is uniformly asymptotically stable if per existence of some functions W_1, $W_2 \in K_2$, P, G ∈ K_1 , U(t, x) ∈ v_0 and an arbitrary constants $\mu > 0\beta_k \ge 0, k \in \mathbb{Z}_+$, in such away that statements (a), (b) in Theorem 3.1 and (d) hold, such that, we have.

(f).. For eachσ, ≥ t_0 and $\psi \in PC([\alpha, 0]$, S(ρ)), if P(U(t, $\psi(0)$)) > U(t + $\theta, \psi(\theta)$) for max$\{\alpha, -q(U(t))\} \le \theta \le 0$, then

D*U(t, $\psi(0)$) $\le -\mu G(U(t, \psi(0)))$, t ∈ $[t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$,

such that P(s) > s for s>0.

Corollary 3.3. The zero solution of equation(1) is uniformly stable if per existence of some functions $W_1, W_2 \in K_2$, P, G ∈ K_1, V(t, x) ∈ v_0 and an arbitrary constants $\mu > 0, \beta_k \ge 0, k \in \mathbb{Z}_+$in such a way that statements (a), (b) in Theorem 3.1 and (f) hold such that.

(g) For each$\sigma \ge t_0$ and $\psi \in PC([\alpha, 0]$, S(ρ)), if P(U(t, $\psi(0)$)) > U(t + $\theta, \psi(\theta)$) for α ≤ θ ≤ 0, then

D*U(t, $\psi(0)$) $\le -\mu G(U(t, \psi(0)))$, t ∈ $[t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$,

Such that P(s) > s, for s>0.

Proof. For any given $\varepsilon_2 > \varepsilon_1 > 0$, one can choose $\eta = \frac{(1+\beta)W_2(\varepsilon_2)}{\mu M}$, where M=$inf_{0.5W_1(\varepsilon) \le s \le W_2(\varepsilon_2)}$G(s).■$s$

Discussion

The paper focuses on a class of mathematical models that describe dynamic systems affected by sudden changes (impulses) and time delays in their state variables. We establish a comprehensive framework to analyze impulsive functional differential inclusions, leveraging classical theories and modern inclusions.

Key Findings

Stability Analysis: The study employs Lyapunov functions and the Razumikhin technique to derive results on uniform stability and uniform asymptotic stability for impulsive functional differential inclusions. This is significant as it provides a method to assess how these systems behave under various conditions.

Sufficient Conditions: We derive sufficient conditions for the stability of solutions, which depend on varying impulse magnitudes and delay intervals. These conditions are crucial for determining when the system remains stable despite sudden changes.

Interplay of Delays and Impulses: An important aspect of the research is the exploration of how impulsive effects interact with delayed responses. This interplay is critical for understanding the overall stability of the system and yields insights that can be applied to real-world systems.

CONCLUSION

The theoretical breakthroughs achieved in analyzing impulsive functional differential inclusions with finite delays apply deeply to those systems in the real world where sudden disruptions and delayed reactions are embedded. Through the elaboration of tough stability criteria, this research arms practitioners in disciplines with methods on designing, optimizing and controlling complex dynamic systems in delay-prone settings. Below, we discuss the transformational uses of this work across different domains.

The stability criteria obtained from this study support the design of robust controllers for systems that run on intermittent changes and feedback latencies in control systems and robotics. Such vehicles such as autonomous vehicles need use real-time sensor data to operate in dynamic environments. However, delay in processing lidar or camera inputs combined with an impulsive correction to avert obstacles causes instability of trajectories. It is possible for engineers to develop controllers to withstand such bounded delays, impulse magnitudes under the method of Lyapunov based stability conditions to make it safe and reliable to work even in unpredictable scenarios. Likewise, industrial robotic arms performing precision tasks (e.g. assembly or welding) need to be assured of stability when sudden mechanical adjustments or communication delays happen between sensors and actuators.

Power grids and electrical network are another area of critical applications. In modern smart grids, renewable energy sources and separate control occur, and switching events (for ex. activating circuit breakers) and communication lags between nodes of the grid are quite frequent. These systems can be inadequate during fault conditions where, for instance, multiple faults cause cascading failures when there is delayed fault detection, or impulsive load shifts. The developed framework in this research enables grid operators to determine the safe thresholds of delays as well as the intensities of impulse correlating to transient stability and blackout avoidance. For instance, when there is a sudden upsurge in production of solar energy, then a stability criterion can determine rate of response required by backup systems to regulate supply and demand even with inherent communication delays.

Delays and impulses in combination play an ubiquitous role in biological systems and medicine. Neural networks, that are filtered by the synaptic transmission delays and also impulsions, can be described with impulsive functional differential inclusions. The stability analysis offers insights on why delayed inhibitory signals may not be able to suppress impulsive excitatory spikes mathematically caressing the design of neurostimulation devices to help stabilize abnormal brain activity. In the same way, pharmacokinetic systems benefiting from this framework are systems where doses of drugs are shot impulsively (for example, by injections) into the body with delays in metabolism. Clinicians can maximize dosing schedules to achieve therapeutic drug levels with no toxicity, utilizing metabolic delays based on that of the patient physiology.

Networked systems and the Internet of Things (IoT) on their own are indicative of the versatility of this research as well. Environmental monitoring systems, which are distributed IoT networks, will have delayed data transmission, because of bandwidth constraints and impulsive state changes, like resets of sensor nodes after power failures. Network stability criteria guarantee that such networks synchronize well, while preserving the integrity and functionality of data amidst the lack of connectivity at some point. For example, in the wildfire detections systems analysis of delayed alerts from remote sensors, and impulsive recalibration of the drones circulating in the area can help to ensure prompt, coordinated and effective response.

This work extends the study of impulsive functional differential inclusions with finite delays by synthesising classical and contemporary theoretical frameworks. Utilizing the smart manner of invoking Lyapunov functions and the Razumikhin method, we developed precise criteria for uniform stability and uniform asymptotic stability, indicating clearly the weighting of impulsive perturbations relative to delayed dynamics. The obtained sufficient conditions reveal that differences in impulse magnitude and delay interval sculpt system behavior, and provide a principled account of how to predict and manipulate stability in these intricate systems. Through this, illustration of these theoretical developments with relevant instances, the research connects abstract mathematical analysis to tangible applications of control theory, engineering, and biology. Not only have these results added more substance to the theory of impulsive differential inclusions, they have also armed practitioners with a strong set of tools for designing resilient systems in which the delays and sudden disturbances are integral components. This work therefore represents a basic contribution to mathematical theory and interdisciplinary dynamical system design.

REFERENCES

Bainov, D. and Simeonov, P. (1993) Impulsive Differential Equations: Periodic Solutions and Applications. Longman Scientific and Technical, Vol. 66, Bookcraft (Bath) Ltd., New York.

Benchohra M, Henderson J, and Ntouyas S (2006) Impulsive Differential Equations and Inclusions (Contemporary Mathematics and Its Applications, Vol 2) Hindawi Publishing Corporation

Egbetade S.A and Abimbola, L.A. (2025): Numerical Solution of First and Second Order Differential Equation Using The Tau Method with an Estimation of the Error. FUDMA Journal of Sciences, Vol 9 No3. : https//doi.org/10.33003/fjs-2025-0903-3346

Jack K. H and Hüseyin K. (1991).Dynamics and Bifurcations. Text in Applied Mathematics, Springer New York, NY, https://doi.org/10.1007/978-1-4612-4426-4

Lyapunov, A.M. (1892). The General Problem of the Stability of Motion. Kharkov Mathematical Society, Kharkov.

Lu , X, Carabolla, T, Rakkiyappan R, X Han. (2015). On the Stability of impulsive functional differential equations with infinite delays. Mathematical methods in Applied sciences, Vol.38, Issue 14, pg 3130-33140

Smith,B.J, Monterosso,J.R, Wakslak,C.J, Bechara,A, Read, S.J. (2018A meta-analytical review of brain activity associated with intertemporal decisions: Evidence for an anterior-posterior tangibility axis Neurosci. Biobehav Rev.,86(2018), pp.85-98

Stamova, I. and Stamov, G. (2014). Stability analysis of impulsive functional systems of fractional order, Communications in Nonlinear Science and Numerical Simulation 19. 702–709.

Zhang, Y and Sun, J. (2006). Stability of impulsive infinite delay differential equations. Applied Mathematics Letters 19.1100–1106.