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CYCLOIDS FOR MINIMAL TIME PROJECTION

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ABSTRACT

The minimum time has historical importance in relation to Fermat's principle, a famous problem that is well known to the readers of general physics mattrers. Our intention in this study is to restrict ourselves to classical Newtonian mechanics, more specifically to a projectile mothion that every college student has certain familiarity. This study established and utilises the Brachistochrone general equation to investigate the minimum time curve travelled by an object (jet) under the influence of gravity. It was found that reaching a maximum altitude within the shortest time possible, a jet fighter must take off with the same speed at a smaller projection angle. This means that travelling directly against gravity takes more time.

Keywords: Brachistochrone, Minimum time, Projection, Parabola special relativity

INTRODUCTION

'The worst of a thief is the one that steals our time', is an expression of wisdom emphasizing the importance of time in our daily life. Aside social affairs, 'time' is a variable in physics-the most precious one that has no reset in biological matters. In 1905, within the context of Special Relativity, Einstein put 'time' on an equal footing with the other spatial variables. The relativity of time is probably the greatest insight that mankind has ever had through Einstein's mind. Recall that twin-paradox as the standard text-book example to justify this basic fact (Socolovsky, 2013). The energy equivalence of mass i.e. $\varDelta E = (\varDelta m)c^2$, that even a layman in the street is aware of, averages as a consequence of intrinsic relativity of time. Since doing experiments with humans has serious biological risks, atomic clocks do the job on our behalf. K. Godel was afraid so much to die that he developed a mathematical/cosmological model in challenge of death (xam & oi Cosmo ogica, 1949). Ironically he presented it as a seventieth birthday gift to Einstein. Today we have many of such similar cosmological models in which the spacetime returns back to its initial time (Mohajan, 2013). At last, in spite of all that highly imaginative mathematical efforts both luminaries passed away.

With General Relativity in 1916, Einstein extended the concept of relativity to cover gravitational observer, both kinematical, dynamical and gravitational are crucial concept verified at large. Each frame/observer carries its own time and these different times are connected by some transformations such as Lorentz on general coordinates (MTW). Proper time, i.e. one's own time in one's own frame, is the only invariant time that does not change when many other things change. This fact makes the starting point for a variational principle in a relativistic (special or general) theory that minimizes the proper time. Mathematically if $(d\tau)$ denotes the infinitesimal proper time interval which is not an exact different, the expression known as action $I = \int_1^2 d\tau$.That is the variation (δI) must be zero for extremality and the second variation $(\delta^2 I)$, must be positive for a minimum. For a detailed explanation of the point we refer to Jacobi's principle of variational calculaus (Akoglu et al., 2017; Gyegwe et al., 2025; Houchmandzadeh, 2020).

In classical Newtonian physics there is no relativity of time. Times are all same absolute as related in the galilean principle of relativity in which spaces change but time not (xam & oi Cosmo ogica, 1949).

Our objective in this study is to restrict ourselves to classical Newtonian mechanics, more specifically to a projectile mothion that every college student has certain familiarity. It is well-known that any such object follows, by virtue of gravity a parabolic trajectory. This is the case when we assume a minimum of action fucntion defined by $I = \int (T - V)dt$, in which $T(= \frac{1}{2}m\vec{v}^2)$and $V(= mgy)$, refer to the kinetic and potential energies, respectively. And obviously t refers to the absolute time of Newton. We intend now at this point to make a diversion by considering a minmal time motion instead of the minimal action. This is naturally related with the well-known Brachistochrone problem (xam & oi Cosmo ogica, 1949), which describes the minimum time curve of descent, i.e. a cycloid in a uniform gravitational field. It will be shown that launching a projectile with an initial speed and angle will be again part of a cycloid, but somewhat modified, depending on the launching angle. In other words we wish to investigate the Fermat's principle for a projectile, which is a mechanical system distinct from a light trajectory that the principle aimed for. What happens to a launched projectile if it is to traverse a path in shortest time?. The question applies for instance to a jet fighter that is to reach a maximum altitude in a minimum time. This is precisely what we intend to investigate in this paper.

MATERIALS AND METHODS

Minimal time versus the action principle

Case I: We consider first a typical projectile motion launched at $x = y = 0$, with an angle θ_o making with the $+x$ axis and with the total initial speed ϑ_o (Fig.1). For simplicity we choose the mass as $m = 1$, so that the action is given by:

$$I = \int L(\dot{x}, \dot{y}; x, y)\, dt$$

$$I = \frac{1}{2}\int [(\dot{x}^2 + \dot{y}^2) - 2gy]dt \qquad (1)$$

in which L is the Lagrangian and a 'dot' means time derivative. The variational principle $\delta I = 0$, as in a more familiar term term, the Euler-Lagrange equation give the differential equations of motion.

$\ddot{x} = 0$ (2)
$\ddot{y} + g = 0$ (3)
Integration of these equations with the initial conditions of a projectile results in:
$x = (v_o \cos\theta_o)t$ (4)
$y = (v_o \sin\theta_o)t - \frac{1}{2}gt^2$ (5)
From which we obtain the orbit equation,
$y = x\tan\theta_o - \frac{gx^2}{2(v_o\cos\theta_o)}$ (6)
This is the well-known parabolic curve of introductory mechanics as shown in Figure1

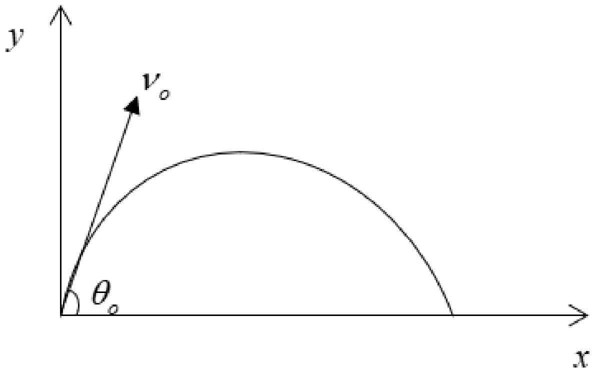

Figure 1: The projectile motion

Case II: Next, let us consider the total conserved energy E (for a unit mass) as
$E = T + V$
$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy$ (7)
$\Rightarrow E = \frac{1}{2}{v_o}^2$
The element of time (dt) is solved from the foregoing as
$dt = \sqrt{\frac{1+x'^2}{{v_o}^2 - 2gy}}$ (8)
Since we are interested in extremizing (in fact minimizing) the time, we choose our action principle as:
$I = \int\sqrt{\frac{1+x'^2}{{v_o}^2 - 2gy}}dy$ (9)
In which $x' = \frac{dx}{dy}$. The resulting equation of variational principle $\delta I = 0$, after a first integral can be expressed in the form:
$\left(\frac{dx}{dy}\right)^2 = \frac{{v_o}^2 - 2gy}{\frac{1}{\alpha} - ({v_o}^2 - 2gy)}$ (10)
in which α is an integration constant. Using the initial conditions for the projectile we obtain:
$\alpha = \frac{\cos\theta_o}{v_o}$ (11)
Let us note that in the typical Brachistochrone problem the mass is released without an initial speed which is require to slide to a point such that it will cover the minimum time. In the present problem the projectile moves against gravity with an arbitrary initial speed (v_o) and angle (θ_o). Complete integral for the Fermat projectile, i.e. of Equation (10) is as follows:
$$\frac{(2x - 1 + \tan^2\theta_o)\tan 2\theta_o + 2\sqrt{1-u}\sqrt{u + \tan^2\theta_o}}{2\sqrt{1-u}\sqrt{u + \tan^2\theta_o}\tan 2\theta_o - (2u - 1 + \tan^2\theta_o)}$$
$$= \tan[2\cos^2\theta_o(\pm\frac{2gx}{{v_o}^2} + \tan\theta_o - \sqrt{1-u}\sqrt{u + \tan^2\theta_o})]$$
$U = 2\sqrt{1-u}\sqrt{u + \tan^2\theta_o}$ (12)
in which we have used the abbreviation:
$u = \frac{2gy}{{v_o}^2}$ (13)
More appropraitely we shorten this expreesion as:
$\frac{W\tan 2\theta_o + U}{U\tan 2\theta_o - W} = \tan\varphi$ (14)
in which
$U = 2\sqrt{1-u}\sqrt{u + \tan^2\theta_o}$
$W = 2u - 1 + \tan^2\theta_o$ (15)
and
$\varphi = \cos^2\theta_o(\pm\frac{4gx}{v^2{}_o} + 2\tan\theta_o - U)$
It can easily be checked that the functions U and W satisfy the constraint without loss of generality from now on we shall choose the (+) sign inside the variable φ.
$U^2 + W^2 = \frac{1}{COS^4\theta_o}$ (16)
Let us also note that $u = \frac{2gy}{{v_o}^2} \leq 1$

Analysis of the Fermat Projectile
Case I: The orbit equation (12), or in abbreviated form equation (14) represents a generalised cycloid of minimal time. It is generalised in the sense that it contains two arbitrary parameters, $v_o$and θ_o. To see the connection with the standard form of the cycloid let us take first $\theta_o = 0$ and see the connection. Equation (12) reduces to:
$\frac{2\sqrt{u(1-u)}}{1-2u} = \tan\varphi$ (17)

with $\varphi = \frac{4gx}{{v_o}^2} - 2\sqrt{u(1-u)}$
solving for $u = \frac{2gy}{{v_o}^2}$ and x yields the standard form of the cycloid as:
$x = \frac{{v_o}^2}{4g}(\varphi + \sin\varphi)$
$y = \frac{{v_o}^2}{4g}(1 - \cos\varphi)$ (18)
depicted in Figure (2).

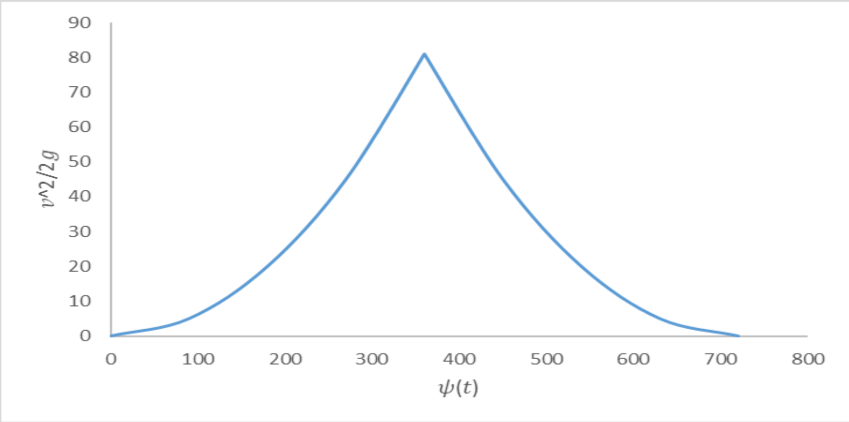

Figure 2: The cycloid curve representing a projectile fired horizontally at $x = 0, y = 0, (\psi)$. In minimal time it moves upward to $y_{max} = \frac{v_o^2}{2g}$, at $x = \frac{v_o^2\pi}{4g}$.

Case II: We are now in a position to express the general form of the cycloid equation for $\theta_o \neq 0$with reference to Equations (14) and (15) it is a good exercise to obtain:

$$x = a[\varphi + sin(\varphi + 2\theta_o) - sin\, 2\,\theta_o]$$
$$y = a[cos\, 2\,\theta_o - cos(\varphi + 2\theta_o)] \quad (19)$$

in which,

$$a = \frac{v_o^2}{4g\, cos^2\, \theta_o} \quad (20)$$

The parameter $\varphi$is proportional to time. In order to see this we use the energy conservation condition: $\dot{x}^2 + \dot{y}^2 = v_o^2 - 2gy$together with the Equation in (19). One obtains easily that

$$\varphi(t) = \frac{2gt}{v_o} cos\, \theta_o \quad (21)$$

At the maximum point ($y_{max}$of the projectile we have the conditions

$$\frac{dx}{d\varphi} = \frac{dy}{d\varphi} = 0 \quad (22)$$

which gives $\varphi + 2\theta_o = \pi$

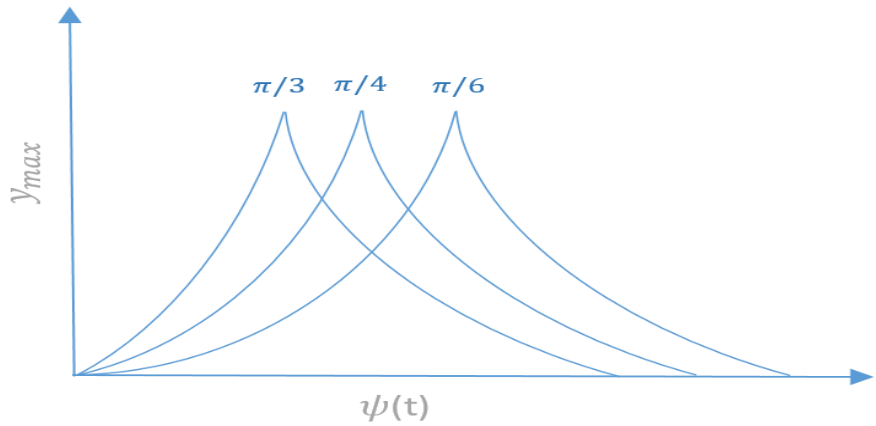

Figure 3: Variation of projection angle with time

Thus for different choices of $\theta_o$we make the plots in Figure (3). It is interesting that as the projection angles changes, and the time to reach that maximum change also depending on angle the y_{max}remains same for the common initial speed once the projectile reaches the maximum altitude in order to follow with the minimal time it must follow the standard cycloid as its next route. It is also interesting that the more steeper the angle of launch is the more time it takes to reach the maximum in the horizontal direction.

CONCLUSION

As representative of the minimum time curve cycloid has a well-known reputation in physics. This originated with the historic problem of Brachistochrone which solves the minimum time of descent far an object under uniform gravitational field. It goes also to the extreme of an expanding model of Friedmann's cosmological model. A fixed point on the rim of a rolling circle determines a cycloid. Different versions of a cycloid such as hypocycloid, epicycloid, prolate etc. were out of our scope in this study. Our idea is to change the initial conditions for the starting point of a cycloid. We consider a projectile motion corresponding to an ordinary angle of projection and initial speed. Minimum action of the projectile determines a parabola whereas. The tip angle and initial speed of a jet fighters, for instance, reaches the maximum altitude in a shortest time along a cycloid. We found the interesting result that reaching to the same maximum height a jet fighter must take off (with the same speed) at a smaller projection angle. This means that climbing directly against gravity takes more time, the message given by Figure 3.

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