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APPROXIMATE SOLUTIONS TO STIFF PROBLEMS OF THREE-STEP LINEAR MULTISTEP METHOD USING HERMIT POLYNOMIALS

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ABSTRACT

Linear multistep method is a problem-solving technique mostly used to find the solution to mathematical problems involving one independent variable mostly called ordinary differential equations. However, this research seeks to carry out a formulation of an efficient numerical scheme for the approximation of first order ordinary differential equation (ODE) has been investigated. The method is a block scheme for 3-step linear multistep method using Hermit polynomials as the basis function. The continuous and discrete multi-step methods (LMM) have been formulated through the technique of collocation and interpolation. Also, numerical examples of ODE'S have been solved and results obtained show that the proposed scheme can be efficient in solving initial value problems of first order ordinary differential equations.

Keywords: Linear multi-step method, Ordinary differential equations, Initial value problems, Hermite polynomials

INTRODUCTION

Background to the study

Linear multistep methods (LMMs) are very popular for solving initial value problems (IVPs) of Ordinary Differential Equations (ODEs). They are also applied to solve higher order ODEs. LMMs are not self-starting hence, need starting values from single-step methods like Euler's Method and Runge-Kutta family of methods Lambart, (1973).

We consider the general form of the first order initial value problems.

$$y'(x) = f\big(x, y(x)\big), y(x_0) = y_0 \quad (1)$$

$$y_k(x) = \sum_{i=0}^{k} c_i \omega_i(x),\ x_n \le x \le x_{n+k} \quad (2)$$

Where

$$\omega_i(x) = x^i, i = 0,1,2, \dots k \quad (3)$$

Substituting (2) into (1) and add $\lambda H_k(x)$ where λ is the perturbed term and $H_k(x)$ is the Hermite polynomial of degree k valid in $x_n \le x \le x_{n+k}$ we have

$$\sum_{i=0}^{k} c_i \omega_i(x), = f(x, y) + \lambda H_k(x) \quad (4)$$

We shall consider cases where $k = 1,2$ and 3 in (2) and (3)

The Hermite polynomial is given by $H_i(x) = \ i = 0,1,2, \dots k$

$$\left.\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \end{aligned}\right\} \quad (5)$$

These polynomials are gotten from the Hermite Rodrigue's formula

$$H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n}\left(e^{x^2}\right) \quad (6)$$

Using the set of polynomials in (5) to formulate the block schemes in the interval $[x_n, x_{n+k}]$, thus, introducing the change of variable to define the Hermite polynomial as

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)} \quad k = 1,2,3 \quad (7)$$

According to Lambert (1973) the general k-step LMM is given as;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad (8)$$

Where α_j and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0$. The LMM in Equation (8) generates discrete schemes which are used to solve first-order ODEs. Other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMMs of the form;

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (9)$$

Where α_j and β_j are expressed as continuous functions of x and are at least differentiable once.

According to Okunuga and Ehigie (2009) the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of LMM. Continuous collocation and interpolation technique is now widely used for the derivation of LMMs, block methods and hybrid methods. Several continuous LMMs have been derived using different techniques and approaches; Alabi (2014) derived continuous solvers of IVPs using Chebyshev polynomial in a multistep collocation technique; Okunuga and Ehigie (2009) derived two-step continuous and discrete LMMs using power series as basis function; Mohammed derived a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods which were directly applied to solve first-order ODEs.

The analytical solution of differential equation may not be easily gotten due to the rate and region of convergence which is always potential in such case numerical methods are very useful. In engineering and physical sciences problems involving first order differential equation can be solved using various methods. One of which is using analytical methods. However, this is tedious and scientists like Adam Moulton (1971) develop method for direct solution of first-order differential ordinary equation could not obtain accurate solution with very low efficiency in comparison to the exact solution of the problem. In this research we will work on a three-step method to solve a stiff problem using hermit polynomials as a bases function in which higher order accuracy will be achieve.

The idea of LMM is one that is beginning to strike the cord in our world today due to its input in the field of study and our lives in general.

The aim of this research paper is to formulate an effective three-step numerical method using the hermit polynomial as a bases function for solution of stiff problems.

In recent years, Edogbanya Helen Olaronke *et al* (2020), proposed a modified Laguerre collocation block method for solving second order ordinary differential equation. Discrete method was given which were used in block and implemented for solving the initial value problems, being the continuous interpolation derived the collocated at grid points. The derived scheme was used to solve some second order differential equation (ODEs) in order to show their validity and accuracy.

There are so many authors who have done work on the direct solution of ordinary differential equation of the form $f(x, y_0, y_1, y_2, \ldots, y_{n.})$, $y(p) = y_0, y(q) = y_1, .. y_{(n-1)} = y_n,$ (10)

Which comprises of Awoyemi (2001), and Kayode (2008). Every one of them worked on the development of several methods for solving equations directly without having to reduce to system of first order differential equations. Awoyemi (2001) developed methods to solve second order initial value problems which are the mathematical formulation for systems without dissipation. Awoyemi (2001) developed Nystrom type technique for initial value problem (IVPs) for the solution of first order differential equations in which the conditions for the determination of the parameters of the systems were discussed. Power series has been used in differential Algebraic systems to solve differential algebraic equations by Haweel (2015). Michael (2016) researched on using power series method as a basic method for solving linear differential equations with variable co-efficient and also on second order differential equations. Familua and Omole, (2017) 5-points mono hybrid point linear multistep method for solving nth order ordinary differential equations using power series functions. Yahaya et al. (2024) examined the development of mathematical model for optimal rice production in Niger state base on variables such as rainfall , temperature, humidity and land area used and production cost using multivariate regressional (MLR) method. Result shows that 96.35% variance in rice production can be explained by the independent variable due to accuracy and high level of yield.

Odekunle *et al* (2013) developed a continuous linear multistep method using interpolation and collocation for the solution of first-order ODE with constant step size; Adesanya *et al* (2012) considered the method of collocation of the differential system and interpolation of the approximate solution to generate a continuous LMM, which is solved for the independent solution to yield a continuous block method; James *et al* (2013) proposed a continuous block method for the solution of second order IVPs with constant step size, the method was developed by interpolation and collocation of power series approximate solution. Anake (2011) developed a new class of continuous implicit hybrid one-step methods capable of solving IVPs of general second order ODEs using the collocation and interpolation techniques of the power series approximate solution. James *et al* (2013) adopted the method of collocation and interpolation of power series approximate solution to generate a continuous LMM.

MATERIALS AND METHODS

Derivation of the Methods

The two-step hybrid block method with second derivative that produces approximations $y_{n+k}$to the first order ordinary differential equations (ODEs)

$$x' = f(t, x) \qquad (11)$$

is given as follows:

$$\sum_{j=0}^{k} \alpha_j x_{n+j} + \sum_{j=1}^{2} \alpha_{vj} x_{n+vj} = h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=1}^{2} \beta_{vj} f_{n+vj}\right) + h^2 \gamma_k f'_{n+k} \qquad (12)$$

α_j, α_{jv} β_j and β_{vj} are constant coefficients.

In order to obtain (12), we approximate the solution by the orthogonal function $X(t)$ of the form

$$X(t) = \sum_{j=0}^{r+s-1} a_j \varphi(t) \qquad (13)$$

where

(i) $t \in [a, b]$

(ii) $\varphi(t)$ is an orthogonal function defined by the Hermit polynomial

(iii) a_j are unknown coefficients to be determined

(iv) r is the number of interpolations for $1 \le r \le k$ and

(v) s is the number of distinct collocation points with $s > 0$

The continuous approximation is constructed by imposing the following conditions

$$X(t_{n+\mu}) = x_{n+\mu}, \{j, v_1, v_2\}, j = 0,1, \ldots, k-1 \qquad (14)$$

$$X'(t_{n+\mu}) = f_{n+\mu}, \{j, v_1, v_2\}, j = 0,1, \ldots, k \qquad (15)$$

$$X''(t_{n+j}) = f'_{n+j}, j = k \qquad (16)$$

where v_1 and v_2 are not integers. Equations (14) – (16) form a nonlinear system of equations in $a_j's$ which is solved using the matrix inversion technique via Maple software. The values of $a_j's$ obtained are then substituted back into (13) to yield the continuous formulation of our proposed method in the form;

$$\sum_{j=0}^{k} \alpha_j(t) x_{n+j} + \sum_{j=1}^{2} \alpha_{vj}(t) x_{n+vj} = h\left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} + \sum_{j=1}^{2} \beta_{vj}(t) f_{n+vj}\right) + h^2 \gamma_k(t) f'_{n+k}, \qquad (17)$$

which upon evaluation at $t = t_{n+k}, k = 2$ gives the discrete two-step second derivative hybrid method. However, we intend to implement our methods in a block form, which shall simultaneously generate approximate solutions to (11). In view of this, evaluating the second derivative of (17) at some required points gives a number of discrete schemes necessary to implement the methods in block form. In what follows, two separate block methods for two-step second derivative block hybrid method will be derived following the above procedures. The distinguishing factor in the two proposed block methods is the choice of $v_i, (i = 1,2)$

Three-step second derivative hybrid block method (TSDHBM3)

In this case, take $H_3(x) = 8x^3 - 12x$ since $k = 3$

We collocate this equation at x_n, x_{n+1}, x_{n+2} and x_{n+3} and solve to have

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)} \qquad (18)$$

$$= \frac{2x_n - (x_{n+3} + x_n)}{x_{n+3} - x_n} \qquad (19)$$

$$= \frac{-x_{n+3} + x_n}{x_{n+3} - x_n} \qquad (20)$$

$$= -1$$

Thus substitute the value of x into $H_3(x) = 8x^3 - 12x$ and obtain $H_3(x) = 4$

Following the same process for x_{n+1} we have

$$= \frac{2x_{n+1} - (x_{n+3} + x_n)}{x_{n+3} - x_n} \qquad (21)$$

Put $x_{n+1} = x_n + h$, $x_{n+3} = x_n + 3h$ and obtain

$$= \frac{2(x_n + h) - (x_n + 3h + x_n)}{x_n + 3h - x_n} \qquad (22)$$

$$= -\frac{1}{3}$$

By substituting the value of x into $H_3(x) = 8x^3 - 12x$ then $H_3(x) = \frac{100}{27}$

Follow the same procedure for x_{n+2} and have

$$x = \frac{2x_{n+2} - (x_{n+3} + x_n)}{x_{n+3} - x_n} \qquad (23)$$

Put $x_{n+2} = x_n + 2h$, $x_{n+3} = x_n + 3h$

and substituting into $H_3(x) = 8x^3 - 12x$ to obtain

$$x = \frac{2(x_n + 2h) - (x_n + 3h + x_n)}{x_n + 3h - x_n} \qquad (24)$$

Substituting the value of x into $H_3(x) = 8x^3 - 12x$ then $H_3(x) = -\frac{100}{27}$

Following the same procedure for x_{n+3} we have

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)} \qquad (25)$$

$$x = \frac{2x_{n+3} - (x_{n+3} + x_n)}{x_{n+3} - x_n} \qquad (26)$$

$x = 1$
Substituting the value of x into $H_3(x) = 8x^3 - 12x$ then we have
$H_3(x) = -4$
Thus recall that $\omega_i(x) = x^i, i = 0,1,2, \ldots k$

$$\left.\begin{array}{l}\omega_0'(x) = 0\\ \omega_1'(x) = 1\\ \omega_2'(x) = 2x\\ \omega_3'(x) = 3x^2\end{array}\right\} \quad (27)$$

The equation $\omega_i(x) = x^i, i = 0,1,2, \ldots k$ reduces to the form
$$f(x,y) = c_1 + 2xc_2 + 3x^2c_3 - \lambda H_3(x) \quad (28)$$
We now collocate the above equation at $x_{n+i} (i = 0,1,2)$ and interpolate (2) at $x = x_n$ to obtain a system of five equations with $c_i (i = 0,1,2,3)$ and parameter λ as
$$y_n = c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3$$
$$f_n = c_1 + 2c_2x_n + 3c_3x_n^2 - 4\lambda$$
$$f_{n+1} = c_1 + 2c_2x_{n+1} + 3c_3x_{n+1}^2 + \frac{100}{27}\lambda$$
$$f_{n+2} = c_1 + 2c_2x_{n+2} + 3c_3x_{n+2}^2 + \frac{100}{27}\lambda$$
$$f_{n+3} = c_1 + 2c_2x_{n+3} + 3c_3x_{n+3}^2 + 4\lambda$$
Solving the system above resulted to
$$\lambda = \frac{9}{128}(f_n - 3f_{n+1} + 3f_{n+2} - f_{n+3})$$
$$c_3 = \frac{1}{12h^2}(f_n - f_{n+1} - f_{n+2} + f_{n+3})$$
$$c_2 = \frac{1}{96h^2}(61hf_n - 63hf_{n+1} - 9hf_{n+2} + 11hf_{n+3} + 24f_nx_n - 24f_{n+1}x_n - 29f_n + 2x_n + 24f_nx_n)$$
$$c_1 = \frac{1}{96h^2}(123h^2f_n - 18h^2f_{n+1} + 18h^2f_{n+2} - 27h^2f_{n+3} + 122hf_nx_n - 126hf_{n+1}x_n - 18hf_{n+2}x_n + 22f_{n+3}x_n + 24f_nx_n^2 - 24f_{n+1}x_n^2 - 24f_{n+2}x_n^2 + 24f_n + 3x_n^2)$$
$$c_0 = \frac{1}{96h^2}(123h^2f_nx_n - 81h^2f_{n+1}x_n + 81h^2f_{n+2}x_n - 27h^2f_{n+3}x_n + 61hf_nx_n^2 - 63hf_{n+1}x_n^2 - 63hf_{n+1}x_n^2 - 9hf_{n+2}x_n^2 + 11hf_{n+3}x_n^2 + 8f_nx_n^3 - 8f_{n+1}x_n^3 - 8f_{n+3}x_n^3 - 96h^2y_n)$$

Again, from (2) we have
$$\bar{y} = c_0 + c_1x + c_2x^2 + c_3x^2$$
The required numerical scheme is then obtained if collocation of the above equation is done at x_{n+1} and substituting for c_0, c_1, c_2 *and* c_3 as
$$y_{n+1} = y_n + \frac{h}{48}(35f_n - 13f_{n+1} + 41f_{n+2} - 15f_{n+3}).$$
Formulating the block schemes for the polynomials of cases $k = 1,2,3$
For $k = 1$ collocate equation (12) at $x = x_{n+1}, x_{n+2}, x_{n+3}$ to obtain
$$\left.\begin{array}{l}y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})\\ y_{n+2} = y_n + h(f_n + f_{n+1})\\ y_{n+3} = y_n + \frac{3h}{2}(f_n + f_{n+1})\end{array}\right\} \quad (29)$$
For $k = 2$ collocate equation (18) at $x = x_{n+1}, x_{n+2}, x_{n+3}$ to obtain
$$\left.\begin{array}{l}y_{n+1} = y_n + \frac{h}{4}(f_n + 2f_{n+1} + f_{n+2})\\ y_{n+2} = y_n + \frac{h}{2}(f_n + 2f_{n+1} + f_{n+2})\\ y_{n+3} = y_n + \frac{3h}{4}(6f_n + 2f_{n+1} + f_{n+2})\end{array}\right\} \quad (30)$$
For $k = 3$ collocate equation (23) at $x = x_{n+1}, x_{n+2}, x_{n+3}$ to obtain
$$\left.\begin{array}{l}y_{n+1} = y_n + \frac{h}{48}(35f_n - 13f_{n+1} + 41f_{n+2} - 15f_{n+3})\\ y_{n+2} = y_n + \frac{h}{48}(33f_n + 13f_{n+1} + 67f_{n+2} - 17f_{n+3})\\ y_{n+3} = y_n + \frac{3h}{8}(f_n + 3f_{n+1} + 3f_{n+2} + f_{n+3})\end{array}\right\} \quad (31)$$

Analysis of the Methods
Consider the analysis of the newly constructed methods such as order, error constant, consistency, convergence and the regions of absolute stability of the methods.
The proposed three-step hybrid block method with second derivative that produces approximations y_{n+k} to the first order ordinary differential equations (ODEs)
$$x' = f(t,x) \quad (32)$$
is given as follows:
$$\sum_{j=0}^{k} \alpha_j x_{n+j} + \sum_{j=1}^{2} \alpha_{vj} x_{n+vj} = h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=1}^{2} \beta_{vj} f_{n+vj}\right) + h^2 \gamma_k f'_{n+k} \quad (33)$$
$\alpha_j, \alpha_{vj}, \beta_j$ and β_{vj} are constant coefficients.
In order to obtain (33), we approximate the solution by the orthogonal function $X(t)$ of the form
$$X(t) = \sum_{j=0}^{r+s-1} \alpha_j \varphi(t) \quad (34)$$
Where
$t \in [a,b]$
$\varphi(t)$is an orthogonal function defined by the Hermit Polynomial.
α_j are the unknown coefficients to be determined?
r is the number of interpolations for $1 \le r \le k$ and
s is the number of distinct collocation points with $s > 0$
The continuous approximation is constructed by imposing the following conditions
$$X(\alpha_{n+\mu}) = x_{n+\mu}, \{j, v_1, v_2, v_3\}, j = 0,1, \ldots, k-1 \quad (35)$$
$$X'(t_{n+\mu}) = f_{n+\mu}, \{j, v_1, v_2, v_3\}, j = 0,1, \ldots, k \quad (36)$$
$$X''(t_{n+\mu}) = f'_{n+j}, \quad j = k \quad (37)$$
$$X'''(t_{n+\mu}) = f''_{n+i}, i = k \quad (38)$$
Where v_1 and v_2 are not integers. Equations (37) to (39) form a nonlinear system of equations in $a_j's$ which is solved using the matrix inversion technique via Maple software. The values of $a_j's$ obtained will be substitute back to (37) to yield the continuous formulation of our proposed method in the form;
$$\sum_{j=0}^{k} \alpha_j(t) x_{n+j} + \sum_{j=1}^{2} \alpha_{vj}(t) x_{n+vj} = h\left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} + \sum_{j=1}^{2} \beta_{vj}(t) f_{n+vj}\right) + h^2 \gamma_k(t) f'_{n+k} + h^3 \beta_v f_{n+v}$$
Which upon evaluation at $t = t_{n+k}, k = 3$ gives the discrete three-step second derivative hybrid method. However, we intend to implement our methods in a block form, which shall simultaneously generate approximate solutions to (34). In view of this, evaluating the second derivative of the above equation at some required points gives a number of discrete schemes necessary to implement the methods in block form. In what follows, three separate block methods for three-step second derivative block hybrid method will be derived following the above procedures. The distinguishing factor in the three proposed block methods is the choice of$v_j, (j = 1,2 \ldots)$.

Numerical Example
Example 1
Consider the following IVP
$y'(t) = -y, y(0) = 1$
With the exact solution $y(t) = e^{-t}$
Example 2
$y'(t) = x(1-y), y(0) = 0$
With the exact solution $y(t) = 1 - e^{-\frac{t^2}{2}}$
Note:
BLS = Block schemes derived in this paper,
LM = Method of Okedayo et al. (2018),
Exact = Exact Solution,

|Exact-BLS| = the absolute value of the exact solution minus computed solution of the method derived in this paper
|Exact-LM| = the absolute value of the exact solution minus computed solution of Okedayo et al. (2018).
The numerical results of these examples are depicted in Tables $1, 2$ with $k = 2$ and $k = 3$ with constant step size of $h = 0.1$ respectively In tables 1 and 2 we presented a comparison of the obtained numerical results using the proposed scheme with the exact solution and Table 1 and 2 presents the comparison of the results obtained from proposed scheme, the exact solution and those numerical results obtained from Okedayo et al. (2018).

Table 1: A comparison of numerical results of proposed Scheme at $k = 2$ with exact solution for example 1

x-value	BLS $k = 2$	Exact	|Exact-BLS|
0.0	1.000000	1.000000	0.000000
0.1	0.905090	0.904837	4.253×10^{-4}
0.2	0.818181	0.818730	5.49×10^{-4}
0.3	0.749092	0.740818	8.274×10^{-3}
0.4	0.675227	0.670320	4.907×10^{-3}
0.5	0.608761	0.606531	2.23×10^{-3}
0.6	0.548760	0.548811	5.1×10^{-3}
0.7	0.504028	0.496585	7.443×10^{-3}
0.8	0.448139	0.449328	1.189×10^{-3}
0.9	0.416232	0.406569	9.663×10^{-3}
1.0	0.373893	0.367879	6.014×10^{-3}

Table 2: A comparison of numerical results of proposed scheme at $k = 3$ with exact solution and LM for Example 2

x-value	BLS $k = 3$	LM	Exact	|Exact-BLS|	|Exact-LM|
0.0	1.000000	1.000000	1.000000	0.000000	0.000000
0.1	0.904808	0.905953	0.904837	2.9×10^{-5}	1.116×10^{-3}
0.2	0.818705	0.820856	0.818730	5.0×10^{-5}	2.126×10^{-3}
0.3	0.740823	0.743857	0.740818	5.0×10^{-5}	3.039×10^{-3}
0.4	0.670304	0.674185	0.670320	1.6×10^{-5}	3.865×10^{-3}
0.5	0.606516	0.611143	0.606531	1.510^{-5}	4.612×10^{-3}
0.6	0.548820	0.554100	0.548811	9.0×10^{-6}	5.289×10^{-3}
0.7	0.496576	0.502486	0.496585	9.0×10^{-6}	5.901×10^{-3}
0.8	0.449321	0.455784	0.449328	7.0×10^{-6}	6.456×10^{-3}
0.9	0.406578	0.413527	0.406569	9.0×10^{-6}	6.958×10^{-3}
1.0	0.367876	0.375290	0.367879	3.0×10^{-6}	7.411×10^{-3}

Tables 1, and 2 shows that the proposed schemes approximate the solutions of initial value problems given in Examples 1 and 2 as the absolute errors are convergent.

CONCLUSION

In this research work, a class of three new block schemes for the approximation of initial value problems of first order ordinary differential equations using Hermite polynomial as a basis function has been obtained. The proposed method was used to solve numerically some initial value problems and the results compared with the exact solutions and the method of Okedayo et al. (2015). From the numerical results, it is observed that the new schemes were capable for solving first order IVPs as generated results compared favorably with the existing method and the exact solutions. The method is very simple to implement.

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