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RETHINKING MULTIGROUP: AN INTRODUCTORY ALTERNATIVE APPROACH IN SINGH'S PERSPECTIVE

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ABSTRACT

The analysis of multigroups—multisets defined over group structures—necessitates robust mathematical frameworks. Singh's dressed epsilon notation offers an elegant approach to this analysis by extending traditional set membership concepts to accommodate multiplicity within sets. This notation introduces a refined membership symbol that conveys additional information about the multiplicity of elements within a multiset. By employing Singh's dressed epsilon method, one can more effectively verify properties of multigroups. This approach not only streamlines the representation of multigroup characteristics but also facilitates deeper insights into their structural properties, thereby advancing the theory.

Keywords: Cardinality, Dressed epsilon, Multigroup Operations, Singh

INTRODUCTION

The concept of multigroup extends classical group theory by incorporating the framework of multisets, allowing for elements to appear multiple times within the algebraic structure. This generalization provides a richer context for modeling and analyzing systems where multiplicity plays a crucial role.

In traditional group theory, a group is defined as a set equipped with a binary operation that satisfies closure, associativity, the existence of an identity element, and the existence of inverses for each element. Multigroups relax certain constraints of these axioms to accommodate the multiplicity inherent in multisets. Specifically, while the binary operation in a group produces a single output for any pair of elements, in a multigroup, this operation can yield multiple outputs, reflecting the possible multiple occurrences of elements. This approach aligns with the natural representation of multisets, where the focus is on the frequency of element occurrences rather than their mere presence or absence.

The development of multigroup theory offers a more nuanced understanding of algebraic structures where element repetition is significant. This has implications for various fields, including combinatorics, computer science, and systems modeling, where the concept of multiplicity is essential. By integrating the principles of multisets into group theory, multigroups provide a robust framework for addressing problems that involve repeated interactions or elements.

In the study of multigroups, traditional analyses often rely on cardinality functions, to represent the number of occurrences of an element within a multiset. However, this approach can become cumbersome, especially when the exact multiplicity of elements is the primary concern Nazmul et al. (2013) and Peter et al. (2024).

To address these challenges, Dasharath Singh introduced the dressed epsilon notation ($\in_+$), which offers a more intuitive and flexible framework for analyzing multigroups (Singh 2006). This notation allows for direct expression of element membership and multiplicity without the need for auxiliary cardinality functions.

The dressed epsilon notation enables concise expressions of membership conditions. For example, stating that an element x appears in a multiset A at least k times is directly written as $x \in_+^n A$, eliminating the need for additional cardinality functions. Utilizing the $\in_+$ and $\in^k$ notations leads to more straightforward and symbolic proofs. This notation aligns closely with classical set theory, facilitating the extension of traditional set-theoretic results to multisets. It allows for the application of established set operations and properties within the context of multisets without necessitating a shift to function-based cardinality approaches.

Theoretical Framework

This study adopts Singh's dressed epsilon notation as an alternative approach to the classical cardinality-based representation of multisets. As opposed to the method employed by Namzul et al. (2013), which primarily relies on counting elements through explicit cardinality functions, our approach focuses on direct membership expressions to characterize multigroup structures.

MATERIALS AND METHODS

The methodology consists of the following key steps:

Reformation of multigroup properties

We systematically re-examine fundamental multigroup properties that were previously established using the cardinality approach. Each of these properties is reinterpreted in the dressed epsilon framework, ensuring that the new notation preserves the logical and structural integrity of the original formulations.

Proof Synthesis

For completeness, we construct new proofs of key multigroup results using Singh's dressed epsilon notation. Where applicable, we reference previously established propositions to ensure coherence and logical continuity. The emphasis is on demonstrating that the dressed epsilon approach is not only valid alternative but also provides additional clarity and flexibility in extending multigroup theory.

Definition of Terms

Definition 1 (Multiset)

Let D be a set. A multiset M over D is a collection of elements from D, where repetitions are allowed. The set D is called the ground set or generic set of the class of all multisets containing elements from D. Different representations of multisets exist (Singh et al., 2007).

Definition 2 (Submultiset)

Let A and B be two multisets. A is called a submultiset of B, written $A \subseteq B$, if:

$\forall z \forall k (z \in^k A \Longrightarrow z \in_+^n B)$ (1)

This means that every element in A appears in B at least as many times as it appears in A (Singh, 2006).

Definition 3 (Union)

Let M and N be two multisets over a ground set D. The union of M and N, denoted $M \cup N$, is the multiset defined by:

$x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = \max(m, n)$. (2)

That is, the multiplicity of each element in the union is the maximum of its multiplicities in M and N.

Definition 4 (Intersection)

Let M and N be two multisets over a ground set D. The intersection of M and N, denoted $M \cap N$, is the multiset defined by:

$x \in^k (M \cap N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = \min(m, n)$. (3)

That is, the multiplicity of each element in the intersection is the mininum of its multiplicities in M and N.

Definition 5 (Sum or Additive Union)

Let M and N be two multisets over a ground set D. The sum (or additive union) of M and N, denoted $M \uplus N$, is the multiset defined by:

$x \in^k (M \uplus N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = m + n$. (4)

That is, the multiplicity of each element in the sum is the total number of times it appears in in both M and N.

See Singh (2006), Singh et. al (2007) and Singh et al. (2008).

Definition 6 Let A, B be multisets. Define $A \circ B$ and A^{-1} as follows:

$x \in^n (A \circ B) \Leftrightarrow$

$x \in^n \bigvee\{y \in^m A \wedge z \in^k B \big| y, z \in X, yz = x\}$. (5)

This is the membership condition. It states that an element x belongs to the multiset $(A \circ B)$ with multiplicity n if and only if there exist elements y from A and z from B such that yz equals x.

$x \in^n A^{-1} \Leftrightarrow x^{-1} \in^n A$. (6)

In other words, an element x belongs to the inverse multiset A^{-1} with multiplicity n if and only if x^{-1} belongs to A with the same multiplicity.

Definition 7 Let X be a group. A multiset G over X is said to be a multigroup over X if the count function G or C_G satisfies the following two conditions:

$x \in^m G \wedge y \in^n G \Longrightarrow xy \in_+^{(m \wedge n)} G \; \forall x, y \in X;$`

$x \in^n G \Longrightarrow x^{-1} \in_+^n G \; \forall x \in X;$`

For example, consider the cyclic group of order 4 $X = \{e, a, a^2, a^3\}$ be the cyclic group of order 4, where $a^4 = e$. Let the multiset $G = \{e, e, e, a, a, a^2, a^2, a^3, a^3\}$ be a multiset over X. The membership conditions are given as follows:

Multiplication conditions:

For all $x, y \in X$, we verify that the multiplication condition

$x \in^m G \wedge y \in^n G \Longrightarrow xy \in_+^{(m \wedge n)} G \; \forall x, y \in X$ (7)

holds:

$ea \in^2 G$, since $a \in^3 G$ and $e \in^3 G$, so $ea = a \in_+^{(3 \wedge 3)} G$.

$ea^2 \in^2 G$, since $a^2 \in^2 G$ and $e \in^3 G$, so $ea^2 = a^2 \in_+^{(3 \wedge 2)} G$.

$ea^3 \in^2 G$, since $a^3 \in^2 G$ and $e \in^3 G$, so $ea^3 = a^3 \in_+^{(3 \wedge 2)} G$.

$aa^2 \in^2 G$, since $a \in^3 G$ and $a^2 \in^2 G$, so $aa^2 = a^3 \in_+^{(3 \wedge 2)} G$.

$aa^3 \in^2 G$, since $a \in^3 G$ and $a^3 \in^2 G$, so $aa^3 = e \in_+^{(3 \wedge 2)} G$.

$a^2a^2 \in^2 G$, since $a^2 \in^2 G$ so $a^2a^2 = e \in_+^{(2 \wedge 2)} G$.

$a^2a^3 \in^2 G$, since $a^2 \in^2 G$ and $a^3 \in^2 G$, so $a^2a^3 = a \in_+^{(2 \wedge 2)} G$.

$a^3a^3 \in^2 G$, since $a^3 \in^2 G$, so $a^3a^3 = a^2 \in_+^{(2 \wedge 2)} G$.

Inversion condition

For all $x \in X$, we verify:

$x \in^n G \Longrightarrow x^{-1} \in_+^n G$ (8)

$a^{-1} = a^3$, and $a^3 \in^2 G \Longrightarrow a^{-1} \in_+^2 G$

$(a^2)^{-1} = a^2 \in_+^2 G$

$(a^3)^{-1} = a$, and $a \in^2 G \Longrightarrow (a^3)^{-1} \in_+^2 G$

$e^{-1} = e$, and $e \in^3 G \Longrightarrow e^{-1} \in_+^3 G$

Since both conditions hold, we conclude that G is a multigroup over X.

RESULTS AND DISCUSSION

Throughout this section, we let X be a group and e be the identity element of X. Also throughout the rest of the paper we denote by $MG(X)$ the set of all multigroups over a group X:

We now prove the following propositions in the sense of Singh's dressed epsilon notation.

Proposition 1 *Let A and B be multisets. Then*

$x \in^n (A \circ B) \Leftrightarrow x \in^n \bigvee_{y \in X}(y \in^m A \wedge y^{-1}x \in^k B)$ (9)

$\Leftrightarrow x \in^n \bigvee_{y \in X}(xy^{-1} \in^m A \wedge y \in^k B), \forall x \in X$ (10)

Proof

By definition,

$x \in^n (A \circ B) \Leftrightarrow$

$x \in^n \bigvee\{y \in^m A \wedge z \in^k B \big| y, z \in X, yz = x\}$. (11)

This means x appears in $(A \circ B)$ exactly n times if there exists elements $y, z \in X$ such that:

y appears in A at least m times, z appears in B at least k times and $yz = x$. Substituting $z = y^{-1}x$ since $yz = x$, we can rewrite z as:

$z = y^{-1}x$. (12)

Thus, the condition becomes: $y \in^m A$ and $y^{-1}x \in^k B$.

Taking the supremum over all possible y, we get:

$x \in^n (A \circ B) \Leftrightarrow x \in^n \; x \in^n \bigvee_{y \in X}(y \in^m A \wedge y^{-1}x \in^k B)$. (13)

Alternatively, taking the supremum over all y:

$\Leftrightarrow x \in^n \bigvee_{y \in X}(xy^{-1} \in^m A \wedge y \in^k B), \forall x \in X$ (14)

Therefore,

$x \in^n (A \circ B) \Leftrightarrow x \in^n \; x \in^n \bigvee_{y \in X}(y \in^m A \wedge y^{-1}x \in^k B) \Leftrightarrow x \in^n \bigvee_{y \in X}(xy^{-1} \in^m A \wedge y \in^k B), \forall x \in X$ (15)

Proposition 2 *For any multiset A*

$(A^{-1})^{-1} = A$ (16)

Proof

By definition, the inverse of A satisfies:

$x \in^n A^{-1} \Leftrightarrow x^{-1} \in^n A$. (17)

This means that x appears in A^{-1} exactly n times if and only if its inverse x^{-1} appears in A exactly n times.

Taking the inverse of A^{-1}, we get

$y \in^m (A^{-1})^{-1} \Leftrightarrow y^{-1} \in^m A^{-1}$. (18)

Since $(y^{-1})^{-1} = y$, it follows that:

$y \in^m (A^{-1})^{-1} \Leftrightarrow y \in^m A$ (19)

Thus, $(A^{-1})^{-1} = A$

Proposition 3 *For any multiset A, B*

$A \subseteq B \Longrightarrow A^{-1} \subseteq B^{-1}$ (20)

Proof

Suppose $x \in^n A \Leftrightarrow x \in^m B$ for some $m \geq n$.

By the definition of the inverse multiset, $x \in^n A^{-1} \Leftrightarrow x^{-1} \in^n A$

Since $A \subseteq B$, we substitute x^{-1} in the inclusion condition:

$x^{-1} \in^n A \Longrightarrow x^{-1} \in^m B$, for some $m \geq n$.

Using the definition of B^{-1}, lwe get

$x \in^n A^{-1} \Longrightarrow x \in^m B^{-1}$, for some $m \geq n$.

Thus, every element in A^{-1} appears in B^{-1} at last as many times. Thus,

$A^{-1} \subseteq B^{-1}$

Proposition 4 *For any family of multisets* A_i
$(\cup_{i\in I} A_i)^{-1} = \cup_{i\in I} {A_i}^{-1}$ (21)
Proof
The union of multisets can be defined as
$x \in^n \cup_{i\in I} A_i \Leftrightarrow x \in^m A_j$ for some $j \in I$, where $m \geq n$
Taking the inverse of the union, we use the definition of the inverse operations:
$x \in^n (\cup_{i\in I} A_i)^{-1} \Leftrightarrow x^{-1} \in^n \cup_{i\in I} A_i$ (22)
Using the definition of union, we substitute:
$x^{-1} \in^n \cup_{i\in I} A_i \Leftrightarrow x^{-1} \in^m A_j$ for some $j \in I$, where $m \geq n$. (23)
Applying the inverse definition again, we get:
$x \in^n {A_j}^{-1}$ for some $j \in I$. (24)
Since this must hold for some j, it follows that:
$x \in^n \cup_{i\in I} {A_i}^{-1}$ (25)
Proposition 5 *For any family of multisets* A_i
$(\cup_{i\in I} A_i)^{-1} = \cup_{i\in I} {A_i}^{-1}$ (26)
Proof:
By the definition of multiset intersection, an element x belongs to $\cap_{i\in I} A_i$ at least
n times if and only if it belongs to every A_i at least n times. That is:
$x \in_+^n \cap_{i\in I} A_i \Leftrightarrow \forall i \in I, x \in_+^n A_i$ (27)
Appling the definition of multiset inversion:
$x \in_+^n {A_i}^{-1} \Leftrightarrow x^{-1} \in_+^n A_i$. (28)
Using this on the left hand side:
$x \in_+^n (\cap_{i\in I} A_i)^{-1} \Leftrightarrow x^{-1} \in_+^n \cap_{i\in I} A_i$ (29)
By the definition of intersection:
$x^{-1} \in_+^n \cap_{i\in I} A_i \Leftrightarrow \forall i \in I, x \in_+^n A_i$ (30)
Applying the inversion to A_i:
$\forall i \in I, x^{-1} \in_+^n A_i \Leftrightarrow \forall i \in I, x \in_+^n {A_i}^{-1}$ (31)
Proposition 6 *For any two multisets* A, B,
$(A \circ B)^{-1} = B^{-1} \circ A^{-1}$ (32)
Proof
An element x belongs to $A \circ B$ exactly n times if and only if there exists elements $y, z \in X$ such that $yz = x$ and:
$x \in^n A \circ B \Leftrightarrow \vee_{y,z\in X, yz=x}(y \in^n A \wedge z \in^n B)$. (33)
Applying the definition of multiset inversion:
$x \in^n (A \circ B)^{-1} \Leftrightarrow \vee_{y,z\in X, yz=x}(y \in^n A \wedge z \in^n B)$.
Applying the inversion property:
$y \in^n A \Leftrightarrow y^{-1} \in^n A^{-1}, z \in^n B \Leftrightarrow z^{-1} \in^n B^{-1}$. (34)
Therefore,
$x \in^n (A \circ B)^{-1} \Leftrightarrow \vee_{y,z\in X, z^{-1}y^{-1}=x}(y^{-1} \in^n A^{-1} \wedge z^{-1} \in^n B^{-1})$. (35)
In other words,
$x \in^n (A \circ B)^{-1} \Leftrightarrow \vee_{u,v\in X, uv=x}(u \in^n A^{-1} \wedge v \in^n B^{-1})$. (36)
which is exactly the definition of :
$x \in^n (B^{-1} \circ A^{-1})$ (37)
Hence,
$(A \circ B)^{-1} = B^{-1} \circ A^{-1}$ (38)
Proposition 7 *For any three multisets* A, B, C,
$(A \circ B) \circ C = A \circ (B \circ C)$ (39)
Proof
By the definition of multiset multiplication, an element x belongs to $A \circ B$ exactly n times if and only if there exists $y, z \in X$ such that $yz = x$ and
$x \in^n A \circ B \Leftrightarrow \vee_{y,z\in X, yz=x}(y \in^n A \wedge z \in^n B)$. (40)
Applying this to $(A \circ B) \circ C$ we have:
$x \in^n (A \circ B) \circ C \Leftrightarrow \vee_{w,v\in X, wv=x}(w \in^n A \circ B \wedge v \in^n C)$. (41)
Expanding $w \in^n A \circ B$:
$x \in^n (A \circ B) \circ C \Leftrightarrow \vee_{w,v\in X, wv=x\ y,z\in X, yz=w}(y \in^n A \wedge z \in^n B \wedge v \in^n C)$. (42)
Since $yz = w$ and $wv = x$, we substitute yz for w:
$x \in^n (A \circ B) \circ C \Leftrightarrow \vee_{y,z,v\in X, (yz)v=x}(y \in^n A \wedge z \in^n B \wedge v \in^n C)$. (43)
Considering that multiplication in groups is associative:
$x \in^n (A \circ B) \circ C \Leftrightarrow \vee_{y,z,v\in X, y(zv)=x}(y \in^n A \wedge z \in^n B \wedge v \in^n C)$. (44)
$x \in^n (A \circ B) \circ C \Leftrightarrow \vee_{y,q\in X, yq=x}(y \in^n A \wedge q \in^n B \circ C)$. (45)
$(A \circ B) \circ C = A \circ (B \circ C)$ (46)
Proposition 8 *For any multigroup* A *over a group* X, *we have*
$(e \in^m A \Rightarrow x \in^k A\ \forall x \in X) \Rightarrow m \geq k$ (47)
Proof
Since A is a multigroup over X, it satisfies the multiplication condition:
$x \in^k A \wedge y \in^n A \Rightarrow xy \in_+^{(n\wedge k)} A$. (48)
Choose $y = x^{-1}$ and from the Inverse condition $x \in^k A \Rightarrow x^{-1} \in_+^k A\ \forall x \in X$;
we get:
$x \in^k A \wedge x^{-1} \in_+^k A \Rightarrow e \in_+^k A$. (49)
It follows that :
$(x \in^k A \wedge x^{-1} \in^m A \Rightarrow e \in^m A) \Rightarrow m \geq k$. (50)
Hence,
$(e \in^m A \Rightarrow x \in^k A\ \forall x \in X) \Rightarrow m \geq k$ (51)
Proposition 9 *For any multigroup* A *over a group* X,
$(x \in^p A \Rightarrow x^n \in^q A\ \forall x \in X) \Rightarrow q \geq p$ (52)
Proof
The proof is by induction on n. The statement is true for $n = 1$ since
$(x \in^p A \Rightarrow x^1 \in^q A\ \forall x \in X) \Rightarrow q = p \Rightarrow q \geq p$ (53)
Suppose it is true for k. That is,
$(x \in^p A \Rightarrow x^k \in^q A\ \forall x \in X) \Rightarrow q \geq p \ldots\ldots\ldots\ldots (i)$
Consider the multiplication condition:
$x \in^p A \wedge x^k \in^q A \Rightarrow xx^k \in_+^{(p\wedge q)} A$.
Since $q \geq p$ from (i), we get:
$x \in^p A \wedge x^k \in^q A \Rightarrow x^{k+1} \in_+^p A \ldots\ldots\ldots\ldots (ii)$
Considering $x^{k+1} \in^q A$ and from the LHS of (ii), it follows that $q \geq p$.
Hence,
$x \in^p A \Rightarrow x^{k+1} \in^q A \Rightarrow q \geq p$ (54)
Thus:
$x \in^p A \Rightarrow x^n \in^q A \Rightarrow q \geq p$ (55)
Proposition 10 *For any multigroup* A *over a group* X,
$x \in^p A \Rightarrow x^{-1} \in^q A \Rightarrow q = p$ (56)
Proof
Let us apply the inverse condition to both x and x^{-1}:
$x \in^p A \Rightarrow x^{-1} \in_+^p A\ \forall x \in X$; (57)
and
$x^{-1} \in^q A \Rightarrow (x^{-1})^{-1} \in_+^q A\ \forall x^{-1} \in X$; (58)
This implies
$x \in^p A \Rightarrow x^{-1} \in_+^p A\ \forall x \in X$; (59)
and
$x^{-1} \in^q A \Rightarrow x \in_+^q A\ \forall x^{-1} \in X$; (60)
It follows that $p = q$.
Proposition 11 *For any multigroup* A *over a group* X,
$A = A^{-1}$ (61)
Proof
We apply the inverse condition to x:
$x \in^n A \Rightarrow x^{-1} \in_+^n A\ \forall x \in X$; (62)
This means that if x belongs to A exactly n times, then its inverse x^{-1} belongs to A at least n times. Hence $A \subseteq A^{-1}$. Similarly applying the inverse condition to x^{-1}, we get $A^{-1} \subseteq A$. It follows that $A = A^{-1}$.

CONCLUSION

In this work, we have demonstrated the effectiveness of Singh's dressed epsilon notation in analyzing multigroups. By utilizing the notations $\in^n$, $\in_+$ and $\in_+^n$ we have been able to express membership in multisets and multigroups in a way that preserves structural clarity. One of the key advantages of this approach is its ability to seamlessly extend classical set-theoretic results to multisets and multigroups while maintaining conceptual and notational coherence. Our results show that operations such as union, intersection, sum, and inverse in multigroups can be elegantly captured. Overall, Singh's dressed epsilon notation offers a rigorous yet intuitive foundation for multiset and multigroup analysis, bridging the gap between classical set theory and non-Cantorian frameworks. The idea is to have a powerful tool that enhances both theoretical exploration and practical applications in multigroup theory, algebraic structures, and related mathematical disciplines.

REFERENCES

Nazmul, S., Majumdar, P., & Samanta, S. K. (2013). On multisets and multigroups. *Ann. Fuzzy Math. Inform*, *6*(3), 643-656.

Singh, D. (2006). *Multiset Theory: A New Paradigm of Science: an Inaugural Lecture*. University Organized Lectures Committee, Ahmadu Bello University.

Singh, D., Ibrahim, A. M., Yohanna, T., & Singh, J. N. (2008). *A systematization of fundamentals of multisets*. Lecturas Matemáticas, 29, 33–48.

Peter, C., Balogun, F. and Adeyemi, O. A. (2024). An exploration of antimultigroup extensions. *FUDMA Journal of Sciences*, *8*(5), 269-273. https://doi.org/10.33003/fjs-2024-0805-2719

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