FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645-2944
Vol. 9 No. 8, August 2025, pp 392 - 405
DOI: https://doi.org/10.33003/fjs-2025-0908-3396

ASSESSING THE IMPACT OF BOOSTER VACCINE AND ISOLATION ON THE TRANSMISSION DYNAMICS OF PERTUSSIS: USING MATHEMATICAL MODELING APPROACH

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Abstract

Pertussis is a highly contagious respiratory disease that is easily prevented by immunisation. The risk of whooping cough is particularly high for neonates. However, due to a lack of vaccinations against pertussis, the disease is still widespread in several nations, particularly in the wake of the COVID-19 pandemic. In this paper, a deterministic mathematical model that describes the transmission dynamics of pertussis for assessing the impact of booster vaccines and isolation was proposed. The model comprises two equilibrium points, the DFE (disease free equilibrium point) and the EEP (endemic equilibrium point), as demonstrated by the determination of the solution's positivity and boundedness as well as the presence of disease equilibria in the mathematical analysis section. It was discovered that if $\mathcal{R}_c < 1$, the DFE is both locally and globally asymptotically stable. The endemic equilibrium point, has been shown to be globally asymptotically stable if the basic reproduction number is greater than unity and $\omega = \psi = \phi_1 = v = \delta = 0$. This has been determined using a nonlinear Lyapunov function of the Go-Volterra type. Sensitivity analysis demonstrates that isolation and vaccine rates are highly sensitive in lowering the control reproduction number. Numerical simulation of model (1) shows that the isolation rate of affected people and booster are crucial parameters for managing pertussis in a community.

Keywords: Pertussis, Booster vaccine, Isolation, Reproduction number, Stability analysis.

INTRODUCTION

Pertussis is a highly contagious respiratory disease that is easily prevented by immunisation. The risk of whooping cough is particularly high for neonates (Surmann *et al.,* 2024). Symptoms include runny nose, nasal congestion, and sneezing in addition to a cough that resembles a "whoop" The bacteria that causes pertussis, commonly referred to as whooping cough, is Bordetella pertussis. The illness mostly impacts the upper respiratory tract and is highly transmissible (Surmann *et al.,* 2024). Even in countries with high vaccination rates, whooping cough, commonly known as pertussis, is a major public health concern and cause of infant mortality worldwide (Korppi, 2013). Most pertussis-related deaths occur in infants younger than 12 months who have not received all or some of the recommended vaccinations against the disease. Additionally, adults and teenagers are also susceptible to the infection, emphasizing the importance of vaccination for all age groups to prevent further transmission and mortality (Pesco *et al.,* 2014). According to WHO estimates, there are 40–50 million cases of whooping cough year, and between 297,000 and 409,000 people die from the disease. Ninety percent of instances take place in low-income nations (Suleman and Riaz, 2020). A report published in 2014 revealed that pertussis affected an astounding 24.1 million people worldwide, resulting in the tragic loss of approximately 160,700 children under the age of five (WHO, 2024). The first known outbreak of pertussis was reported in France in 1578, and the disease has since caused significant health crises around the world. One notable example occurred in Cape Town in 1947, where the whooping cough outbreak claimed the lives of 107 people, demonstrating the devastating impact of this illness on communities (Butt, 2023). It was widely used in various European nations by the 1600s. (Suleman and Riaz, 2020). Approximately 100,000 cases of pertussis were reported annually prior to the

wP vaccination being implemented in England and Wales in 1957 (Pesco *et al.,* 2014). There was an epidemic in Israel, the United States, and Ireland at the beginning of 2010. More than 10,000 cases of whooping cough were recorded in California in 2014, making it the largest outbreak since 1947 (Butt, 2023).

The whooping cough Bacteria can spread readily via the air from one person to another. Tiny bacterial particles may be released by an infected individual when they cough or sneeze. Other people then inhale the germs (CDC, 2022). It also spreads when people often interact or share breathing spaces, such when a baby is placed on your chest (CDC, 2022). The initial symptoms typically appear seven to 10 days after infection. They include coughing, runny nose, and low-grade fever, which typically develops into a hacking cough and whooping (hence the name whooping cough). Convulsions and brain damage are rare side effects, however pneumonia is a very prevalent one (WHO, 2022).

The cornerstone of medical treatment for cases of pertussis is supportive care, while antibiotics may be helpful. This treatment eliminates the organism from secretions, which lowers communicability and may alter the course of the disease if initiated early. Antibiotics such as azithromycin, clarithromycin, and erythromycin are recommended. Additionally, trimethoprim-sulfamethoxazole can be used. All close contacts of an individual with pertussis should receive an antibiotic that effectively combats the infection, regardless of their age or level of immunisation (NCDC, 2011). The major strategy for preventing pertussis is vaccination. In various countries throughout the 1950s, diphtheria-tetanus-pertussis (DTP) was introduced as the main vaccine during the first year of life, marking the beginning of mass immunisation. Deaths and the prevalence of illness have dropped by more than 90% (Freitas *et al.,* 2011). According to WHO guidelines, the first dose should be administered as early as six weeks of age, and the second and third doses should be given four to eight weeks apart, or ten to fourteen and fourteen to eighteen weeks, respectively. A booster dose is recommended, ideally during the second year of life. Later in life, further booster doses may be required based on the local epidemiology (WHO,2022).

Over the years mathematical models play a vital role in studying the dynamics of infectious diseases such as (Ibrahim *et al.,* 2025), (Andrawus *et al.,* 2024a), (Andrawus *et al.,* 2024b), (Mustapha *et al.,* 2023) and (Ochieng, 2025).

Many Mathematical models have been proposed to study the dynamics of pertussis. Few researchers developed mathematical models incorporating booster vaccine in studying pertussis among are; (Rozenbaum *et al.,* 2012), (Pesco *et al.,* 2013) and (Gillooly *et al.,* 2016) . In addition researchers ignored the impact of isolation in controlling the spread of pertussis. Therefore in this research we proposed a model that incorporating both booster vaccine and isolation in controlling the spread of pertussis in the society.

MODEL FORMULATION

The total human population denoted by $N(t)$ at time t is subdivided into seven subgroups, namely: susceptible unvaccinated $S(t)$, susceptible vaccinated, $V(t)$, exposed $E(t)$, mildly infected $M(t)$, severely infected $I(t)$, isolated $J(t)$ and recovered individuals $R(t)$.

$$N(t) = S(t) + V(t) + E(t) + M(t) + I(t) + J(t) + R(t)$$

With parameter π as the recruitment rate, susceptible unvaccinated individuals can acquire pertusiss infection either by direct contact through inhalation of infectious droplets produced during coughing or sneezing or indirect with surface or object contaminated with bacteria. The exposed individual may either progress to infectious mild or severe class at the rates $\alpha_1\alpha_2$ and $\alpha_1(1-\alpha_2)$. The infected mild progresses to severe at the rate η and recovered naturally at the rate τ_1. Infected severe are isolated at the rate γ. The pertussis-induced mortality rate for infected severe and isolated individuals is denoted by δ, while the natural death rate of the whole classes is represented by μ.

$$\begin{aligned}
\frac{dS}{dt} &= (1-p)\Pi + \omega V - (\lambda + \psi + \mu)S,\\
\frac{dV}{dt} &= p\Pi + \psi S - (\theta\lambda + \omega + v + \mu)V,\\
\frac{dE}{dt} &= \lambda S + \theta\lambda V - (\alpha_1 + \mu)E,\\
\frac{dM}{dt} &= \alpha_1\alpha_2 E - (\eta + \phi_1 + \mu)M,\\
\frac{dI}{dt} &= \alpha_1(1-\alpha_2)E + \eta M - (\gamma + \mu + \delta)I,\\
\frac{dJ}{dt} &= \gamma I - (\phi_2 + \mu + \delta)J,\\
\frac{dR}{dt} &= \phi_1 M + \phi_2 J + vV - \mu R.
\end{aligned} \tag{1}$$

. Where

$$\lambda = \frac{\beta(M + \xi I)}{N}$$

THEORETICAL ANALYSIS OF THE MODEL

Boundedness and Positivity of Solution

If a solution to a system of equations is bounded, it is said to exist. As a result, the state variables' solution set must be bounded. Furthermore, the system (1) must always have a positive solution, hence we asserted the following theorem:

Theorem 1 *The model* (1) *can be solved for all* $t > 0$ *if the solution begins and stays in the positive invariant*

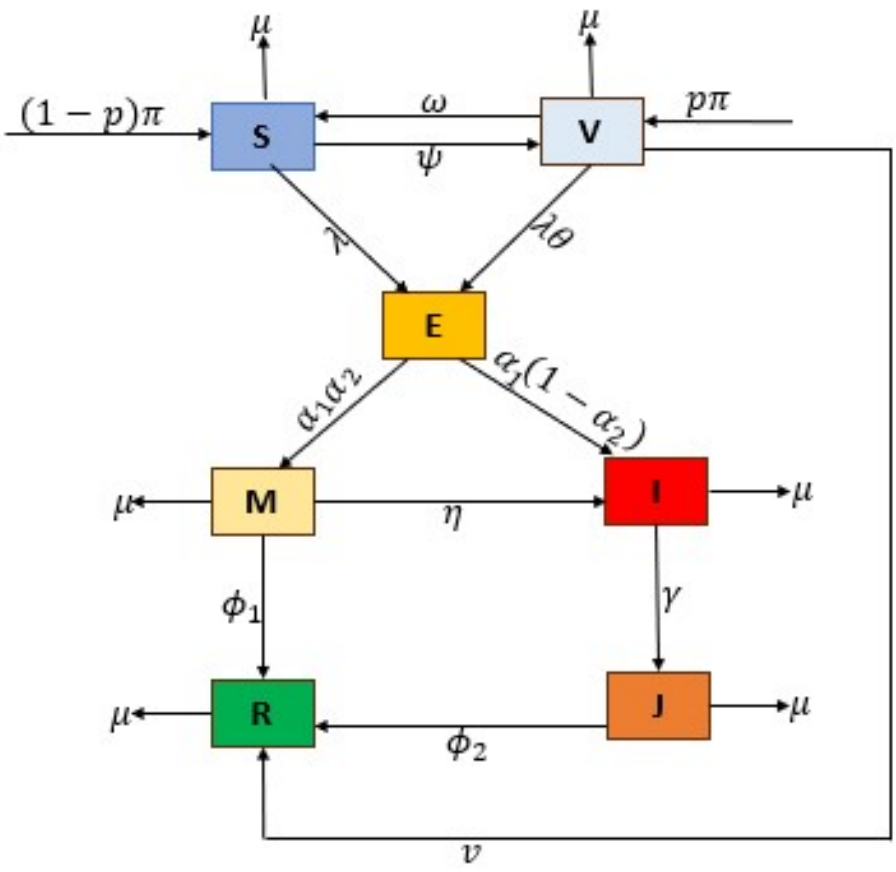

Figure 1: Schematic diagram representing the structure of model (1). Arrows denote the transitions between compartments with associated expressions indicating the per capita transition rates.

set G, which is defined as follows:

$$G = \left\{ (S(t), V(t), E(t), M(t), I(t), J(t), R(t)) \in R_+^7 : N \leq \frac{\Pi}{\mu} \right\}. \quad (2)$$

Proof 1 *System (1) added together yields the over all population change:*

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dV}{dt} + \frac{dE}{dt} + \frac{dM}{dt} + \frac{dI}{dt} + \frac{dJ}{dt} + \frac{dR}{dt},$$

$$\frac{dN}{dt} = \pi - \mu N - \delta(I + J),$$

$$\frac{dN}{dt} + \mu N \leq \pi$$

$$N(t) =\leq \frac{\Pi}{\mu} + \left(N_0 - \frac{\Pi}{\mu} \right) e^{-}\mu t$$

We discovered that:
if $N_0 \leq \frac{\Pi}{\mu}$ *then*$N(t) \leq \frac{\Pi}{\mu}$*, Accordingly, the solution stayed in area M if the starting population was smaller than the carrying capacity.*
if $N_0 > \frac{\Pi}{\mu}$*, that is, if the initial population exceeds the carrying capacity* $\frac{\Pi}{\mu}$*, which suggests that the initial solutions are not located in area G, then* $N(t) > \frac{\Pi}{\mu}$*, for all* $t > 0$*, but* $\lim_t \to \infty N(t) = \frac{\Pi}{\mu}$ *The first solution eventually enters G, as demonstrated by this. Because all of the solutions outside of area G are eventually drawn into G and the solution of the system (1) with beginning conditions in G for any time* $t > 0$*, G is positively invariant and draws all of the system's initial solutions.*

Pertussis Free Equilibrium Point

If there is no pertussis in the community, the model system (1) has a disease-free equilibrium ϵ^0. By solving the noninfected classes and setting the infection classes and the right-hand sides of the system of equation (1) to zero, one can analytically ascertain this equilibrium:

Table 1: Interpretation of the state variables and parameters used in the model (1).

Variable	Description
N	Total population
S	Susceptible unvaccinated individuals
V	Susceptible vaccinated individuals
E	Exposed individuals
M	Infected mild individuals
I	Infected severe individuals
J	Isolated individuals
R	Recovered individuals
Parameter	**Description**
Π	Per ca pita birth rate
p	A proportion of birth or immigrant who are vaccinated with first dose
ψ	Vaccine coverage
ω	Vaccine wining rate
μ	Natural mortality rate
β	Effective contact rate
θ	Reduction risk of infection
α_1	Progression rate of exposed individuals to mild and severe infectious compartment
α_2	Fraction of exposed individuals that are mild
η	Progression rate of mild infectious to severe infectious compartment
γ	Isolation rate
ϕ_1	Recovery rates of infectious mild individuals due to treatment
ϕ_2	Recovery rates of isolated individuals due to treatment
δ	Pertussis induced death rate
ξ	Modification parameter due to reduced infectiousness of severe individuals
v	booster vaccine dose

$$\epsilon^0 = \left(S^0, V^0, E^0, M^0, I^0, J^0, R^0\right)$$

$$S^0 = \frac{\left((1-p)\, m_2 + p\omega\right)\Pi}{m_1\, m_2 - \omega\, \psi}, \tag{3}$$

$$V^0 = \frac{\left((1-p)\, \psi + m_1 p\right)\Pi}{m_1\, m_2 - \omega\, \psi}, \tag{4}$$

$$R^0 = \frac{\left((1-p)\, \psi + m_1 p\right) v\Pi}{\mu(m_1\, m_2 - \omega\, \psi)}), \tag{5}$$

$$\left(E^0, M^0, I^0, J^0\right) = (0, 0, 0, 0).$$

Where

$$m_2 = \psi + \mu, m_3 = \omega + v + \mu.$$

Reproduction Number

The model's reproduction number, or threshold parameter, is determined by applying the next-generation matrix technique $\mathcal{R}_0 = \rho(\mathcal{V}_1 V_2^{-1})$ as in (Ibrahim *et al.,* 2025) and (Andrawus *et al.,* 2024a) where ρ is the spectral radius or the dominant eigenvalue of the matrix $\mathcal{V}_1\mathcal{V}_2^{-1}$ The matrix for newly introduced infection terms linearized at the disease-free equilibrium is denoted by $\mathcal{V}_1$. On the other hand, matrix $\mathcal{V}_2$ represents the remaining transition terms.

$$\mathcal{V}_1 = \begin{bmatrix} 0 & \frac{\beta\left(S^0+\theta\, V^0\right)}{N^0} & \frac{\xi\,\beta\left(S^0+\theta\, V^0\right)}{N^0} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\mathcal{V}_2 = \begin{bmatrix} m_3 & 0 & 0 & 0 \\ -\alpha_1\alpha_2 & m_4 & 0 & 0 \\ -\alpha_1(1-\alpha_2) & -\eta & m_5 & 0 \\ 0 & 0 & -\gamma & m_6 \end{bmatrix} \tag{7}$$

$$\mathcal{V}_2^{-1} = \begin{bmatrix} {m_3}^{-1} & 0 & 0 & 0 \\ \frac{\alpha_1\,\alpha_2}{m_3\,m_4} & {m_4}^{-1} & 0 & 0 \\ \frac{\alpha_1\,(\eta\,\alpha_2 - m_4\,\alpha_2 + m_4)}{m_3\,m_4\,m_5} & \frac{\eta}{m_4\,m_5} & {m_5}^{-1} & 0 \\ \frac{\gamma\,\alpha_1\,(\eta\,\alpha_2 - m_4\,\alpha_2 + m_4)}{m_3\,m_4\,m_5\,m_6} & \frac{\gamma\,\eta}{m_4\,m_5\,m_6} & \frac{\gamma}{m_5\,m_6} & {m_6}^{-1} \end{bmatrix} \tag{8}$$

$$\mathcal{V}_1\mathcal{V}_2^{-1} = \begin{bmatrix} \frac{\beta\left(V^0\theta + S^0\right)\alpha_1\,(\eta\,\xi\,\alpha_2 - m_4\,\xi\,\alpha_2 + m_4\,\xi + m_5\,\alpha_2)}{N^0 m_3\,m_4\,m_5} & \frac{\beta\left(V^0\theta + S^0\right)(\eta\,\xi + m_5)}{N^0 m_4\,m_5} & \frac{\xi\,\beta\left(V^0\theta + S^0\right)}{N^0 m_5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

where $m_3 = \alpha_1 + \mu, m_4 = \eta + \phi_1 + \mu, m_4 = \gamma + \mu + \delta, m_5 = \phi_2 + \mu + \delta$. To determine the eigenvalues of the matrix $\mathcal{V}_1\mathcal{V}_2^{-1}$ We utilize the $\det(\mathcal{V}_1\mathcal{V}_2^{-1} - YI) = 0$ where the eigenvalues are represented by Y. (10) calculates the eigenvalues as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\left(\theta\,V^0 + S^0\right)\alpha_1\,\beta\,(((\eta - m_4)\xi + m_5)\alpha_2 + \xi\,m_4)}{N^0 m_4\,m_3\,m_5} \end{bmatrix} \tag{10}$$

The control reproduction number, or dominant eigenvalue, can be found as follows:

$$\mathcal{R}_c = \frac{\beta\alpha_1\,\left(S^0 + \theta\,V^0\right)((\eta - m_4)\alpha_2\xi + m_5\alpha_2 + m_4\xi)}{N^0 m_4 m_3 m_5}. \tag{11}$$

Substituting (3), (4) and (5) into (11), we obtain

$$\mathcal{R}_c = \frac{\beta\mu\alpha_1[(1-p)(m_2 + \psi\theta) + p(\omega + m_1\theta)]((\eta - m_4)\alpha_2\xi + m_5\alpha_2 + m_4\xi)}{m_3 m_4 m_5[\mu m_2 + \psi(\mu + v)]}. \tag{12}$$

Epidemiological Interpretation of $\mathcal{R}_c$: Here, $\mathcal{R}_c$ is interpreted as the number of susceptible individuals produced by pertussis infected during the entire infections period in a community with the presence of booster vaccine and isolation.

Local Stability of Pertussis Free Equilibrium

A system is considered locally asymptotically stable when a small disruption has no effect on its equilibrium state. Thus, the condition where a small number of illnesses will not result in a bigger outbreak is known as a locally asymptotically stable pertussis-free equilibrium. This requirement is satisfied mathematically if all of the (1) eigenvalues of the linearized system have negative negative real components. This is relevant to the following Theorem (2).

Theorem 2 *The pertussis-free equilibrium(PFE) ϵ^0, of the model (1), is locally-asymptotically stable (LAS) in G if $\mathcal{R}_c < 1$, and unstable if $\mathcal{R}_c > 1$.*

Proof 2 *The Jacobian matrix at the pertussis-free equilibrium is calculated as follows in order to linearize system (1):*

$$J(\epsilon^0) = \begin{bmatrix} -\psi - \mu & \omega & 0 & -\beta A_1 & -\beta\xi A_1 & 0 & 0 \\ \psi & -m_2 & 0 & -\theta\beta A_2 & -\theta\beta\xi A_2 & 0 & 0 \\ 0 & 0 & -m_3 & \beta A_1 + \theta\beta A_2 & \beta\xi A_1 + \theta\beta\xi A_2 & 0 & 0 \\ 0 & 0 & \alpha_1\alpha_2 & -m_4 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1(1-\alpha_2) & \eta & -m_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & -m_6 & 0 \\ 0 & v & 0 & \phi_1 & 0 & \phi_2 & -\mu \end{bmatrix}. \tag{13}$$

where, $A_1 = \frac{\mu((1-p)m_2+p\omega)}{m_1m_2-\psi\omega}, A_2 = \frac{\mu((1-p)\psi+m_1p)}{m_1m_2-\psi\omega}$.
Reducing equation (13) to row-echelon form yields:

$$J(\epsilon^0) = \begin{bmatrix} n_{11} & n_{12} & 0 & n_{14} & n_{15} & 0 & 0 \\ 0 & \frac{n_{22}\,n_{11}-n_{21}\,n_{12}}{n_{11}} & 0 & \frac{n_{24}\,n_{11}-n_{21}\,n_{14}}{n_{11}} & \frac{n_{25}\,n_{11}-n_{21}\,n_{15}}{n_{11}} & 0 & 0 \\ 0 & 0 & n_{33} & n_{34} & n_{35} & 0 & 0 \\ 0 & 0 & 0 & \frac{n_{44}\,n_{33}-n_{43}\,n_{34}}{n_{33}} & -\frac{n_{43}\,n_{35}}{n_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_{77} \end{bmatrix}. \tag{14}$$

Where

$$\begin{aligned} n_{11} = -(\psi+\mu), n_{12} = \omega, n_{14} = -\beta\xi A_1, n_{15} = -\beta A_1, n_{21} = \psi, n_{22} = -m_2, n_{24} = -\theta\beta A_2, \\ n_{25} = -\theta\beta\xi A_2, n_{33} = -m_3, n_{34} = \beta A_1 + \theta\beta A_2, n_{35} = \beta\xi A_1 + \theta\beta\xi A_2, n_{43} = \alpha_1\alpha_2, n_{44} = -m_4, \\ n_{53} = \alpha_1(1-\alpha_2), n_{54} = \eta, n_{55} = -m_5, n_{65} = \gamma, n_{66} = -m_6, n_{72} = v, n_{74} = \phi_1, n_{76} = \phi_2, n_{77} = -\mu, \\ K = \frac{n_{33}\,n_{44}\,n_{55} - n_{34}\,n_{43}\,n_{55} + n_{35}\,n_{43}\,n_{54} - n_{35}\,n_{44}\,n_{53}}{n_{44}\,n_{33} - n_{43}\,n_{34}} \end{aligned} \tag{15}$$

The eigenvalues are calculated using the maple software as follows,

$$\begin{bmatrix} n_{77} \\ n_{66} \\ n_{33} \\ \frac{n_{44}\,n_{33}-n_{43}\,n_{34}}{n_{33}} \\ \frac{n_{33}\,n_{44}\,n_{55}-n_{34}\,n_{43}\,n_{55}+n_{35}\,n_{43}\,n_{54}-n_{35}\,n_{44}\,n_{53}}{n_{44}\,n_{33}-n_{43}\,n_{34}} \\ n_{11} \\ \frac{n_{22}\,n_{11}-n_{21}\,n_{12}}{n_{11}} \end{bmatrix}. \tag{16}$$

Clearly, $\lambda_1, \lambda_2, \lambda_3$ *and* λ_6 *are all negatives from (15). Now for the remaining eigenvalues after substitution we have,*
λ_4 *is negative if and only if*

$$\frac{m_3m_4 - \alpha_1\alpha_2\beta\,(A_1+\theta A_2)}{-m_2} < 0 \tag{17}$$

$$\iff m_3m_4 - \alpha_1\alpha_2\beta\,(A_1+\theta A_2) < 0. \tag{18}$$

$$\iff \alpha_1\alpha_2\beta\,(A_1+\theta A_2) < m_3m_4 \tag{19}$$

$$\iff \frac{\alpha_1\alpha_2\beta\,(A_1+\theta A_2)}{m_3m_4} < 1 \tag{20}$$

λ_4 *is negative.*

λ_5 *is negative if and only if*

$$-m_3m_4m_5 + \frac{\beta\mu\alpha_1[(1-p)(m_2+\psi\theta)+p(\omega+m_1\theta)]((\xi-m_4)\alpha_2\xi+m_4\alpha_2+m_4\xi)}{[\mu m_2+\psi(\mu+v)]} < 0 \tag{21}$$

$$\iff \frac{\beta\mu\alpha_1[(1-p)(m_2+\psi\theta)+p(\omega+m_1\theta)]((\eta-m_4)\alpha_2\xi+m_5\alpha_2+m_4\xi)}{[\mu m_2+\psi(\mu+v)]} < m_3m_4m_5. \tag{22}$$

$$\iff \frac{\beta\mu\alpha_1[(1-p)(m_2+\psi\theta)+p(\omega+m_1\theta)]((\eta-m_4)\alpha_2\xi+m_5\alpha_2+m_4\xi)}{m_3m_4m_5[\mu m_2+\psi(\mu+v)]} < 1 \tag{23}$$

$$\frac{\beta\mu\alpha_1[(1-p)(m_2+\psi\theta)+p(\omega+m_1\theta)]((\eta-m_4)\alpha_2\xi+m_5\alpha_2+m_4\xi)}{m_3m_4m_5[\mu m_2+\psi(\mu+v)]} = \mathcal{R}_c < 1 \tag{24}$$

λ_5 *is also negative.*
λ_7 *is negative if and only if*

$$\frac{m_2(\psi+\mu),-\psi\omega}{-(\psi+\mu)} < 0 \tag{25}$$

$$\iff m_2(\psi+\mu)-\psi\omega < 0 \tag{26}$$

$$\iff m_2(\psi+\mu) < \psi\omega \tag{27}$$

$$\iff \frac{m_2(\psi+\mu)}{\psi\omega} < 1. \tag{28}$$

λ_7 *is also negative.*

This shows that all the eigenvalues are negative if $\mathcal{R}_c < 1$ and unstable otherwise. This proved theorem (2).

A community will not have a significant outbreak of pertussis if $\mathcal{R}_c < 1$, as demonstrated by epidemiological evidence in Theorem (2). Furthermore, the disease can be managed if the control reproduction number is kept under one and there are few people with pertussis.

Global Stability of Pertussis Free Equilibrium

Global stability refers to a dynamical system's capacity to remain stable in the face of larger disturbances. Thus, it is crucial to ascertain whether or not the pertussis-free equilibrium will remain stable in spite of the more significant disturbances. As a result, we proposed the following theorem on system stability (1).

Theorem 3 *The pertussis-free equilibrium of the system* (1) *is globally asymptotically stable in* G *if* $\mathcal{R}_c < 1$ *and unstable if* $\mathcal{R}_c > 1$.

Proof 3

$$\begin{bmatrix} \dot{E} \\ \dot{M} \\ \dot{I} \\ \dot{J} \end{bmatrix} = (\mathcal{V}_1-\mathcal{V}_2)\begin{bmatrix} E \\ M \\ I \\ J \end{bmatrix} - \mathcal{X}\begin{bmatrix} E \\ M \\ I \\ J \end{bmatrix}. \tag{29}$$

$$(\mathcal{V}_1-\mathcal{V}_2) = \begin{bmatrix} m_3 & \frac{\beta(S^0+\theta V^0)}{N^0} & \frac{\xi\beta(S^0+\theta V^0)}{N^0} & 0 \\ \alpha_1\alpha_2 & -m_4 & 0 & 0 \\ \alpha_1(1-\alpha_2) & \eta & -m_5 & 0 \\ 0 & 0 & \gamma & -m_6 \end{bmatrix} \tag{30}$$

where $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are the same matrices in in equations* (31) *and* (7)

$$\mathcal{X} = \begin{bmatrix} 0 & \frac{\beta\left(S^0+\theta V^0\right)}{N^0} - \frac{\beta\left(S+\theta V\right)}{N} & \frac{\xi\,\beta\left(S^0+\theta V^0\right)}{N^0} - \frac{\xi\,\beta\left(S+\theta V\right)}{N} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

The fact that $S^* \geq V$ *implies that matrix* $\mathcal{X}$ *is nonnegative. Thus,* (29) *can be expressed as*

$$\begin{bmatrix} \dot{E} \\ \dot{M} \\ \dot{I} \\ \dot{J} \end{bmatrix} \leq (\mathcal{V}_1 - \mathcal{V}_2) \begin{bmatrix} E \\ M \\ I \\ J \end{bmatrix}. \tag{32}$$

The condition $\rho(\mathcal{V}_1\mathcal{V}_2^{-1}) < 1$ *which hold if* $\mathcal{R}_c < 1$ *(as shown in the theorem* (2) *proof), is identical to* $(\mathcal{V}_1 - \mathcal{V}_2)$ *with eigenvalues that have negative real portions. All of the eigenvalues of* $(\mathcal{V}_1 - \mathcal{V}_2)$ *have negative real portions when* $\mathcal{R}_c < 1$*. As a result, the linearized subsystem* (29) *is stable when* $\mathcal{R}_c < 1$*. This implies that* $E(t), M(t), I(t), J(t) \to (0, 0, 0, 0)$ *as* $t \to \infty$
By substituting $E(t) = M(t) = I(t) = J(t) = 0$ *into the uninfected compartments* $(S(t), V(t), R(t))$ *Since the resulting equations reduce to* (3) *through* (5)*, it has been established that the entire model system* (1) *is globally stable.*

Pertussis Endemic Equilibrium Point

An equilibrium state occurs when a disease has spread and persisted in a society for an extended period of time. The state variables of the model (1) are difficult to solve explicitly, but we can derive their implicit solutions in terms of λ and demonstrate that the equilibrium state exists. Let $S^{**}, V^{**}, E^{**}, M^{**}, I^{**}, J^{**}$ be the state variables in the endemic equilibrium state, where λ^* is the infection force. The state variable solutions are as follows:

$$\begin{aligned}
S^{**} &= -\frac{\Pi\,(\lambda^* p\theta - \lambda^*\theta + m_2 p - \omega p - m_2)}{\lambda^{*2}\theta + \lambda^* m_1\theta + \lambda^* m_2 + m_1 m_2 - \omega\psi} \\
V^{**} &= \frac{\Pi\,(\lambda p + m_1 p - p\psi + \psi)}{\lambda^2\theta + \lambda m_1\theta + \lambda m_2 + m_1 m_2 - \omega\psi} \\
E^{**} &= \frac{\lambda^*\Pi\,(m_1 p\theta - p\psi\theta + \lambda^*\theta - m_2 p + p\omega + \psi\theta + m_2)}{m_3\left(\lambda^{*2}\theta + \lambda^*\theta m_1 + m_2\lambda^* + m_2 m_1 - \psi\omega\right)} \\
M^{**} &= \frac{\alpha_1\alpha_2\lambda^*\Pi\,(m_1 p\theta - p\psi\theta + \lambda\theta - m_2 p + \omega p + \psi\theta + m_2)}{m_4 m_3\,(\lambda^{*2}\theta + \lambda^* m_1\theta + \lambda^* m_2 + m_1 m_2 - \omega\psi)} \\
I^{**} &= \frac{\alpha_1\,(\eta\alpha_2 - m_4\alpha_2 + m_4)\lambda^*\Pi\,(m_1 p\theta - p\psi\theta + \lambda^*\theta - m_2 p + \omega p + \psi\theta + m_2)}{m_5 m_4 m_3\,(\lambda^{*2}\theta + \lambda^* m_1\theta + \lambda^* m_2 + m_1 m_2 - \omega\psi)} \\
J^{**} &= \frac{\gamma\alpha_1\,(\eta\alpha_2 - m_4\alpha_2 + m_4)\lambda\Pi\,(m_1 p\theta - p\psi\theta + \lambda^*\theta - m_2 p + \omega p + \psi\theta + m_2)}{m_5 m_6 m_4 m_3\,(\lambda^{*2}\theta + \lambda^* m_1\theta + \lambda^* m_2 + m_1 m_2 - \omega\psi)}
\end{aligned} \tag{33}$$

It has been observed that R is not included in the above equation ; this is acceptable as the immunity acquired by those recovered causes them to be eliminated from the population. A person may also be recruited as a new susceptible if their immunity wanes. Endangemicity will not affect an individual who has immunity because he will not contract the infection. However, someone may be recruited as susceptible if their immunity wanes.

Pertussis Existance of EE point

To determine the existence of endemic equilibrium, we used **Descarte's rule of sign** where the number of roots of a polynomial equation is equal to the number of changes of sign for details see (Mustapha *et al.*, 2023). Now an endemic state force of infection is given by

$$\lambda^* = \frac{\beta(M^{**} + \xi I^{**})}{N^{**}} \tag{34}$$

where,

$$N^{**} = S^{**} + V^{**} + E^{**} + M^{**} + I^{**} + J^{**}. \quad (35)$$

After solving the equation and substituting $M^{*}*$, $I^{*}*$, and $N^{*}*$ in (34), we obtain $\lambda^{*} = 0$ (which is comparable to the stable DFE) and the quadratic equation that follows.

$$Z_1\lambda^{*^2} + Z_2\lambda^{*} + Z_3 = 0 \quad (36)$$

where,

$$\begin{aligned}
Z_1 =& m_3m_4m_5(m_4 + \alpha_1(1-\alpha_2)) + (\alpha_1(1-\alpha_2)\eta + \alpha_1\alpha_2m_4)(m_6+\gamma)\\
Z_2 =& m_3m_4m_5m_6[(1-p)\theta + p] + (m_5m_6 + \alpha_1(1-\alpha_2))[(1-p)(m_2+\theta\psi) + p(m_1\theta+\omega)]\\
&+ (\alpha_1(1-\alpha_2)\eta + \alpha_1\alpha_2m_4)[(m_6+\gamma)((1-p)(m_2+\theta\psi)\\
&+ p(m_1\theta+\omega))] - \beta\theta m_6(\alpha_1(1-\alpha_2)m_5 + \alpha_1(1-\alpha_2)\eta\xi + \alpha_1\alpha_2m_4\xi)\\
Z_3 =& m_3m_4m_5[\mu m_2 + \psi(\mu+v)][1-\mathcal{R}_c].
\end{aligned} \quad (37)$$

Clearly, $Z_1 > 0$ since all the parameters are positive. So there are four cases to be considered depending on the sign of Z_2 and Z_3.

Theorem 4 *The system (1) has:*

I. no endemic equilibrium if $Z_2 > 0$ and $Z_3 > 0$ $\iff \mathcal{R}_c < 1$.

II. a unique endemic equilibrium if $Z_2 < 0$ and $Z_3 < 0 \iff \mathcal{R}_c > 1$.

III. a unique endemic equilibrium if $Z_2 > 0$ and $Z_3 < 0 \iff \mathcal{R}_c > 1$.

IV. two positive equilibrium if $Z_2 < 0$ and $Z_3 > 0$. $\iff \mathcal{R}_c < 1$ and $Z_2^2 - 4Z_1Z_3 > 0$.

The following theorem was established based on items 2 and 3 of theorem (4) for reference.

Theorem 5 *The system (1) has a unique positive endemic equilibrium if $\mathcal{R}_c > 1$.*

Global Stability of EE Point

Theorem 6 *Suppose $\delta = \omega = \psi = v = \phi_1 = 0$ then the endemic equilibrium is globally asymptotically stable if $\mathcal{R}_c > 1$ and unstable if $\mathcal{R}_c < 1$.*

Proof 4 *Let F be a Lyapunov function of the Goh-Volterra type in the manner described.*

$$\begin{aligned}
F =& \left(S - S^{**} - S^{**}ln\frac{S}{S}^{**}\right) + \left(V - V^{**} - V^{**}ln\frac{V}{V}^{**}\right) + \left(E - E^{**} - E^{**}ln\frac{E}{E}^{**}\right)\\
&+\frac{(\alpha_1+\mu)}{\alpha_1}\left(M - M^{**} - M^{**}ln\frac{M}{M}*^{*}\right) + \frac{(\alpha_1+\mu)(\eta+\mu)}{\alpha_1\eta}\left(I - I^{**} - I^{**}ln\frac{I}{I}^{**}\right)\\
&+\frac{(\alpha_1+\mu)(\eta+\mu)(\gamma+\mu)}{\alpha_1\gamma\eta}\left(J - J^{**} - J^{**}ln\frac{J}{J^{**}}\right).
\end{aligned} \quad (38)$$

When (38) is differentiated in relation to time, we obtain

$$\begin{aligned}
\dot{F} =& \left(1 - \frac{S^{**}}{S}\right)\dot{S} + \left(1 - \frac{V^{**}}{V}\right)\dot{V} + \left(1 - \frac{E^{**}}{E}\right)\dot{E} + \frac{(\alpha_1+\mu)}{\alpha_1}\left(1 - \frac{M^{**}}{M}\right)\dot{M}\\
&\frac{(\alpha_1+\mu)(\eta+\mu)}{\alpha_1}\left(1 - \frac{I^{**}}{I}\right)\dot{I} + \frac{(\alpha_1+\mu)(\eta+\mu)(\gamma+\mu)}{\alpha_1\gamma\eta}\left(1 - \frac{J^{**}}{J}\right)\dot{J}
\end{aligned} \quad (39)$$

with

$$N = \frac{\Pi}{\mu} \quad (40)$$

As we modify the force of the infection, we have

$$\bar{\lambda} = \bar{\beta}(M + \xi I) \quad (41)$$

where

$$\bar{\beta} = \beta\frac{\Pi}{\mu} \quad (42)$$

When (1) is substituted with (39), we obtain

$$\begin{aligned}\dot{F} &= \left(1-\frac{S^{**}}{S}\right)((1-p)\pi-\lambda S-\mu S)+\left(1-\frac{V^{**}}{V}\right)(p\pi-\alpha\lambda V-\mu V)\\ &+\left(1-\frac{E^{**}}{E}\right)(\lambda S+\alpha\lambda V-(\alpha_1+\mu)E)+\frac{(\alpha_1+\mu)}{\alpha_1}\left(1-\frac{M^{**}}{M}\right)(\alpha_1 E-(\eta+\mu)M)\\ &+\frac{(\alpha_1+\mu)(\eta+\mu)}{\alpha_1\eta}\left(1-\frac{I^{**}}{I}\right)(\eta M-(\gamma+\mu)I)\\ &+\frac{(\alpha_1+\mu)(\eta+\mu)(\gamma+\mu)}{\alpha_1\gamma\eta}\left(1-\frac{J^{**}}{J}\right)(\gamma I-(\phi_2+\mu)J)\end{aligned} \tag{43}$$

With relationships

$$\begin{aligned}(1-p)\Pi &= \lambda^{**}S^{**}+\mu S^{**},\\ p\Pi &= \theta\lambda^{**}V^{**}+\mu V^{**},\\ (\alpha_1+\mu)E^{**} &= \lambda^{**}S^{**}+\theta\lambda^{**}V^{**},\\ (\eta+\mu)M^{**} &= \alpha_1 E^{**},\\ (\gamma+\mu)I^{**} &= \eta M^{**},\\ (\phi_2+\mu)J^{**} &= \gamma I^{**}.\end{aligned} \tag{44}$$

We may simplify by changing the relations in (44) to (43).

$$\begin{aligned}\dot{F} &\le \mu S^{**}\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)+\mu V^{**}\left(2-\frac{V}{V^{**}}-\frac{V^{**}}{V}\right)\\ &+\lambda S^{**}\left(6-\frac{S^{**}}{S}-\frac{SE^{**}}{S^{**}E}-\frac{EM^{**}}{E^{**}M}-\frac{MI^{**}}{M^{**}I}-\frac{IJ^{**}}{I^{**}J}-\frac{J}{J^{**}}\right)\\ &+\theta\lambda V^{**}\left(6-\frac{V^{**}}{V}-\frac{VE^{**}}{V^{**}E}-\frac{EM^{**}}{E^{**}M}-\frac{MI^{**}}{M^{**}I}-\frac{IJ^{**}}{I^{**}J}-\frac{J}{J^{**}}\right)\end{aligned} \tag{45}$$

Next, we use the connection between the arithmetic and geometric means to obtain

$$\begin{aligned}&\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)\le 0,\left(2-\frac{V}{V^{**}}-\frac{V^{**}}{V}\right)\le 0,\left(6-\frac{S^{**}}{S}-\frac{SE^{**}}{S^{**}E}-\frac{EM^{**}}{E^{**}M}-\frac{MI^{**}}{M^{**}I}-\frac{IJ^{**}}{I^{**}J}-\frac{J}{J^{**}}\right)\le 0,\\ &\left(6-\frac{V^{**}}{V}-\frac{VE^{**}}{V^{**}E}-\frac{EM^{**}}{E^{**}M}-\frac{MI^{**}}{M^{**}I}-\frac{IJ^{**}}{I^{**}J}-\frac{J}{J^{**}}\right)\le 0.\end{aligned} \tag{46}$$

Hence,
$F^{'}\le 0$. *Strict equality* $F^{'}=0$ *is only true at* $S=S^{**},V=V^{**},E=E^{**},M=M^{**},I=I^{**}$, *and* $J=J^{**}$. *The model 1 exhibits only one invariant set, which is the endemic equilibrium* ϵ^{*}. *Applying the Lasalle invariance principle, the outcome is as follows see (Lasalle, 1976). For the model (1), this means that the endemic equilibrium (EE)* ϵ^{*} *is globally asymptotically stable (GAS).*

Sensitivity analysis

We analyzed the sensitivity of the fundamental reproduction number $\mathcal{R}_c$ in relation to certain important associated parameters of pertussis transmission dynamics, including isolation and booster vaccine, using the forward sensitivity index approach. The easiest way to stop the spread of pertussis infections is to reduce the basic reproduction number, which is a parameter-dependent output (P. Van De Driessche and Watmough, 2002) . Since some parameters are extremely sensitive, some are only marginally sensitive, and some have no relative sensitivity at all, some parameter modifications might not have an equivalent impact on the outcomes (Ibrahim *et al.,* 2025). Thus, the normalized sensitivity Hence the normalized sensitivity index $\chi_{\eta}^{\mathcal{R}_c}$ is given as,

$$\chi_{\eta}^{\mathcal{R}_c}=\frac{\eta}{\mathcal{R}_c}\times\frac{\partial\mathcal{R}_c}{\partial\eta} \tag{47}$$

We get the sensitivity status of each parameter with respect to $\mathcal{R}_c$ in the table (2).

Figure (2) shows the most sensitive parameters for increasing the control reproduction number are

Table 2: Forward Normalized Sensitivity Indices

Parameter	Elasticity Indices	Values of the Elasticity index
θ	$\eta_{\theta}^{\mathcal{R}_c}$	0.2105
ξ	$\eta_{\xi}^{\mathcal{R}_c}$	0.6415
ψ	$\eta_{\psi}^{\mathcal{R}_c}$	-0.8113
ω	$\eta_{\omega}^{\mathcal{R}_c}$	0.0509
v	$\eta_{v}^{R_c}$	-0.3248
β	$\eta_{\beta}^{R_c}$	1.0000
η	$\eta_{\eta}^{R_c}$	0.0041
α_1	$\eta_{\alpha_1}^{R_c}$	0.1254
α_2	$\eta_{\alpha_2}^{R_c}$	0.3142
γ	$\eta_{\gamma}^{R_c}$	-0.8128
ϕ_1	$\eta_{\phi_1}^{R_c}$	-0.3159
δ	$\eta_{\delta}^{R_c}$	-0.0571

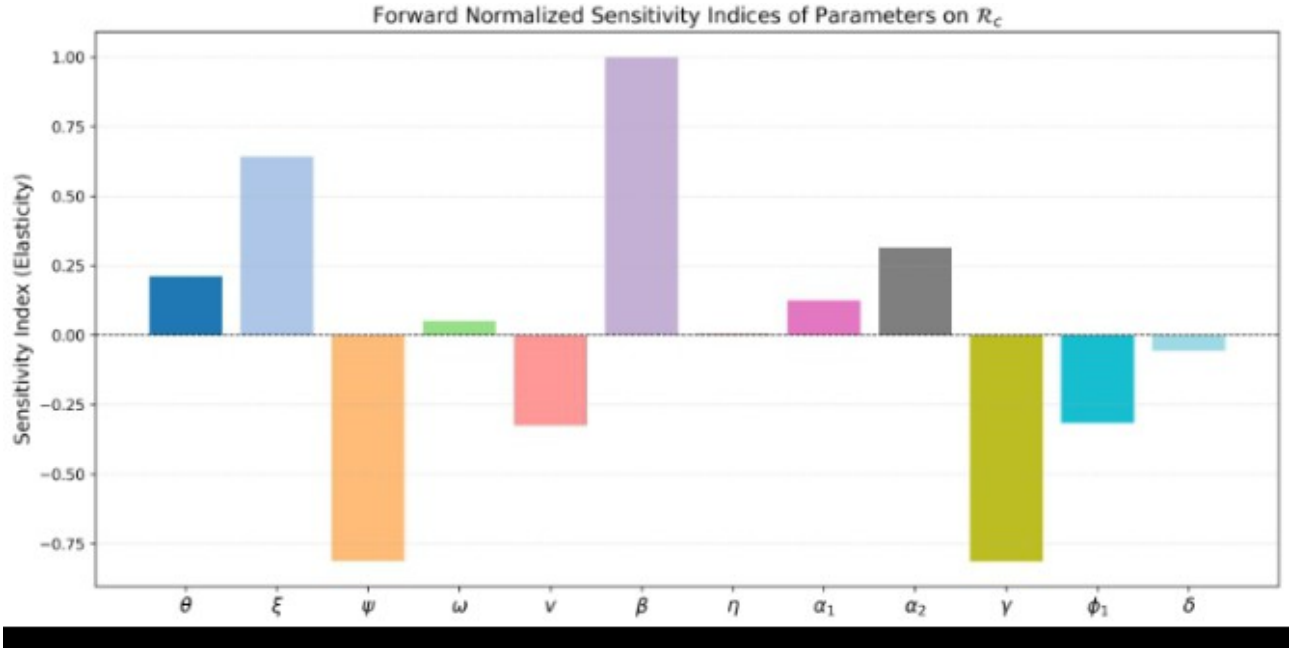

Figure 2: Bar chart graph showing the elasticity indices versus parameters

contact rate β and modification parameter due to the reduction of infectious severe individuals ξ while the most sensitive parameters for decreasing the control reproduction number are isolation γ and vaccine rate ψ. This shows that isolation and vaccination have a strong impact on reducing the spread of pertussis in a community.

NUMERICAL SIMULATIONS

Numerical simulations of the proposed model (1) provide valuable insights into the dynamics of the disease by simulating the interaction of the various factors that contribute to its spread and control. Some of the parameters used in the simulations are derived from existing literature, while others are assumed, and two are control parameters that vary between 0 and 1.

Impact of Isolation

Both Figures (3) and (4) illustrate the effectiveness of increasing the isolation rate in controlling the spread of the disease described by the proposed model (1), with an increase in the isolation rate leading to a significant reduction in the number of severely infected individuals and a corresponding increase in the number of isolated individuals.

Impact of Vaccination

Figure (6, 7) shows the impact of vaccination rate on the susceptible un-vaccinated and vaccinated respectively. The figures shows a significant decrease in the number of susceptible and an increases in the number of vaccinated individuals as the parameter ψ increases.

Table 3: Ranges and baseline values of parameters of model (1).

Parameter	Ranges (Baseline)	Unit	Reference
λ	0.0006	per year	(Johnson and Edogbanya, 2024)
Π	0.019	per year	(Johnson and Edogbanya, 2024)
p	0.06	per year	Fitted
ψ	0.003264	per year	Fitted
ω	0.77657	per year	Fitted
μ	0.012	per year	Fitted
δ	0.053	per year	(Johnson and Edogbanya, 2024)
β	0.58	per year	(Johnson and Edogbanya, 2024)
η	0.0071	per year	Fitted
θ	0.8315045	per year	Fitted
γ	0.2758	per year	Fitted
ϕ_1	0.6009	per year	Fitted
ϕ_2	0.0836	per year	Fitted
α_2	0.0301	per year	Fitted
α_1	0.1840	per year	Fitted
ξ	0.598350	per year	Fitted
v	0.012781	per year	Fitted

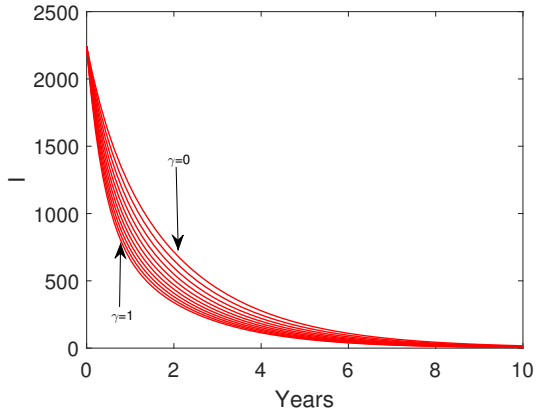

Figure 3: Impact of isolation rate on severely infectious individuals

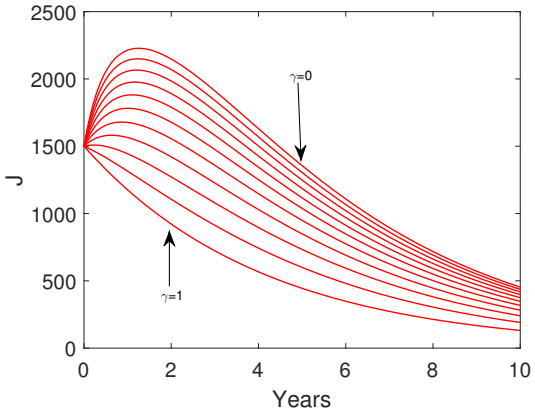

Figure 4: Impact of isolation rate on Isolated individuals

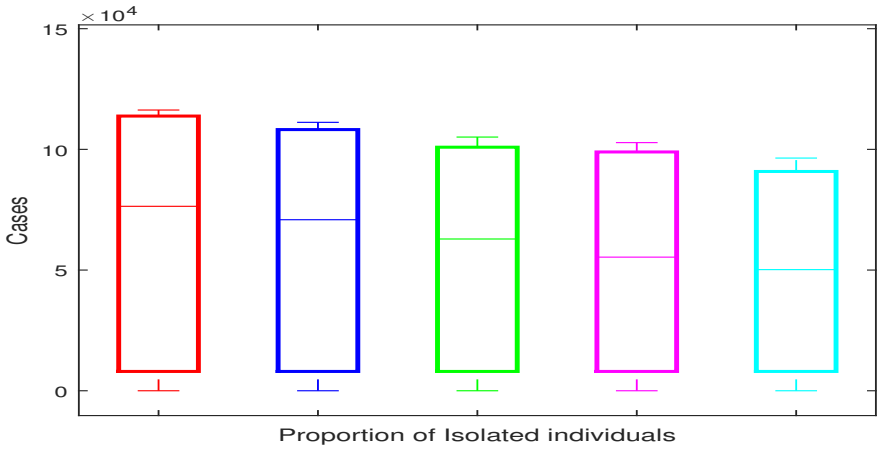

Figure 5: Box plotting illustrating the effectiveness of isolation

DISCUSSION

Figures (6) and (7) shows the impact of vaccination. Figure (6) shows that as the vaccine rate increases, the number of unvaccinated significantly decreases while figure (7) demonstrates that as the vaccine rate increases, the number of vaccinated individuals significantly increases, indicating that vaccine is an effective strategy for controlling pertussis in a community. The effectiveness of isolation as a control measure for pertussis is clearly depicted in Figures (3)

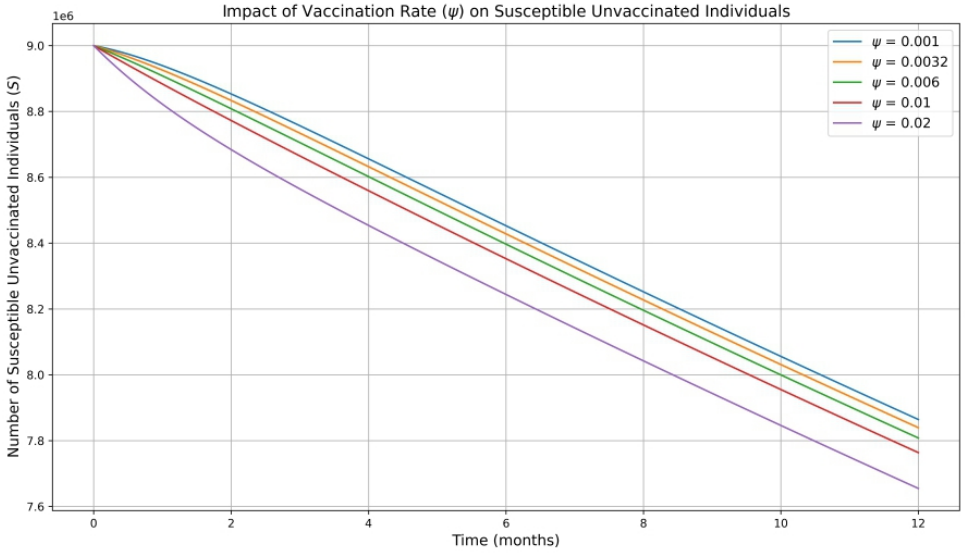

Figure 6: Pattern of Susceptible Individuals with differents values of ψ

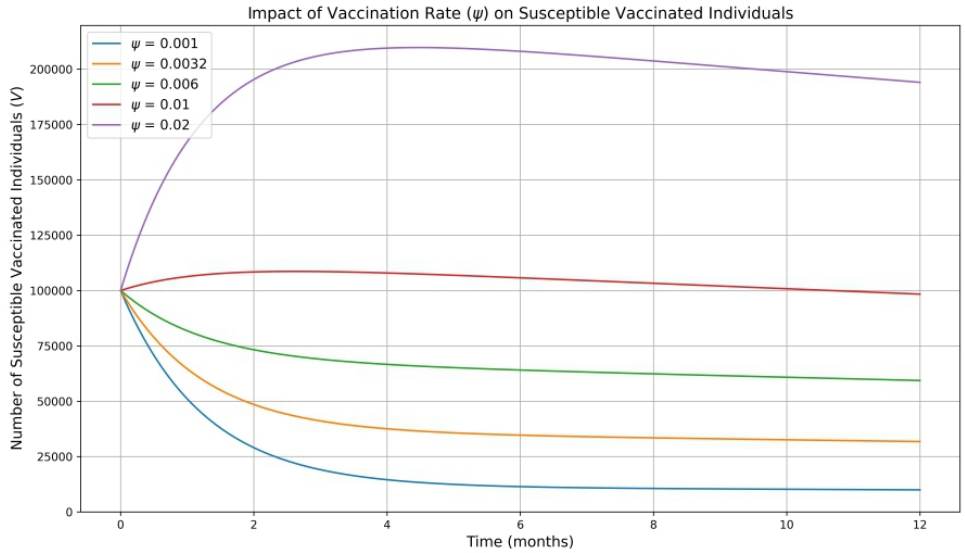

Figure 7: Pattern of Vaccinated Individuals with differents values of ψ

and (4). Figure (3) shows that as the isolation rate increases, the number of severely infected individuals decreases significantly. Figure (4) demonstrates that as the isolation rate increases, the number of isolated individuals increases significantly, indicating that isolation is an effective strategy for controlling the spread of the disease. Figure (5) provides a clear visualization of the effectiveness of the isolation strategy in controlling pertussis in the environment, with the number of cases shown to decrease over time. This visual representation reinforces the importance of isolation in reducing the spread of the disease and highlights the effectiveness of the strategy in controlling the disease.

CONCLUSION

An epidemiological compartmental model consisting of seven classes susceptible unvaccinated, susceptible vaccinated, exposed, mildly, severely, isolated, and recovered was used to predict the dynamics of pertussis. Numerical simulations and a thorough mathematical study of the model were performed. The model comprises two equilibria, the DFE and the EEP, as demonstrated by the mathematical analysis section's determination of the solution's positivity and boundedness as well as the presence of disease equilibria. Whenever $\mathcal{R}_c < 1$, the DFE is found to be asymptotically stable both locally and globally. Global asymptotic stability of endemic equilibrium point EEP has been ascertained using the nonlinear Lyapunov function of Go–Volterra type, which reveals that, the EEP is globally asymptotically stable if the control reproduction number is greater than unity and $\omega = \psi = \phi_1 = \upsilon = \delta = 0$. Numerical simulation indicates that the isolation rate of infected individuals and the booster vaccine are critical factors in societal control

of pertussis. Sensitivity analysis demonstrates that isolation and booster vaccinations are highly sensitive in lowering the control reproduction number. The box plot also demonstrates how crucial isolation is to containing pertussis in the environment. Therefore, we recommend that the isolation of infected individuals and the promotion of vaccination in the general public be prioritized.

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