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A FOUR-STEP BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULAE FOR THE SOLUTION OF GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

Ordinary Differential Equations (ODEs) play a crucial role in modeling various real-world phenomena across physics, engineering, and the applied sciences. Many of these equations, especially second-order ODEs, arise in fields such as mechanics, fluid dynamics, and electrical circuit analysis. Traditional numerical methods like single-step and multi-step techniques have been extensively explored for solving these equations. However, stiff and non-stiff problems often require more efficient and stable numerical schemes. Backward Differentiation Formulae (BDF) are implicit multi-step methods well known for their stability properties, making them suitable for solving stiff ODEs. Hybrid and block approaches have been introduced to enhance the accuracy, efficiency, and convergence of numerical methods. The block method enables the simultaneous solution of multiple points within a single step, improving computational efficiency, while the hybrid approach incorporates additional off-step points to increase accuracy. In this paper, the block hybrid Backward Differentiation formulae (BHBDF) for the step number k=4 was developed. For this purpose, power series was employed as the basis function for the development of schemes in a collocation and interpolation techniques at some selected grid and off- grid points which gave rise to continuous schemes and were further evaluated at those points to produce discrete schemes combined together to form block methods. Analysis of the basic properties of the discrete schemes investigated showed consistency, zero stability and convergence of the proposed block methods. Tested problems were solved to examine the efficiency and accuracy of the proposed method. The results showed that the proposed methods with relatively small errors performed favorably in comparison with the existing methods.

Keywords: Backward Differentiation Formula, Block, Collocation, Ordinary Differential Equation, Stiff

INTRODUCTION

An equation that establishes a connection between an unknown function and one or more of its derivatives is known as a differential equation. Stated differently, it refers to the connection that exists between one or more independent variables and a dependent variable (Dahlquist, 1956).

Milne (1953) suggested block methods for solving ordinary differential equations. Due to his shortcomings, which include poor performance, a low order of accuracy, and an error term, hybrid approaches were introduced. As demonstrated in Dahlquist, hybrid approaches were first developed to get over the zero-stability barrier that existed in block methods (Dahlquist, 1956).

Besides the ability to change step size, the other benefit of these methods is utilizing data off-grid points which contribute to the accuracy of the methods. This paper presents a four-step block backward differentiation formula for the numerical solution of stiff second-order differential equations. The basic properties of the method such as zero stability, order, consistency, and convergence were examined. Several numerical problems will be solved and comparison will be made with other methods to show the efficiency of the proposed. This paper considers an approximate method for the solution of stiff differential equation of second-order initial value problem of the form,

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \delta_0 \qquad (1)$$

Where f satisfies a Lipchitz condition as given in Henrici (1962)

In this paper we develop a continuous hybrid second derivative block backward differentiation formula based on interpolation and collocation for the solution of stiff ordinary differential equations with constants step size

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

Where α_j and β_j are continuous coefficients to be determined h is the step size, k is the step number, m is the order of the differential equation.

MATERIALS AND METHODS

Derivation of the Numerical Scheme

We seek an approximation of the form;

$$Y(x) = \sum_{j=0}^{i+c-1} \alpha_j x^j \qquad (3)$$

Where i is the interpolation points, c is the collocation points and α_j are unknown coefficients to be determined. Then, we take

$$Y(x) = y_{n+j}, j = 0,1,2, \dots, k-1 \qquad (4)$$

$$Y''(x_{n+k}) = f_{n+k} \qquad (5)$$

To derive 4SBHBDF, we take $t = 6, c = 1$ and $x \in [x_n, x_{n+4}]$. Therefore, (3) becomes;

$$Y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6$$
$$Y'(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + 5\alpha_5 x^4 + 6\alpha_6 x^5$$
$$Y''(x) = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + 20\alpha_5 x^3 + 30\alpha_6 x^4 \qquad (6)$$

Interpolating (4) at $x_{n+i}; i = 0, \frac{1}{2}, 1, \frac{3}{2} 2, 3$ and collocate (5) at $x_{n+i}; i = 4$. This results in a system of equations;

$$Y''(x_{n+4}) = 2\alpha_2 + 6\alpha_3 x_{n+4} + 12\alpha_4 x_{n+4}^2 + 20\alpha_5 x_{n+4}^3 + 30\alpha_6 x_{n+4}^4$$
$$Y(x_{n+i}) = \alpha_0 + \alpha_1 x_{n+i} + \alpha_2 x_{n+i}^2 + \alpha_3 x_{n+i}^3 + \alpha_4 x_{n+i}^4 + \alpha_5 x_{n+i}^5 + \alpha_6 x_{n+i}^6$$

$$\begin{aligned}
Y(x_n) &= \alpha_0 + \alpha_1 x_n + \alpha_2 x_n^2 + \alpha_3 x_n^3 + \alpha_4 x_n^4 + \alpha_5 x_n^5 + \alpha_6 x_n^6\\
Y\left(x_{n+\frac{1}{2}}\right) &= \alpha_0 + \alpha_1 x_{n+\frac{1}{2}} + \alpha_2 x_{n+\frac{1}{2}}^2 + \alpha_3 x_{n+\frac{1}{2}}^3 + \alpha_4 x_{n+\frac{1}{2}}^4 + \alpha_5 x_{n+\frac{1}{2}}^5 + \alpha_6 x_{n+\frac{1}{2}}^6\\
Y(x_{n+1}) &= \alpha_0 + \alpha_1 x_{n+1} + \alpha_2 x_{n+1}^2 + \alpha_3 x_{n+1}^3 + \alpha_4 x_{n+1}^4 + \alpha_5 x_{n+1}^5 + \alpha_6 x_{n+1}^6\\
Y\left(x_{n+\frac{3}{2}}\right) &= \alpha_0 + \alpha_1 x_{n+\frac{3}{2}} + \alpha_2 x_{n+\frac{3}{2}}^2 + \alpha_3 x_{n+\frac{3}{2}}^3 + \alpha_4 x_{n+\frac{3}{2}}^4 + \alpha_5 x_{n+\frac{3}{2}}^5 + \alpha_6 x_{n+\frac{3}{2}}^6\\
Y(x_{n+2}) &= \alpha_0 + \alpha_1 x_{n+2} + \alpha_2 x_{n+2}^2 + \alpha_3 x_{n+2}^3 + \alpha_4 x_{n+2}^4 + \alpha_5 x_{n+2}^5 + \alpha_6 x_{n+2}^6\\
Y(x_{n+3}) &= \alpha_0 + \alpha_1 x_{n+3} + \alpha_2 x_{n+3}^2 + \alpha_3 x_{n+3}^3 + \alpha_4 x_{n+3}^4 + \alpha_5 x_{n+3}^5 + \alpha_6 x_{n+3}^6
\end{aligned} \tag{7}$$

$D\psi = Y$

where

$\psi = \left(\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \alpha_2, \alpha_3, \beta_4\right)^T$, $Y = \left(y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+3}, f_{n+4}\right)^T$ and the matrix W of the proposed method is expressed as

$$W = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6\\
1 & \left(x_n+\frac{1}{2}h\right) & \left(x_n+\frac{1}{2}h\right)^2 & \left(x_n+\frac{1}{2}h\right)^3 & \left(x_n+\frac{1}{2}h\right)^4 & \left(x_n+\frac{1}{2}h\right)^5 & \left(x_n+\frac{1}{2}h\right)^6\\
1 & (x_n+h) & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 & (x_n+h)^6\\
1 & \left(x_n+\frac{3}{2}h\right) & \left(x_n+\frac{3}{2}h\right)^2 & \left(x_n+\frac{3}{2}h\right)^3 & \left(x_n+\frac{3}{2}h\right)^4 & \left(x_n+\frac{3}{2}h\right)^5 & \left(x_n+\frac{3}{2}h\right)^6\\
1 & (x_n+2h) & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 & (x_n+2h)^6\\
1 & (x_n+3h) & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 & (x_n+3h)^6\\
0 & 0 & 2 & 6(x_n+4h) & 12(x_n+4h)^2 & 20(x_n+4h)^3 & 30(x_n+4h)^4
\end{bmatrix} \tag{8}$$

Solving (8) using matrix inversion method with the aid of Maple 2017 software to obtain the following continuous coefficients;

$$\begin{aligned}
\alpha_0 = {}& \frac{1}{45378}\frac{1}{h^6}(45378h^6 + 213165h^5x_n + 368068h^4x_n^2 + 304235h^3x_n^3 + 127982h^2x_n^4 + 26020hx_n^5 + 1992x_n^6)\\
&- \frac{1}{45378}\frac{1}{6}((213165h^5 + 736136h^4x_n + 912705h^3x_n^2 + 511928h^2x_n^3 + 130100hx_n^4 + 11952x_n^5)x)\\
&+ \frac{1}{45378}\left(\frac{368068h^4 + 912705h^3x_n + 767892h^2x_n^2 + 260200hx_n^3 + 29880x_n^4}{h^6}\right)x^2\\
&+ \frac{1}{45378}\left(\frac{304235h^3 + 511928h^2x_n + 260200hx_n^2 + 39840x_n^3}{h^6}\right)x^3\\
&- \frac{1}{45378}\left(\frac{63991h^2 + 65050hx_n + 14940x_n^2}{h^6}\right)x^4\\
&- \frac{2}{22689}\left(\frac{6505h + 2988x_n}{h^6}\right)x^5\\
&+ \frac{332}{7563}\frac{x^6}{h^6}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\alpha_{\frac{1}{2}} = {}& \frac{16}{37815}\frac{x_n(25164h^5 + 67860h^4x_n + 68295h^3x_n^2 + 31970h^2x_n^3 + 6921hx_n^4 + 550x_n^5)}{h^6}\\
&+ \frac{16}{37815}\frac{1}{h^6}((25164h^5 + 135720h^4x_n + 204885h^3x_n^2 + 127880h^2x_n^3 + 34605hx_n^4 + 3300x_n^5)x)\\
&- \frac{16}{2521}\frac{(4524h^4 + 13659h^3x_n + 12788h^2x_n^2 + 4614hx_n^3 + 550x_n^4)}{h^6}x^2\\
&+ \frac{16}{7563}\frac{(13659h^3 + 25576h^2x_n + 13842hx_n^2 + 2200x_n^3)}{h^6}x^3\\
&- \frac{16}{7563}\frac{(6394h^2 + 6921hx_n + 1650x_n^2)}{h^6}x^4\\
&+ \frac{16}{12605}\frac{(2307h + 1100x_n)}{h^6}x^5\\
&- \frac{1760}{7563}\frac{x^6}{h^6}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\alpha_1 = {}& \frac{1}{5042}\frac{1}{h^6}(x_n(56412h^5 + 208476h^4x_n + 248597h^3x_n^2 + 128823h^2x_n^3 + 29700hx_n^4 + 2452x_n^5))\\
&- \frac{3}{5042}\frac{1}{6}((18804h^5 + 138984h^4x_n + 248597h^3x_n^2 + 171764h^2x_n^3 + 49500hx_n^4 + 4904x_n^5)x)\\
&+ \frac{3}{5042}\frac{(69492h^4 + 248597h^3x_n + 257646h^2x_n^2 + 99000hx_n^3 + 12260x_n^4)}{h^6}x^2
\end{aligned}$$

$$
-\frac{1}{5042}\frac{(248597h^3+515292h^2x_n+297000hx_n^2+49040x_n^3)}{h^6}x^3
+\frac{3}{5042}\frac{(42941h^2+49500hx_n+12260x_n^2)}{h^6}x^4
-\frac{6}{2521}\frac{(2475h+1226x_n)}{h^6}x^5
+\frac{1226}{2521}\frac{x^6}{h^6} \quad (11)
$$

$$
\alpha_{\frac{3}{2}} =
-\frac{16}{22689}\frac{x_n(10668h^5+42964h^4x_n+57719h^3x_n^2+32774h^2x_n^3+8041hx_n^4+690x_n^5)}{h^6}
+\frac{16}{22689}\frac{1}{h^6}((10668h^5+85928h^4x_n+173157h^3x_n^2+131096h^2x_n^3+40205hx_n^4+4140x_n^5)x)
-\frac{16}{22689}\frac{(42964h^4+173157h^3x_n+196644h^2x_n^2+80410hx_n^3+10350x_n^4)}{h^6}x^2
+\frac{16}{22689}\frac{(57719h^3+131096h^2x_n+80410hx_n^2+13800x_n^3)}{h^6}x^3
-\frac{16}{22689}\frac{(32774h^2+40205hx_n+10350x_n^2)}{h^6}x^4
+\frac{16}{22689}\frac{(8041h+4140x_n)}{h^6}x^5
-\frac{3680}{7563}\frac{x^6}{h^6} \quad (12)
$$

$$
\alpha_2 =
\frac{1}{15126}\frac{1}{h^6}(x_n(36801h^5+154260h^4x_n+220407h^3x_n^2+134984h^2x_n^3+35172hx_n^4+3136x_n^5))
-\frac{1}{15126}\frac{1}{6}((36801h^5+308520h^4x_n+661221h^3x_n^2+539936h^2x_n^3+175860hx_n^4+18816x_n^5)x)
+\frac{1}{5042}\frac{(51420h^4+220407h^3x_n+269968h^2x_n^2+117240hx_n^3+15680x_n^4)}{h^6}x^2
-\frac{1}{15126}\frac{(220407h^3+539936h^2x_n+351720hx_n^2+62720x_n^3)}{h^6}x^3
+\frac{2}{7563}\frac{(33746h^2+43965hx_n+11760x_n^2)}{h^6}x^4
-\frac{2}{2521}\frac{(2931h+1568x_n)}{h^6}x^5
+\frac{1568}{7563}\frac{x^6}{h^6} \quad (13)
$$

$$
\alpha_3 =
-\frac{1}{226890}\frac{1}{h^6}(x_n(33756h^5+146860h^4x_n+222785h^3x_n^2+148745h^2x_n^3+43204hx_n^4+4140x_n^5))
+\frac{1}{226890}\frac{1}{6}((33756h^5+293720h^4x_n+668355h^3x_n^2+594980h^2x_n^3+216020hx_n^4+24840x_n^5)x)
-\frac{1}{45378}\frac{(29372h^4+133671h^3x_n+178494h^2x_n^2+86408hx_n^3+12420x_n^4)}{h^6}x^2
+\frac{1}{45378}\frac{(44557h^3+118996h^2x_n+86408hx_n^2+16560x_n^3)}{h^6}x^3
-\frac{1}{45378}\frac{(29749h^2+43204hx_n+12420x_n^2)}{h^6}x^4
+\frac{2}{113445}\frac{(10801h+6210x_n)}{h^6}x^5
-\frac{46}{2521}\frac{x^6}{h^6} \quad (14)
$$

$$
\begin{aligned}
\beta_4 = \frac{1}{5042}\,&\frac{x_n\left(18h^5+81h^4x_n+130h^3x_n^2+95h^2x_n^3+32hx_n^4+4x_n^5\right)}{h^4}\\
&-\frac{1}{2521}\,\frac{\left(9h^5+81h^4x_n+195h^3x_n^2+190h^2x_n^3+80hx_n^4+12x_n^5\right)x}{h^4}\\
&+\frac{1}{5042}\,\frac{\left(81h^4+390h^3x_n+570h^2x_n^2+320hx_n^3+60x_n^4\right)x^2}{h^4}\\
&-\frac{5}{2521}\,\frac{\left(13h^3+38h^2x_n+32hx_n^2+8x_n^3\right)x^3}{h^4}+\frac{5}{5042}\,\frac{\left(19h^2+32hx_n+12x_n^2\right)x^4}{h^4}\\
&-\frac{4}{2521}\,\frac{\left(4h+3x_n\right)x^5}{h^4}+\frac{2}{2521}\,\frac{x^6}{h^4}
\end{aligned}
\tag{15}
$$

The values of the continuous coefficients are then substituted in to the proposed method in (3) to obtain

$$y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}}+\alpha_1(x)y_{n+y_{n+1}}+\alpha_{\frac{3}{2}}(x)y_{n+\frac{3}{2}}+\alpha_2(x)y_{n+2}+\alpha_3(x)y_{n+3}+h^2\beta_4(x)f_{n+4} \tag{16}$$

Expressing (17) further gives the continuous form of the 4SHBDF with 2-step off- step interpolation point as

$$
\begin{aligned}
y(x) &= f_n+4\left(\frac{1}{5042}\frac{1}{h^6}\left(x_n(18h^5+81h^4x_n+130h^3x_n^2+95h^2x_n^3+32hx_n^4+4x_n^5)\right)\right.\\
&-\frac{1}{2521}\left(\frac{1}{h^6}(9h^5+81h^4x_n+195h^3x_n+190h^2x_n^3+80hx_n^4+12x_n^5)x\right)\\
&+\frac{1}{5042}\left(\frac{1}{h^6}(81h^4+390h^3x_n+570h^2x_n^2+320hx_n^3+60x_n^4)x^2\right)\\
&+\frac{5}{2521}\left(\frac{1}{h^6}(13h^3+38h^2x_n+32hx_n^2+8x_n^3)x^3\right)\\
&+\frac{5}{5042}\left(\frac{1}{h^6}(19h^2+32hx_n+12x_n^2)x^4\right)\\
&-\frac{4}{2521}\left(\frac{1}{h^6}(4h+3x_n)x^5\right)+\frac{2}{2521}\left(\frac{x^6}{h^6}\right)\\
&+y_n\left(\frac{1}{45378}\left(\frac{1}{h^6}(45378h^6)\right)+213165h^5x_n+368068h^4x_n^2+304235h^3x_n^3+127982h^2x_n^4+26020hx_n^5+1992x_n^6\right)\\
&-\frac{1}{45378}\left(\frac{1}{h^6}(213165h^5+736136h^4x_n+912705h^3x_n^2+511928h^2x_n^3+130100hx_n^4+11952x_n^5)x\right)\\
&+\frac{1}{45378}\left(\frac{1}{h^6}(368068h^4+912705h^3x_n+767892h^2x_n^2+260200hx_n^3+29880x_n^4)x^2\right)\\
&+\frac{1}{45378}\left(\frac{1}{h^6}(304235h^3+511928h^2x_n+260200hx_n^2+39840x_n^3)x^3\right)\\
&+\frac{1}{22689}\left(\frac{1}{h^6}(63991h^2+65050hx_n+14940x_n^2)x^4\right)-\frac{2}{22689}\left(\frac{1}{h^6}(6505h+2988x_n)x^5\right)\\
&+y_{n+1}\left(\frac{1}{5042}\left(\frac{1}{h^6}(x_n(56412h^5+208476h^4x_n+248597h^3x_n^2+128823h^2x_n^3+29700hx_n^4+2452x_n^5))\right)\right)\\
&-\frac{3}{5042}\left(\frac{1}{h^6}((18804h^5+138984h^4x_n+248597h^3x_n^2+171764h^2x_n^3+49500hx_n^4+4904x_n^5)x)\right)\\
&+\frac{3}{5042}\left(\frac{1}{h^6}(69492h^4+248597h^3x_n+257646h^2x_n^2+99000hx_n^3+12260x_n^4)x^2\right)\\
&-\frac{1}{5042}\left(\frac{1}{h^6}(248597h^3+515292h^2x_n+297000hx_n^2+49040x_n^3)x^3\right)\\
&+\frac{3}{5042}\left(\frac{1}{h^6}(42941h^2+49500hx_n+12260x_n^2)x^4\right)-\frac{6}{2521}\left(\frac{1}{h^6}(2475h+1226x_n)x^5\right)
\end{aligned}
\tag{17}
$$

Evaluate (18) at $x=x_{n+4}$, gives the discrete scheme as

$$y_{n+4}=-\frac{18515}{7563}y_n+\frac{38144}{2521}y_{n+\frac{1}{2}}-\frac{95480}{2521}y_{n+1}-\frac{66710}{2521}y_{n+2}+\frac{41608}{7563}y_{n+3}+\frac{356608}{7563}y_{n+\frac{3}{2}}+\frac{420}{2521}h^2f_{n+2} \tag{18}$$

To obtain the sufficient schemes required, we obtain the first derivative of (18) and evaluate the continuous function at $x=x_n, x=x_{n+\frac{1}{2}}, x=x_{n+1}, x=x_{n+\frac{3}{2}}, x=x_{n+2}, x=x_{n+3}$ and $x=x_{n+4}$

$$hz_{n+4}=-\frac{175261}{45378}y_n+\frac{892864}{37815}y_{n+\frac{1}{2}}-\frac{146742}{2521}y_{n+1}-\frac{578929}{15126}y_{n+2}+\frac{669398}{113445}y_{n+3}+\frac{1607104}{22689}y_{n+\frac{3}{2}}+\frac{1163}{2521}h^2f_{n+4}$$

$$hz_{n+3}=-\frac{5235}{7563}y_n+\frac{55008}{12605}y_{n+\frac{1}{2}}-\frac{58275}{5042}y_{n+1}-\frac{26055}{2521}y_{n+2}+\frac{62759}{25210}y_{n+3}+\frac{118880}{7563}y_{n+\frac{3}{2}}+\frac{45}{2521}h^2f_{n+4}$$

$$hz_{n+2}=\frac{1525}{15126}y_n-\frac{27136}{37815}y_{n+\frac{1}{2}}+\frac{5724}{2521}y_{n+1}+\frac{43195}{15126}y_{n+2}+\frac{3556}{37815}y_{n+3}-\frac{34816}{7563}y_{n+\frac{3}{2}}-\frac{3}{2521}h^2f_{n+4}$$

$$hz_{n+1}=\frac{2023}{22689}y_n-\frac{35936}{37815}y_{n+\frac{1}{2}}-\frac{3747}{5054}y_{n+1}-\frac{3305}{7563}y_{n+2}+\frac{4591}{226890}y_{n+3}+\frac{45856}{22689}y_{n+\frac{3}{2}}-\frac{1}{2521}h^2f_{n+4}$$

$$hz_n=-\frac{23685}{5054}y_n-\frac{134208}{12605}y_{n+\frac{1}{2}}-\frac{28206}{2521}y_{n+1}-\frac{12267}{5054}y_{n+2}+\frac{5626}{37815}y_{n+3}+\frac{56896}{7563}y_{n+\frac{3}{2}}-\frac{9}{2521}h^2f_{n+4}$$

$$hz_{n+\frac{3}{2}}=-\frac{887}{15126}y_n+\frac{5913}{12605}y_{n+\frac{1}{2}}-\frac{39861}{20168}y_{n+1}+\frac{8739}{10084}y_{n+2}-\frac{8147}{302520}y_{n+3}+\frac{1831}{2521}y_{n+\frac{3}{2}}+\frac{9}{20168}h^2f_{n+4}$$

$$hz_{n+\frac{1}{2}} = -\frac{2840}{7563}y_n - \frac{86401}{37815}y_{n+\frac{1}{2}} + \frac{84825}{20168}y_{n+1} + \frac{18485}{30252}y_{n+2} - \frac{10217}{302520}y_{n+3} + \frac{56896}{7563}y_{n+\frac{3}{2}} + \frac{15}{20168}h^2 f_{n+4} \quad (19)$$

Where z is the first derivative of y.

$$\left.\begin{aligned}
y_{n+3} &= \frac{64139}{92191}y_n - \frac{400032}{92191}y_{n+\frac{1}{2}} - \frac{1010475}{92191}y_{n+1} + \frac{679401}{92191}y_{n+2} - \frac{1261792}{92191}y_{n+\frac{3}{2}} - \frac{81}{3179}h^2 f_{n+4} + \frac{22689}{92191}h^2 f_{n+3} \\
y_{n+\frac{1}{2}} &= \frac{10433}{14364}y_n - \frac{1145}{2128}y_{n+1} - \frac{473}{1064}y_{n+2} + \frac{1637}{57456}y_{n+3} + \frac{4409}{3591}y_{n+\frac{3}{2}} - \frac{31}{44688}h^2 f_{n+4} - \frac{2521}{11172}h^2 f_{n+\frac{1}{2}} \\
y_{n+1} &= -\frac{785}{24597}y_n + \frac{4384}{8199}y_{n+\frac{1}{2}} - \frac{193}{8199}y_{n+2} - \frac{23}{24597}y_{n+3} + \frac{12832}{24597}y_{n+\frac{3}{2}} + \frac{1}{24597}h^2 f_{n+4} - \frac{2521}{24597}h^2 f_{n+1}
\end{aligned}\right\} \quad (20)$$

$$y_{n+2} = -\frac{5353}{15471}y_n + \frac{1312}{573}y_{n+\frac{1}{2}} - \frac{8269}{15471}y_{n+3} + \frac{88736}{15471}y_{n+\frac{3}{2}} + \frac{19}{3438}h^2 f_{n+4} + \frac{2521}{3438}h^2 f_{n+2}$$

The equations (18) – (20) are the proposed 4-Step Block Hybrid Backward Differentiation Formulae (BHBDF) for solving second order ordinary differential equations.

Zero Stability

Since the root of the first characteristic polynomial satisfies $|r_j| \leq 1$,then $r = (0,0,0,0,0,0,1)$ or 1 therefore the newly develop block is zero-stable.

Consistency

The newly develop block scheme is consistent since $p \geq 1$

Convergence

According to Awari (2017). The new block scheme is convergent, since is both consistent and zero-stable

Numerical examples

Problem 1: see Hussaini (2021)

Linear System of Second Order Initial Value problem (IVP)

$$\frac{d^2y_1}{dx^2} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{d^2y_2}{dx^2} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$y_1(0) = 1, y_1'(0) = 2, y_2(0) = 1, y_2'(0) = 2$, h=0.01

Exact Solution: $y_1(x) = e^{2x}$, $y_2(x) = e^{2x}$

Problem 2: See Badmus and Yahay (2019)

Constant Coefficient Linear Type

$$y'' = \frac{6}{x^2}y' - \frac{4}{x}y$$

$y(1) = -1, y'(1) = 1, h = \frac{1}{320}$

Exact solution as: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$

RESULTS AND DISCUSSION

Table 1: Comparisons of Errors in the solutions of Problem 1

X	Hussani and Rahina (2021)	Absolute Error in (New Method)
0.0	0.000000E-00	0.000000E-00
0.1	3.782543E-11	1.584762E-14
0.2	1.496751E-10	3.538009E-14
0.3	3.704399E-10	5.409330E-14
0.4	7.379615E-10	7.645367E-14
0.5	1.312451E-09	1.218539E-13
0.6	2.160988E-09	1.698903E-13
0.7	3.382452E-09	2.166456E-13
0.8	5.094917E-09	2.660267E-13
0.9	7.464829E-09	3.349836E-13
1.0	1.066781E-08	4.057153E-13

Table 2: Comparisons of Errors in the solutions of Problem 2

X	Badmus and Yahaya (2019)	Absolute Error in (New Method)
1.000000	0.000000E-00	0.000000E-00
1.003125	2.902001E-15	1.000000E-17
1.006250	6.233869 E-15	2.000000E-17
1.009375	9.585251E-15	3.000000E-17
1.0125625	1.306081E-14	4.000000E-17
1.015625	1.526144E-14	5.000000E-17
1.018750	1.716720E-14	6.000000E-17
1.021875	1.943949E-14	7.000000E-17
1.025000	2.0173730E-14	8. 000000E-17
1.028125	2.414736-E14	9. 000000E-17
1.031259	2.55-703 E-14	1. 000000E-16

Discussion

The proposed method (*4SBHBDF)* in *Table 1* has the smallest error values, making it the most accurate and efficient method. The error values of the method are significantly smaller than the other method, indicating better convergence. The other method Hussaini and Rahina (2021), have larger error values indicating less accuracy and efficiency. Hence the table show that the new method outperforms the other in term of accuracy efficiency and convergence.

While *Table 2* show the error for each method at different *x* values. The newly developed method has the smallest error values making it the most accurate and efficient method. Therefore, it is concluded that the new method is the most accurate and fastest converging method.

CONCLUSION

In conclusion, the proposed block hybrid backward differentiation formula with step number *k*=4 has shown to be a reliable and efficient method for solving stiff ODEs. Its ability to produce accurate results with smaller errors makes it a promising approach for solving complex problems. The results of this study demonstrate the potential of BHBDF to be a valuable tool for numerical analysis, and its efficiency and accuracy make it a suitable alternative to existing methods.

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