FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 9 No. 4, April, 2025, pp 74 - 77
DOI: https://doi.org/10.33003/fjs-2025-0904-3309

SINGH'S PERSPECTIVE ON MULTISET THEORY - A CONSERVATIVE EXTENSION OF STANDARD SET THEORY

***Chinedu Peter and Balarabe Abdullahi**

Department of Mathematics, Federal University, Dutsin-Ma, Nigeria

*Corresponding authors' email: macpee3@yahoo.com

ABSTRACT

In this paper, we demonstrate that Singh's approach to defining membership—through its novel yet underutilized notation—allows for a seamless and unambiguous analysis of multisets alongside their set counterparts. This stands in contrast to the widely used cardinality-based approach prevalent in existing literature. The Singh's dressed epsilon approach ($\in_+, \in^k$ and $\in_+{}^k$) provides a membership-based way of handling multisets, in contrast to the cardinality-based approach that relies on counting functions ($C_{M(x)}$ or $M(x)$).

Keywords: Cardinality, Dressed epsilon, Set, Singh, Multiset

INTRODUCTION

In Lake (1976), functions are taken as primitive, though Lake acknowledges that an axiomatization independent of functions might be desirable. Such an axiomatization could be conveniently expressed using $x \in_z y$, which represents "x belongs to y precisely z times." This approach was later adopted by Blizard (1989), the most recent and comprehensive work in this direction. Unlike many other formalizations of non-Cantorian set theory, Blizard (1989) develops a formal theory of multisets as a conservative extension of standard set theory. The symbol $\in_+$ was first introduced by Singh and Singh (2003). It was also used in Peter and Singh (2011) and Peter et al. (2024) or the establishment of some concepts in graph theory using multiset approach. See Singh (2006) for details.

For any element x occurring in a multiset A, meaning $m_A(x) > 0$, we write $x \in_+ A$. This symbol, known as the dressed epsilon, is a binary predicate representing "belongs to at least once," in contrast to $\in$, which denotes "belongs to only once" in the context of a set. Consequently, $m_A(x) = 0$ implies $x \notin A$, while $x \in_+{}^k A$ signifies that x belongs to A at least k times. In contrast, $x \in^k A$ explicitly states that x belongs to A exactly k times. A multiset over any ground set D is called empty, denoted by $\emptyset$ or [ ], if $m_\emptyset(x) = 0$ for all $x \in D$.

Singh himself emphasized that the introduction of $\in_+$ greatly enhances the language of multiset, citing an example: $A \subseteq B$ stands for $\forall z \forall k(z \in^k A \rightarrow z \in_+{}^k B$. The dressed epsilon notation allows us to focus on individual elements and their occurrences rather than treating multisets as functions. It aligns with how we naturally think about membership—whether an element appears in a multiset and how many times—rather than computing its frequency separately.

Proofs using $\in_+, \in^k$ and $\in_+{}^k$ tend to be more direct and symbolic, avoiding the extra layer of defining and tracking cardinality functions. This makes proofs cleaner and easier to follow, especially when dealing with set operations like unions, intersections, and inverse images.

For instance, instead of writing as in the following Equation 1,

$$C_{f^{-1}(N)}(x) = C_N\big(f(x)\big). \quad (1)$$

we simply state:

$$x \in^n f^{-1}(N) \text{ if and only if } f(x) \in^n N. \quad (2)$$

Singh's dressed epsilon naturally handles nonexistent elements. The dressed epsilon notation in Equation 2 can express uncertainty or partial membership more naturally. Using $\in_+$ we can say that an element appears at least once in a multiset, avoiding explicit cases for elements with zero occurrences. Instead of checking

$$C_M(x) > 0. \quad (3)$$

We simply write

$$x \in_+ M. \quad (4)$$

Many set-theoretic results generalize smoothly to multisets using $\in^n$ notation, making it easier to extend classical set results. When dealing with function compositions, images, and inverse images, the approach translates naturally, whereas cardinality-based proofs often require breaking things into case-by-case counting arguments. Using the case of inverse image of an intersection for instance:

$$x \in^n f^{-1}[\cap_{i \in I} M_i] \text{ if and only if } f(x) \in^n \cap_{i \in I} M_i. \quad (5)$$

The statement in Equation 5 is direct and avoids needing to track explicit summations over count functions.

The approach aligns with classical Set Theory notation. That is, it is more in line with standard set membership logic than the function-based cardinality approach.

This makes it easier to integrate multisets into general set-theoretic reasoning without switching between different notations.

MATERIALS AND METHODS

This study builds on standard set theory and explores Singh's dressed epsilon notations as a conservative extension to multiset theory. Unlike traditional approaches that rely on explicit cardinality functions, this framework allows direct membership-based expressions, providing an alternative yet structurally consistent foundation for multisets within standard set theoretic principles.

Se systematically reinterpret fundamental properties of multisets using the dressed epsilon framework. The translation from cardinality-based expressions to membership-based formulations is rigorously examined to ensure logical equivalence and consistency with classical multiset operations.

Key theorems in multiset theory are revisited and re-proven within the new framework. Previously established results from the cardinality approach are reconstructed using Singh's notation, ensuring that the conservative extension preserves the essential axioms and properties of standard set theory while offering an alternative perspective.

Definition of Terms

Definition 1 (Multiset) Let D be a set. A multiset M over D is a collection of elements of D with repetitions allowed. The set D is called the ground or generic set of the class of all multisets containing elements from D. Different representations of multiset exist.

Definition 2 (Submultiset) Let A and B be two multisets. A is a submultiset of B, written $A \subseteq B$ if $\forall z \forall k(z \in^k A \rightarrow z \in_+{}^k B)$.

Definition 3 (Union) Let M and N be two multisets over a ground set D. $A \cup B$ is the multiset defined by $x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = \max(m, n)$

Definition 4 (Intersection) Let M and N be two multisets over a ground set D. $A \cup B$ is the multisetdefined by $x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = \min(m, n)$

Definition 5 (Sum or additive union) Let M and N be two multisets over a ground set D. $A \cup B$ is the multisetdefined by $x \in^k (M \cup N) \Leftrightarrow x \in^m M$ and $x \in^n N$ with $k = m + n$.

See Singh et al. (2006) and Singh et al. (2008)

Definition 6 (Hierarchical decomposition of multisets) Let M be a mutiset over a set X, then the set $M_r = \{x \in X : x \in_+{}^r M\}$ is called r-level reference of M where r is the position of the reference set when all the reference sets (the empty set inclusive) are arranged in a descending order using the non-proper containment relation $\supseteq$. In this case, the set M_r for each r is known as an r-*reference set*.

Definition 7 (Hierarchical decomposition of multisets) Let M be a mutiset over a set X, then the set $M_r = \{x \in X : x \in_+{}^r M\}$ is called r-level reference of M where r is the position of the reference set when all the reference sets (the empty set inclusive) are arranged in a descending order using the non-proper containment relation $\supseteq$. In this case, the set M_r for each r is known as an r-*reference set*.

Example 8 Let $M = \{a, a, a, a, b, b, c, c, c\}$, then the level r reference sets are:

$M_1 = \{a, b, c\}, M_2 = \{a, b, c\}, M_3 = \{a, c\}, M_4 = \{a\}, M_5 = \emptyset$ where $r \leq 5$. Thus, $\{a, b, c\}$ is both a level 1 and a level 2 reference set.

Definition 9 (**$\boldsymbol{n}$-regular multiset**)

A multiset M over a based set X is called n-regular multiset of a set $S \subseteq X$ if $x \in^n M$ for all $x \in X$. In other words, a multiset whose objects have the same multiplicity n is called an n-regular multiset of the root set of the multiset.

Definition 10 Let X and Y be two nonempty sets and let $f: X \rightarrow Y$ be a mapping. The image of a multiset M under f, is denoted by $f(M)$ or $f[M]$, is defined by:

$y \in^n f(M)$ if and only if $\exists x \in_+ M$ such that $f(x) = y$ and $n = \bigvee_{f(x)=y} (x \in^n M)$. (6)

In Equation 6 above, it means that the multiplicity of y in $f(M)$ is determined by taking the least upper bound (join operation, $\vee$) over the multiplicities of all x in M that are mapped to y by f.

Definition 11 Let X and Y be two nonempty sets and $f: X \rightarrow Y$ be a mapping. The inverse image of a multiset N under f, is denoted by $f^{-1}(N)$, is defined by:

$x \in^n f^{-1}(N)$ if and only if $f(x) \in^n N$ (7)

Equation 7 states that an element x appears exactly n times in $f^{-1}(N)$ if and only if $f(x)$ appears exactly n times in N.

RESULTS AND DISCUSSION

As opposed to the method used in Nazmul et al. (2013), we have been able to employ Singh's dressed epsilon notation to establish various results in multiset theory. Our use of dressed epsilon notation offers a more direct and expressive way to describe multiset membership and repetition. This alternative framework not only reproduces known results but also provides a clearer and more intuitive foundation for further generalizations and new findings in multiset analysis.

Proposition 1. *A subsequent reference set is contained or is equal to its preceding reference set.*

Proof.

Let $x \in M_r$. Then $x \in_+{}^r M$. In particular $x \in^{r-1} M$. Therefore, $x \in M_{r-1}$. □

Proposition 2. *Let M and N be multisets over a base set X such that $M \subseteq N$, then $M_r \subseteq N_r$ where M_r and N_r are the r^{th} reference sets of M and N, respectively.*

Proof.

Suppose $M \subseteq N$ and let $x \in M_r$. We want to show that $x \in N_r$. Since $M_r \subseteq M$ then $x \in M$, which implies $x \in N$ by hypothesis. Moreover, $x \in^m M$ and $x \in^n N \Longrightarrow m \leq n$. Hence $x \in N_t$ for some $t \geq r$. Thus, $x \in N_r$. Therefore, $M_r \subseteq N_r$. □

Proposition 3. *Let M and N be multisets over a base set X. Then $(M \cap N)_r = M_r \cap N_r$ where M_r and N_r are the r^{th} reference sets of M and N, respectively.*

Proof.

Let $x \in (M \cap N)_r$. Then x belongs to the r^{th} reference set of $M \cap N$. Thus, x belongs to the r^{th} reference set of M and x belongs to the r^{th} reference set of N. That is, $x \in M_r$ and $x \in N_r$. Which means $x \in M_r \cap N_r$. Hence, $(M \cap N)_r \subset M_r \cap N_r$.

Conversely, Let $x \in M_r \cap N_r$. Then $x \in M_r$ and $x \in N_r$. That is x belongs to the r^{th} reference set of M and x belongs to the r^{th} reference set of N. Thus, x belongs to the r^{th} reference set of $M \cap N$. Thus, $x \in (M \cap N)_r$. Hence, $M_r \cap N_r \subset (M \cap N)_r$.

It follows from the two assertions that $(M \cap N)_r = M_r \cap N_r$. □

Proposition 4. *Let M and N be multisets over a base set X, then $(M \cup N)_r = M_r \cup N_r$ where M_r and N_r are the r^{th} reference sets of M and N, respectively.*

Proof.

Let $x \in (M \cup N)_r$. Then x belongs to the r^{th} reference set of $M \cup N$. Thus, x belongs to the r^{th} reference set of M or x belongs to the r^{th} reference set of N. That is, $x \in M_r$ or $x \in N_r$. Which means $x \in M_r \cup N_r$. Hence, $(M \cup N)_r \subset M_r \cup N_r$.

Conversely, Let $x \in M_r \cup N_r$. Then $x \in M_r$ or $x \in N_r$. That is x belongs to the r^{th} reference set of M or x belongs to the r^{th} reference set of N. Thus, x belongs to the r^{th} reference set of $M \cup N$. Thus, $x \in (M \cup N)_r$. Hence, $M_r \cup N_r \subset (M \cup N)_r$.

It follows from the two assertions that $(M \cup N)_r = M_r \cup N_r$. □

Corollary 5. (This is a corollary from the pair-wise equality theorem for ordered multisets. See Peter and Singh (2013)). *Let M and N be multisets over a base set X, then $M = N$ if and only if $M_r = N_r$, where M_r and N_r are the r^{th} reference sets of M and N, respectively.*

Proof.

Let M and N be multisets over a domain set X and suppose $M_i = N_j$ for all $i = j$. Since M_i are the references of M, an element x of M belongs to M_i for some i. Since only unidentical elements belong to any set-based reference of a multiset, only one occurrence of x belongs to a reference containing x of M. Thus, $M(x) = \alpha_{M(x)}$ where $\alpha_{M(x)}$ denotes the number of references of M containing x. By hypothesis, N_j contains x for some j and only one occurrence of x belongs to N_j. Thus, $M(x) = N(x) \forall x \in X$. Therefore, $M = N$.

The Converse follows from the Pairwise Equality Theorem in Peter and Singh (2013).

Proposition 6. (First reference theorem) *If M_r, $r \in \mathbb{N}$ are the reference sets of a multiset M over a base set S, then $C_M(x) = \sum_{n\in\mathbb{N}} \chi_{M_r}(x)$, where χ_{M_r} is the characteristric function of M_r.*

Proof.

By the definition of reference sets of a multiset, the number r of references M_r containing x is the same as the cardinality of x in M.

That is, $x \in^r M$ holds.

Moreover, $\chi_{M_r}(x) = 1$ and $r(\chi_{M_r}(x)) = r = \sum_{r\in\mathbb{N}} \chi_{M_r}(x)$

Hence, $C_M(x) = \sum_{n\in\mathbb{N}} \chi_{M_r}(x)$

Proposition 7. (Second reference theorem) *A multiset is the union of all*

Let N_r be the r-regular multisets of the reference sets M_r, $r \in \mathbb{N}$ of a multiset M over a base set S, then $M = \bigcup_{r\in\mathbb{N}} N_r$.

Proof.

Let $x \in_+ M$. By definition of N_r, $x \in N_r$ for some $r \in \mathbb{N}$. In particular, $x \in N_r$ implies $x \in^r M_r$. This implies $x \in_+ \bigcup_{r\in\mathbb{N}} M_r$

Conversely, Let $x \in_+ \bigcup_{r\in\mathbb{N}} M_r$. Then $x \in^r M_r$ for some $r \in \mathbb{N}$. Thus, $x \in N_r$ implies $x \in N_{r-1}$. Thus, This implies that $x \in N_r$ for some $r \in \mathbb{N}$. Since $M = \bigcup_{r\in\mathbb{N}} A_r$. $x \in_+ M$.

Proposition 8 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then $M_1 \subseteq M_2$ implies $f(M_1) \subseteq f(M_2)$.*

Proof

We need to show that if

$x \in^n M_1 \Longrightarrow x \in^m M_2$ for some $m \geq n$, (8)

then

$y \in^n f(M_2)$ for some $m \geq n$. (9)

By definition, $M_1 \subseteq M_2$ means

$\forall x \in_+ M_1, x \in_+ M_2$ and $x \in^n M_1 \Longrightarrow$
$x \in^m M_2$ for some $m \geq n$ (10)

The image of M under f is given by:

$y \in^n f(M)$ if and only if $\exists x \in^n M$ such that $f(x) = y$ (11)

Suppose $y \in^n f(M_1)$. By the definition of multiset image, this means $\exists x \in^n M_1$ such that $f(x) = y$.

Since $M_1 \subseteq M_2$, we know that: $x \in^m M_2$ for some $m \geq n$.

Thus, $y \in^m f(MM_2)$ for some $m > n$.

Thus, $y \in_+ f(M_1) \Longrightarrow y \in_+ f(M_2)$.

Proposition 9. *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then $f[\bigcup_{i\in I} M_i] \subseteq \bigcup_{i\in I} f[M_i]$.*

Proof

We need to show that for any $y \in Y$

$y \in_+ f[\bigcup_{i\in I} M_i] \Longrightarrow y \in_+ \bigcup_{i\in I} f(M_i)$ (12)

Suppose $y \in_+ f[\bigcup_{i\in I} M_i]$. By the definition of image, $\exists x \in_+ \bigcup_{i\in I} M_i$ such that $f(x) = y$.

By the definition of union, this means

$\exists i \in I, x \in_+ M_i$.

Applying the definition of image to each M_i:

$y \in_+ f(M_i)$ for some $i \in I$. (13)

Thus,

$y \in_+ \bigcup_{i\in I} f(M_i)$ (14)

This proves

$[\bigcup_{i\in I} M_i] \subseteq \bigcup_{i\in I} f(M_i)$ (15)

Proposition 10 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then. $M_1 \subseteq M_2$ implies $f^{-1}[M_1] \subseteq f^{-1}[M_2]$*

Proof

We need to show that if $y \in^n M_1 \Longrightarrow y \in^m M_2$, then $x \in^n f^{-1}[M_1] \Longrightarrow x \in^n f^{-1}[M_2]$ for some $m \geq n$.

By definition $M_1 \subseteq M_2$ means $\forall y \in_+ M_1, y \in_+ M_2$ and $y \in^n M_1 \Longrightarrow y \in^m M_2$ for some $m \geq n$.

Using Singh's notation, the inverse image under f is defined as: $x \in^n f^{-1}[M]$ if and only if $f(x) \in^n M$.

Suppose $M_1 \subseteq M_2$, we have:

$f(x) \in^m M_2$ for some $m \geq n$. (16)

Thus, $x \in_+ f^{-1}[M_1] \Longrightarrow x \in_+ f^{-1}[M_2]$. Hence the proof.

Proposition 11 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y, then $f^{-1}[\bigcup_{i\in I} M_i] = \bigcup_{i\in I} f^{-1}[M_i]$.*

Proof

We need to show that for every $x \in_+ X$:

$x \in_+ f^{-1}[\bigcup_{i\in I} M_i]$ if and only if $x \in_+ \bigcup_{i\in I} f^{-1}[M_i]$ (17)

By definition, the inverse image under f is:

$x \in_+ f^{-1}(M)$ if and only if $f(x) \in_+ M$ (18)

An element belongs to the union of multisets if and only if it belongs to at least one of the multisets in the family. Using Singh's notation:

$y \in_+ \bigcup_{i\in I} M_i$ if and only if $\exists i \in I, y \in_+ M_i$ (19)

Suppose $x \in_+ f^{-1}[\bigcup_{i\in I} M_i]$. By definition

$f(x) \in_+ \bigcup_{i\in I} M_i$ (20)

By the definition of union:

$\exists i \in I, f(x) \in_+ M_i$ (21)

By the inverse image definition

$x \in_+ f^{-1}(M_i)$ (22)

Thus,

$x \in_+ \bigcup_{i\in I} f^{-1}[M_i]$ (23)

Conversely, suppose $x \in_+ \bigcup_{i\in I} f^{-1}[M_i]$ (24)

By the definition of union:

$\exists i \in I, x \in_+ f^{-1}(M_i)$ (25)

By the definition of inverse image again:

$x \in_+ f^{-1}[\bigcup_{i\in I} M_i]$ (26)

Hence the proof.

Proposition 12 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then $f^{-1}[\bigcap_{i\in I} M_i] = \bigcap_{M_i} f^{-1}[M_i]$.*

Proof

We need to show that for any element $x \in X$, $x \in_+ f^{-1}[\bigcap_{i\in I} M_i]$ if and only if $x \in_+ \bigcap_{M_i} f^{-1}[M_i]$.

We Note that $x \in_+ f^{-1}[M_i]$ if and only if $f(x) \in_+ M$. Moreover, $y \in_+ \bigcap_{i\in I} M_i$ if and only if $\forall i \in I, y \in_+ M_i$. Now suppose $x \in f^{-1}[\bigcap_{i\in I} M_i]$. By definition of the inverse image: $f(x) \in_+ \bigcap_{i\in I} M_i$.

By the definition of intersection:

$\forall i \in I, f(x) \in_+ M_i$ (27)

Appplying the inverse image to each M_i:

$\forall i \in I, x \in_+ f^{-1}(M_i)$ (28)

Thus,

$x \in_+ \bigcap_{M_i} f^{-1}[M_i]$ (29)

Conversely, suppose $x \in_+ f^{-1}[\bigcap_{i\in I} M_i]$. This means:

$\forall i \in I, x \in_+ f^{-1}[M_i]$ (30)

By the definition of inverse image:

$\forall i \in I, f(x) \in_+ M_i$ (31)

By the definition of intersection:

$f(x) \in_+ \bigcap_{i\in I} M_i$ (32)

Applying the definition of inverse image again:

$x \in_+ f^{-1}[\bigcap_{i\in I} M_i]$ (33)

Having shown both directions, the result follows.

Proposition 13 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then $f(M) \subseteq N$ implies $M \subseteq f^{-1}(N)$.*

Proof

The assumption $f(M) \subseteq N$ means $\forall y \in Y, y \in^n f(M) \Longrightarrow y \in^m N$ with $n \leq m$ where $m, n \in \mathbb{N}$. That is, every $\boldsymbol{y}$ in $f(M)$ appears in $\boldsymbol{N}$ at least as many times. Also, $M \subseteq f^{-1}(N)$ means the inverse image of N consists of all $x \in X$ such that $f(x) \in_+ N$. This means $f(x)$ appears at least once in N. Since x contributes to $f(x)$ in $f(M)$, and $f(M) \subseteq N$, it follows that

x must appear in $f^{-1}(N)$ as many times as in M. Thus, we have $x \in^n M \Longrightarrow x \in^n f^{-1}(N)$.

Proposition 14 *Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping from X to Y. Then $g(f((M)) = (gf)(M)$.*

Proof

Suppose $z \in^n g(f(M))$. By definition,

$$\bigvee_{g(y)=z} y \in^n f(M). \tag{34}$$

Expanding $y \in^n f(M)$ using the definition of image:

$$\bigvee_{g(y)=z} \bigvee_{f(x)=y} x \in^n M. \tag{35}$$

Rearranging the logical structure:

$$\bigvee_{g(f(x))=z} x \in^n M. \tag{36}$$

Since $g(f(x)) = (g \circ f)(x)$, we obtain:

$$z \in^n (g \circ f)(M). \tag{37}$$

Conversely, Suppose $z \in^n (g \circ f)(M)$. By the definition,

$$\bigvee_{(g \circ f)(x)=z} x \in^n M. \tag{38}$$

Expanding $(g \circ f)(x)$ as $g(f(x))$, we get:

$$\bigvee_{g(f(x))=z} x \in^n M. \tag{39}$$

Rewriting this using the definition of image,

$$\bigvee_{g(y)=z} \bigvee_{f(x)=y} x \in^n M. \tag{40}$$

Since the left hand side of this matches the definition of $g(f(M))$, we get

$$z \in^n g(f(M)). \tag{41}$$

Proposition 4.15 *Let X and Y be nonempty sets and let $f: X \to Y$ and $g: Y \to Z$ be mappings. Then for any multiset N_j in Z,*

$$f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j). \tag{42}$$

Proof.

We need to show that for all $x \in X$,

$$x \in^n f^{-1}[g^{-1}(N_j)] \text{ if and only if } x \in^n [gf]^{-1}(N_j) \tag{43}$$

By the definition of inverse mapping,

$$y \in^m g^{-1}(N_j) \text{ if and only if } g(y) \in^m N_j \tag{44}$$

Applying f^{-1} to both sides,

$x \in^n f^{-1}[g^{-1}(N_j)]$ if and only if $f(x) \in^n g^{-1}(N_j)$

Expanding $g^{-1}(N_j)$ in terms of N_j:

Since $f(x) \in^n g^{-1}(N_j)$, by the definition of inverse image, this implies

$g(f(x)) \in^n N_j$

This is the definition of $(gf)^{-1}(N_j)$.

This means $x \in^n [gf]^{-1}(N_j)$.

CONCLUSION

The Singh's dressed epsilon approach provides a more intuitive, concise, and natural way to reason about multisets. It avoids unnecessary complications with counting functions, allowing proofs to stay focused on membership properties rather than arithmetic. This makes it particularly useful when working with set operations, functions, and inverse images in multiset theory.

REFERENCES

Blizard, W. D. (1989). Multiset theory. *Notre Dame Journal of formal logic*, *30*(1), 36-66. https://doi.org/10.1305/ndjfl/1093634995

Lake, J. (1976). Sets, fuzzy sets, multisets and functions. *Journal of the London Mathematical Society*, *2*(3), 323-326. https://doi.org/10.1112/jlms/s2-12.3.323

Nazmul, S., Majumdar, P., & Samanta, S. K. (2013) On multisets and multigroups. *Ann. Fuzzy Math. Inform*, *6*(3), 643-656.

Peter, C., Balogun, F., and Adeyemi, O. A. (2024) An Exploration of Antimultigroup Extensions, FUDMA Journal of Sciences, *Vol. 8 No. 5, pp 269 - 273 Authors: Chinedu Peter | Funmilola Balogun | Omotosho Adewumi Adeyemi* https://doi.org/10.33003/fjs-2024-0805-2719

Peter, C., & Singh, D. (2013). Grid ramification of set-based multiset ordering. *Asian Journal of Fuzzy and Applied Mathematics*, *1*(3).

Singh, D., & Singh, J. N. (2003). Some combinatorics of multisets. *International Journal of Mathematical Education in Science and Technology*, *34*(4), 489–499. https://doi.org/10.1080/0020739031000078721.

Singh, D. (2006). *Multiset Theory: A New Paradigm of Science: an Inaugural Lecture*. University Organized Lectures Committee, Ahmadu Bello University.

Singh, D., Ibrahim, A., Yohanna, T., and Singh, J. (2007). An overview of the applications of multisets. Novi Sad Journal of Mathematics, 37(3):73- 92.

Singh, D., Ibrahim, A. M., Yohanna, T., & Singhy, J. N. (2008). A systematization of fundamentals of multisets. *Lecturas Matematicas*, *29*(1), 33-48