FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 9 No. 3, March, 2025, pp 119 - 121
DOI: https://doi.org/10.33003/fjs-2025-0903-3346

NUMERICAL SOLUTION OF FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS USING THE TAU METHOD WITH AN ESTIMATION OF THE ERROR

***[1]Egbetade, Samuel Adeleye and [2]Abimbola Latifat Adebisi**

[1]Department of Mathematics & Statistics, The Polytechnic, Ibadan, Oyo State
[2]Department of Mathematics & Statistics, Abiola Ajimobi Technical University, Ibadan,Oyo State

*Corresponding authors' email: adeleyeegbetade@gmail.com Phone: +2348081894274

ABSTRACT

The Subject of Numerical methods is an important aspect of ordinary differential equations. It is useful in providing solutions to a wide variety of complex differential equations arising from engineering, physical and biological sciences, health and other allied disciplines which are difficult to tackle by exact methods. Numerical approximations of differential equations of one and higher order have been provided using Euler Method, Tau Method, Runge-Kutta Method, Adams-Bashforth Method, Milne Simpson predictor-corrector Method, Adams-Moulton linear multistep method and a host of others. In this paper, we discussed the application of tau method for solving first and second order initial value problems of ordinary differential equations. Numerical examples are given for the sake of illustration of the method. To validate the accuracy of the method, we compare the approximate solutions obtained with exact solutions. By estimating the error, it is observed that the error decreases as the order of Tau approximations increases. This showed the performance and computational efficiencies of tau method.

Keywords: Tau method, Numerical Solution, Differential Equation, Error, Accuracy

INTRODUCTION

Many differential equations from modern science and engineering fields cannot be solved directly by exact methods (Ramos,2019). Due to the inability of most exact methods to solve complex differential equations, much attention has been focused on numerical or approximate methods (Egbetade 2006). The necessity for producing solutions to many stiff (complex) differential equations has continued to give rise to a variety of numerical methods. Most of these methods are chosen from the point of view of speed, convenience and accuracy. (Okor and Nwachukwu, 2022).

Islam (2015) used Euler Method and Runge-Kutta Method of fourth order to provide solutions to initial value problems of ordinary differential equations. Numerical solutions of the two methods were compared and the results showed that Runge-Kutta technique gave more accurate results.

In a paper by Mamadu and Njoseh (2016), a tau-collocation approach for solving first and second order ordinary differential equations was presented. Numerical evidences of the method for linear initial value problem in first and second order differential equations showed that the method is adequate and effective.

Mustapha and Hamza (2017), employed a complex algebra system (CAS) with the aid of Maple software to derive a numerical skill for solving initial value problems using Adams-Bashforth Method. The results showed a remarkable efficiency in the derivation of higher order Adams-Bashforth Method.

Hossen et al. (2019) performed a computational assessment of solutions by modified Euler and Runge-Kutta Methods for initial value problems of ordinary differential equations. Authors concluded that both methods produced numerical solutions that are accurate when compared with the exact solutions of the problems considered.

Kedir and Geleta (2021) discussed Adams-Bashforth scheme using the fourth order Runge-Kutta and Euler method to find the solution of initial value problems. Different model examples were tested for numerical experimentation with the derived scheme. Results validate the applicability of the method and they concluded that Runge-Kutta Method is more stable and converges faster that Euler Method.

Tadema (2024) employed Milne Simpson predictor-corrector technique and fourth order Adams-Bashforth-Moulton predictor-corrector Method to solve second order initial value problems of ordinary differential equations. Numerical results showed that both methods are highly efficient and particularly suitable for approximate solutions of initial value problems.

Over the past decades, several authors have reported results of experiments on polynomial approximations of linear and non-linear differential equations using the Tau method (Lanczos, 1938; Ortiz, 1969; Freilich and Ortiz, 1982; Adeniyi and Edungbola, 2007; Egbetade et al., 2008).

Furthermore, the Tau method has been applied to give solution to differential algebraic equations (Haghani et al., 2011), neural delay differential systems (Hafshejani et al., 2011), fractional partial differential equations (Vanani and Aminataei, 2011) and many red life problems which are modeled by various differential equations (Hosseni et al., 1991).

The objective of this paper is to obtain numerical solution of 1st and 2nd order linear differential equations. Error estimation of our approximations for the considered numerical problems will be carried out to determine the accuracy of the tau method.

The Tau Method

Accurate approximate solution of ordinary differential equations can be obtained by the Tau method which was originally formulated by Lanczos (1938) for the sake of completeness, we review briefly the procedure of the method (Lanczos, 1938; Ortiz, 1969; Egbetade, 1991).

The method solves the class of mth order linear differential equation

$$P_m(x) = P_m(x)\frac{d^m y(x)}{dx^m} + \cdots + P_0(x)y(x) = Q(x), [0,1] \quad (1)$$

a polynomial solution of the form

$$y_n(x) = \sum_{i=0}^{n} a_i x^i \quad (2)$$

The coefficients $a_i, i = 0(1)n$ are to be determined in such a way that $y_n(x)$ is the exact solution of the perturbed equation

$$P_m(x)\frac{d^m y(x)}{dx^m} + \cdots + P_0(x)y(x) = Q(x) + H_n(x) \quad (3)$$

where

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{m+s-1}\mathrm{T}_{n-m+i+1}, [0, \infty] \quad (4)$$

is the perturbation term

The parameter, $\tau_i, i = 1(1)m + s$ are free tau parameters to be determined.

$$\mathrm{T}_r(x) = Cos[r\ cos^{-1}(2x - 1)] = \sum_{i=0}^{r} C_i^{(r)}\, x^r, \quad (5)$$

is the Chebychev polynomial valid in the interval [0,1].

The tau parameters in equation (4) and the coefficients $a_i, i = 0(1)n$ in equation (2) are obtained by equating corresponding coefficients of powers of x in equation (4) together with the associated conditions of equation (1). By substituting the tau parameters and the coefficients a_i into equation (2), we obtain the tau approximate solution $y_n(x)$ of equation (1)

The error of Tau method is calculated by error

$$= |y_n(x) - y(x)| \quad (6)$$

where $y(x)$ is the exact solution of equation (1). A tau program which directly implements the tau system and the error estimate was developed for the differential equations under consideration.

Numerical Examples

Example 1

$$2(1 + x)y'(x) + y(x) = 0,\ 0 \le x \le 1 \quad (7)$$

$$y(0) = 1 \quad (8)$$

For this problem, the exact solution is given by $y(x) = (1 + x)^{-1/2}$

Solving equation (7) by tau method, we seek an approximate polynomial solution

$$y_n(x) = \sum_{i=-0}^{n} a_i x^i \quad (9)$$

Now $y_n(x)$ satisfies the perturbed equation

$$2(1 + x)y'_n(x) + y_n(x) = \tau_i \mathrm{T}_n \quad (10)$$

From equation (9)

$$y'_n(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} \quad (11)$$

For the case n=3, we obtain

$$2(1 + x)(a_1 + 2a_2x + 3a_3x^2) + (a_0 + a_1x + a_2x^2 + a_3x^3) = \tau_1\mathrm{T}_3(x) \quad (12)$$

By substituting Chebyshev polynomial T3(x) into the right-hand side of equation (12), we have

$$(a_{11} + 2a_1)(3a_1 + 4a_2)x + (5a_2 + 6a_3)x^2 + 7a_3x^3 = -\tau_1 + (18\tau_1)x + (-48\tau_1)x^2 + (32\tau_1)x^3 \quad (13)$$

Equating corresponding coefficients of powers of x in equation (13), we have the following set of equations:

$$\left.\begin{aligned} a_0 &= 2a_1 = \tau_1 \\ 3a_1 + 4a_2 &= 18\tau_1 \\ 5a_2 + 6a_3 &= -48\tau_1 \\ 7a_2 &= 32\tau_1 \end{aligned}\right\} \quad (14)$$

The solution of these equations in terms of τ_1 gives

$$\left.\begin{aligned} a_3 &= \tfrac{32}{7}\tau_1 \\ a_2 &= \tfrac{-528}{35}\tau_1 \\ a_1 &= \tfrac{-494}{35}\tau_1 \\ a_1 &= \tfrac{-\tau_1 - 1}{2}\ (using\ the\ initial\ conditions\ a_0 = 1) \end{aligned}\right\} \quad (15)$$

By solving equation (15) we have $\tau_1 = \frac{35}{953}$

Then, we have the following results

$a_0 = 1$

$a_1 = -494D$

$a_2 = -528D$

$a_3 = 160D$

where $D = \frac{1}{953}$

Thus, from equation (9) we have

$$y_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$= 1 + D(-494x = 528x^2 + 160x^3)$$

Evaluating this solution $x = 0.3$ gives

$$y_3(0.3) = 0.8771651$$

Table 1: Tau and Exact solution to example 1

Value of x	Solution		Error
	Tau	**Exact**	
0.0	1.0000000	1.0000000	0.0
0.1	0.9187677	0.9534626	4.69×10^{3}
0.2	0.9129813	0.9128709	1.10×10^{4}
0.3	0.8771651	0.8770580	1.07×10^{-4}
0.4	0.8451868	0.8451542	3.26×10^{-5}
0.5	0.8165271	0.8164966	3.05×10^{-5}
0.6	0.7905849	0.7905694	1.55×10^{-5}
0.7	0.7669682	0.7669649	3.30×10^{-6}
0.8	0.74535581	0.7453559	2.20×10^{-6}
0.9	0.7254771	0.7254763	8.0×10^{-7}
1.0	0.7071074	0.7071068	6.0×10^{-8}

Example 2

$$y''(x) - 2y(x) = 0,\ 0 \le x \le 1 \quad (16)$$

$$y(0) = 1 \quad (17)$$

The exact solution is $y(x) = \frac{1}{2}(e^{\sqrt{2x}} + e^{-\sqrt{2x}})$

Tau and exact solutions in the interval $0 \le x \le 1$ are given in Table 2

Table 2: Tau and Analytic Solutions to Example 2

Value of x	Solution		Error
	Tau	Exact	
0.0	1.0000000	1.0000000	0.0
0.1	1.0332501	1.0100245	2.32×10^{-2}
0.2	1.0817584	1.0402673	4.15×10^{-2}
0.3	1.0945677	1.0913563	3.21×10^{-3}
0.4	1.1649541	1.1643095	6.51×10^{-4}
0.5	1.2608097	1.2605874	3.22×10^{-4}
0.6	1.3821278	1.3821175	1.03×10^{-5}
0.7	1.5313488	1.5313364	1.25×10^{-6}
0.8	1.71123299	1.7112322	7.7×10^{-6}
0.9	1.9254095	1.9254091	6.0×10^{-7}
1.0	2.17811574	2.1781570	4.0×10^{-7}

Discussion of Result

From the numerical problems solved, results were presented in Tables 1 and 2. In both tables, the error estimate of the tau method was shown. It could be seen that the values generated by tau method are very close to the exact solutions. These results are in agreement with numerical results of previous authors reported in the literature.

CONCLUSION

Approximate solutions of linear 1st and 2nd order differential equations has been presented using the tau method. The method was shown to be computationally efficient and accurate since the tau approximations are very close to exact solutions. The higher the order of tau approximant, the smaller the error of the approximation. This is a desirable advantage of the tau method.

REFERENCES

Adeniyi, R. B. and Edungbola, O. (2007). On the recursive formulation of the Tau method for a class of over determined 1st order differential equations. Abacus 34(2B), 249-261.

Egbetade, S. A.(1991). On the Recursive Formulation of the Tau Method. B.Sc Dissertation (unpublished). University of Ilorin, Ilorin.

Egbetade, S. A. (2006). A computational error estimates of the tau method for linear initial value problems. Ife Journal of science 8(2), 105-110

Egbetade, S. A., Ajila, R.A. Oladeji, F. A. and Odetunde, O. S. (2008). On the comparison of the Tau and Euler methods for the numerical solution to initial value problems. Int. J. Numer. Maths 2(1), 42-55.

Freilich, J. H. and Ortiz, E. L. (1982). Numerical solution of systems of ordinary differential equations with the Tau method; An error analysis. Mathematics of Computation 39(16), 467 - 479.

Hafshejani, J. S., Vanani, S. K. and Esmaily, J. (2011). Operational Tau Approximation for Neutral Delay differential systems. J. Appl. Sciences 11(14), 2585 - 2591

Haghani, F. K., Vanani, S. K. and Hafshejani, J.S. (2011). Tau approximate solutionnof Differential Algebraic Equations. World App.Sciences Journal 13(Special Issue of Applied Math), 18 - 26.

Hosseni, M. Abadi, A. and Ortiz, E. L. (1991). A Tau method based on non-uniform space time elements for the numerical simulation of solitons. Computers Math Applic 22 (9), 7 -19.

Hossen, M. Ahmed, Z., Kabir, R. and Hossen, Z.(2019). A computational investigation on numerical solution of initial value problems using modifier Euler and Runge-Kutta method. IOSR-JM 12, 2278-2298.

Islam M.A (2015). Accurate analysis of numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE). *IOSR Journal of Mathematics* 11(3), 18-23.

Kedir, A.K. and Geleta ,K. (2021). Adams-Bashforth-Moulton predictor -corrector method for solving first order initial value problems and it's error analysis. *Indian Journal of Advanced Mathematics(*IJAM) 10,12-21.

Lanczos, C. (1938). Trigonometric interpolation of empirical and analytical functions. J. Math.Phys 17,123-199.

Mamodu, J. E. and Njoseh, I. N. (2016). Tau-Collocation Approximation Approach for Solving 1st and 2nd Order Ordinary Differential Equations J.of App.Maths and Physics 4, 383-390

Mustapha, A. and Hamza, M. M (2017). Derivation for the numerical solution of ordinary differential equations via computer Algebra. Journal of Applied and Computational Mathematics 3, 1-16

Tadema, A.T. (2024). Numerical solution of second order differential equations. CAND Journal 3(4), 277-290

Ortiz , E. L. (1969). The Tau method SIAM J. Numer.Anal 6, 480-492.

Vanani, S. K., Aminataei, A. (2011). Tau approximate solution of fractional partial differential equations. Computers and Mathematics with Applications. 62,1075 - 1083.