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MOTIONS AROUND THE OUT-OF-PLANE EQUILIBRIUM POINTS FOR BINARY LALANDE 21258, BD+195116, ROSS 614, 70 OPHIUCHI AND 61 CYGNI SYSTEMS

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ABSTRACT

This study explores the orbital behaviour surrounding out-of-plane equilibrium points (OEPs) within the circular restricted three-body problem (CR3BP) framework, with a particular emphasis on binary star systems where the primary stars are represented as oblate and radiating entities. The research centres on the stability (Lyapunov-wise) of two pairs of OEPs, $L_{6,7}(x_0, 0, \pm z_0)$and$L_{8,9}(x_0, 0, \pm z_0)$, respectively, which are influenced by the oblateness and radiation pressure coefficients of the primary stars. By applying the theoretical framework to five specific binary systems—Lalande 21258, BD+195116, Ross 614, 70 Ophiuchi, and 61 Cygni—we assess the stability properties of these equilibrium points. Our findings indicate that the OEPs exhibit instability across all five systems, as evidenced by the positive real parts of the complex roots linked to their perturbations. This instability implies that any perturbations will amplify over time, resulting in significant deviations from the equilibrium states. The implications of this research are significant for the design of satellite constellations and the planning of space missions, as a thorough understanding of the stability of these equilibrium points is essential for successful mission execution and orbital insertion strategies. This work contributes to the wider domain of celestial mechanics by deepening our comprehension of dynamical behaviours in intricate binary systems.

Keywords: Stability, Binary systems, OEPs, CRTBP

INTRODUCTION

The examination of equilibrium points in orbital systems is a fundamental aspect of celestial mechanics, essential for comprehending the stability and dynamics of celestial bodies. Within the frameworks of the classical circular restricted three-body problem (CRTBP) (see Szebehely (1967a), Szebehely (1967b)), and the elliptical restricted three-body problem (ERTBP), the characteristics of equilibrium points situated in the orbital plane have been extensively studied. Notably, the collinear equilibrium points denoted as $L_i, i = 1,2,3$, where gravitational forces are aligned, are recognized as unstable, while the triangular equilibrium points denoted by$L_{4,5}$ generally exhibit stability for the mass ratio $\mu, 0 < \mu \leq \frac{1}{2}$. Numerous adaptations of these models have been investigated, incorporating factors such as perturbations from Coriolis and centrifugal forces, the non-sphericity of the primary bodies (including oblateness and triaxiality), as well as additional influences like radiation pressure and drag forces (Szebehely, 1967; Chernikov, 1970; Bhatnagar and Hallan, 1978; Schuerman, 1980; Kunitsyn and Tureshbaev, 1985; Elipe, 1992; Ragos et al., 1995; Singh and Ishwar, 1999; Singh and Umar, 2012; Sharma et al., 2001; Papadakis, 2005; AbdulRaheem and Singh, 2006; Singh and Begha, 2011; Singh and Taura, 2013; Jain and Aggarwal, 2015; Idrisi and Jain, 2016); Singh and Taura (2012).

In particular, when analyzing the plane that is perpendicular to the motion of the primary body, out-of-plane equilibrium points (OEPs) can be identified. The emergence of these points may be attributed to photogravitational effects or the significant oblateness of the primary bodies. The existence of OEPs was questioned by Todoran (1993), who challenged earlier assertions made by Radzevskii (1950) and other scholars. Nevertheless, Ragos and Zagouras (1993) refuted Todoran's claims, affirming the existence of these equilibrium points. Roman (2001) further explored OEPs in relation to radiation pressure within the context of the photogravitational restricted three-body problem (RTBP), identifying a pair of OEPs $L_6(-0.055; 0; +1.07)$ and$L_7(-0.055; 0; -1.07)$ in the binary RW-Monocerotis system.

Douskos and Markellos (2006) discovered the presence of OEPs when analyzing scenarios involving one or both primaries emitting radiation, as well as situations where one primary is oblate and the other radiates. Their findings provided numerical proof that the OEPs are not stable. However, Wu et al. (2018) argued that the explanation given by Douskos and Markellos (2006) did not align with the common physical perspective, which suggests that gravitational force is the sole force acting in the z-direction.

Singh and Umar (2012) investigated the elliptic restricted three-body problem (ERTBP) featuring an oblate primary and a luminous secondary, concluding that OEPs exhibited instability across a range of parameter combinations. Huda et al. (2015) reported periodic behaviour of OEPs influenced by the radiation from the primary and the oblateness of the secondary. Abouelmagd and Mostafa (2015) delved into OEPs and restricted regions within the restricted three-body problem (RTBP) characterized by non-isotropic mass distributions, emphasizing their significance for small celestial bodies and spacecraft dynamics. Suraj et al. (2018) analyzed OEPs in a photogravitational RTBP framework involving heterogeneous spheroids. Idrisi and Ullah (2021, 2022) studied OEPs considering albedo effects and within the context of the restricted six-body problem, respectively, while Leke and Singh (2023) concentrated on OEPs in extra-solar planetary systems. Most recently, Idrisi and Ullah (2024) discovered unstable symmetric OEPs in the circular restricted three-body problem (CRTBP), particularly under conditions of significant oblateness.

This study aims to explore the dynamics of the infinitesimal body around OEPs in the CRTBP framework, specifically when both primaries exhibit radiative and oblate characteristics. The analysis is applied to particular binary star systems—Lalande 21258, BD+195116, Ross 614, 70 Ophiuchi, and 61 Cygni—offering valuable insights into the behaviour and stability of these out-of-plane equilibrium points.

In this analytical framework, the symbols m_1, m_2 and represent the masses of the larger and smaller primary bodies, respectively, while m_3 denotes the mass of the infinitesimal body. The model assumes that the two primary celestial entities revolve in circular orbits around their mutual centre of mass. Meanwhile, the infinitesimal body moves within the same plane as the primaries' motion, without affecting their orbital paths.

The mass parameter, denoted as μ, is defined by the equation $\mu = \frac{m_2}{m_1+m_2}$. The distance between the primary bodies is adopted as the reference unit of length, with the gravitational constant normalized to $G = 1$. The unit of mass is selected such that $m_1 + m_2 = 1$, resulting in dimensionless masses for the primary bodies expressed as $m_1 = 1 - \mu$ and $m_2 = \mu$, respectively.

The present investigation utilizes the synodic coordinate framework, in which the infinitesimal body's location is characterized by $P(x, y, z)$, while the primary and secondary bodies' positions are represented by $P_1(\mu, 0,0)$ and $P_2(-(1-\mu), 0,0)$, respectively (refer to Fig. 1).

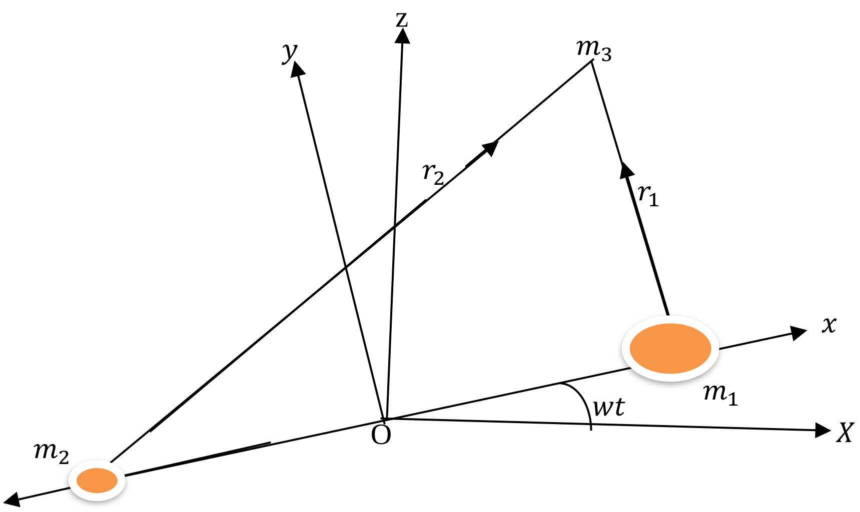

Figure 1: The configuration of the rotating coordinate system for the Restricted Three-Body Problem (RTBP), where the masses are represented by radiating oblate primaries and an infinitesimal body

Thus, the equations of motion of the infinitesimal body in the dimensionless synodic coordinate system with radiation pressure parameters q_1 and $q_2 (q_i \le 1, i = 1,2)$ and oblateness parameters A_1 and $A_2 (A_i << 1,\ i = 1,2)$ (Singh & Ishwar (1999)) are

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y, \quad \dot{z} = \Omega_z, \tag{1}$$

where

$$\Omega = \frac{1}{2}n^2(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3} - \frac{3(1-\mu)z^2 A_1 q_1}{2r_1^5} - \frac{3\mu A_2 q_2 z^2}{2r_2^5}, \ \cdots \tag{2}$$

$$\Omega_x = n^2 x + \frac{15A_1(1-\mu)q_1(x-\mu)z^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1(x-\mu)}{2r_1^5} - \frac{(1-\mu)q_1(x-\mu)}{r_1^3} + \frac{15A_2\mu q_2(x+1-\mu)z^2}{2r_2^7} - \frac{3A_2\mu q_2(x+1-\mu)}{2r_2^5} - \frac{\mu q_2(x+1-\mu)}{r_2^3}, \tag{3}$$

$$\Omega_y = n^2 y + \frac{15A_1(1-\mu)q_1 yz^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1 y}{2r_1^5} - \frac{(1-\mu)q_1 y}{r_1^3} + \frac{15A_2\mu q_2 yz^2}{2r_2^7} - \frac{3A_2\mu q_2 y}{2r_2^5} - \frac{\mu q_2 y}{r_2^3}, \tag{4}$$

$$\Omega_z = \frac{15A_1(1-\mu)q_1 z^3}{2r_1^7} - \frac{9A_1(1-\mu)q_1 z}{2r_1^5} - \frac{(1-\mu)q_1 z}{r_1^3} + \frac{15A_2\mu q_2 z^3}{2r_2^7} - \frac{9A_2\mu q_2 z}{2r_2^5} - \frac{\mu q_2 z}{r_2^3}, \tag{5}$$

and the distances between the infinitesimal body and primary and secondary bodies are given as $r_1^2 = (x-\mu)^2 + y^2 + z^2$, and $r_2^2 = (x-\mu+1)^2 + y^2 + z^2$, respectively.

The mean motion for each of the systems is given by

$$n = \sqrt{1 + \frac{3}{2}(A_1 + A_2)},$$

and the Jacobian integral of motion is represented as

$$C = n^2(x^2+y^2) + \frac{2q_1(1-\mu)}{r_1} + \frac{2q_2\mu}{r_2} + \frac{A_1 q_1(1-\mu)}{r_1^3} + \frac{\mu q_2 A_2}{r_2^3} - \frac{3A_1 q_1(1-\mu)z^2}{r_1^5} - \frac{3\mu A_2 q_2 z^2}{r_2^5} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{6}$$

In Table 1, we present the physical parameters of the binary systems. The parameters M_A and M_B are the masses of the more massive and less massive stars in each binary system as compared to the mass of the Sun. The symbol μ as shown earlier is the mass parameter. The luminosity of the binary systems denoted by L_A and L_B respectively are obtained from the relation (Mia and Kushvah, (2016))

$$\frac{L}{L_S} \approx \left(\frac{M}{M_S}\right)^{3.9},$$

Where L_S and M_S are the luminosity and mass of the Sun.

Radiation pressure has had a key effect on the formation of stars and the shaping of clouds of dust and gases on a wide range of scales. The mass reduction factor is represented as $q_i = 1 - \frac{F_P}{F_g}, i = 1,2$ (F_P and F_g are the radiation pressure and the gravitational attraction forces being exerted by the binary systems on objects around them) or $q_i = 1 - \beta, i = 1,2$ or based on the Stefan-Boltzmann's law (Xuetang and Lizhong, (1993)) as

$$q_i = 1 - \frac{A\kappa L}{a\rho M},\ i = 1,2\ ,$$

Where M, L and k, and are the mass, luminosity, and radiation pressure efficiency factor of a star. Also, α and ρ are the radius and density of the dust grain particles moving in the binary systems while $A = \frac{3}{16\pi cG}$ is a constant with c and G as the speed of light and Gravitational constant.

The values of the luminosity and mass reduction factor $q_i = 1,2$ have been obtained by computing in the *C.G.S.* system of unit, using $L_s = 3.846\times10^{33} erg/s$, $c = 3\times10^{10} cm/s$, $G = 6.67384\times10^{-8} cm^3 g^{-1} s^{-2}$, $M_s = 1.989\times10^{33} g$ and $\kappa = 1$. Also, we have assumed the values for the radius and density of the dust grain particles as $a = 2\times10^{-2} cm$ and $\rho = 1.4 g/cm^3$ ((Xuetang and Lizhong (1993)). Arbitrary values are been used for the oblateness coefficients A_1 and A_2 as shown in Table 1.

Table 1: Physical Parameters of the Five Binary Systems

Parameters	*Lalande* 21258	$BD+19$ 5116	*Ross* 614	70 Ophiuchi	61 Cygni
$M_A(M_S)$	0.48	0.33	0.17	1.02	0.7
$M_B(M_S)$	0.1	0.16	0.1	0.64	0.63
μ	0.1724	0.3265	0.3704	0.3855	0.4739
$L_A(L_S)$	0.00637	0.00265	0.000492	0.47	0.0887
$L_B(L_S)$	0.0000344	0.00037	0.0000288	0.0895	0.0414
q_1	0.972692	0.983475	0.994045	0.0518079	0.73925
q_2	0.999292	0.995241	0.999407	0.712233	0.864775
A_1	0.10	0.12	0.14	0.16	0.18
A_2	0.11	0.13	0.15	0.17	0.19

The OEPs are obtained by solving equations (3) and (5) to get

$$x_0 = (\mu-1) - \frac{3\sqrt{3}(\mu-1)(2+3A_1)(1-q_1)}{2q_2\mu} A_2^{3/2} - \frac{9\sqrt{3}(\mu-1)(2+6q_1+45A_1q_1)}{4q_2\mu} A_2^{5/2} + O(A_2^3), \quad (7)$$

and

$$z_0 = \pm\sqrt{3}A_2^{1/2} + \frac{9(\mu-1)q_1(2+9A_1)}{4q_2\mu} A_2^2 - \frac{63\sqrt{3}(\mu-1)^2(2+3A_1)^2(1-q_1)^2}{16q_2^2\mu^2} A_2^{5/2} + O(A_2^3). \quad (8)$$

Stability of the Out-of-plane Equilibrium Points

Variational Equations

To study the motion of the infinitesimal body near any of the out-of-plane equilibrium points $L_{6,7}$ and $L_{8,9}$, we transfer the origin to $L_6(x_0, 0, z_0)$ and $L_8(x_0, 0, z_0)$ such that the newly defined coordinate system ξ, η and ζ are parallel to the axes Ox , Oy and Oz . Then we have

$$x = x_0 + \xi\ ,\ y = \eta \text{ and } z = z_0 + \zeta\ . \quad (9)$$

By using system (9) in Eqns. (1), and expanding Ω in Taylor series around the OEPs and taking only linear terms in ξ, η and ζ , the equations of motion (1) become

$$\begin{aligned} &\ddot{\xi} - 2n\dot{\eta} = \xi\Omega^0_{xx} + \zeta\Omega^0_{xz}, \\ &\ddot{\eta} + 2n\dot{\xi} = \eta\Omega^0_{yy}, \\ &\ddot{\zeta} = \xi\Omega^0_{zx} + \zeta\Omega^0_{zz}, \end{aligned} \quad (10)$$

where

$$\Omega_{xx}^{0}=n^{2}-\frac{105A_{1}(1-\mu)q_{1}(x_{0}-\mu)^{2}z_{0}{}^{2}}{2r_{10}^{9}}+\frac{15A_{1}(1-\mu)q_{1}(x_{0}-\mu)^{2}}{2r_{10}^{7}}+\frac{15A_{1}(1-\mu)q_{1}z_{0}{}^{2}}{2r_{10}^{7}}-\frac{3A_{1}(1-\mu)q_{1}}{2r_{10}^{5}}$$
$$+\frac{3(1-\mu)q_{1}(x_{0}-\mu)^{2}}{r_{10}^{5}}-\frac{(1-\mu)q_{1}}{r_{10}^{3}}-\frac{105A_{2}\mu q_{2}(x_{0}+1-\mu)^{2}z_{0}{}^{2}}{2r_{20}^{9}}+\frac{15A_{2}\mu q_{2}(x_{0}+1-\mu)^{2}}{2r_{20}^{7}}$$
$$+\frac{15A_{2}\mu q_{2}z_{0}{}^{2}}{2r_{20}^{7}}-\frac{3A_{2}\mu q_{2}}{2r_{20}^{5}}+\frac{3\mu q_{2}(x_{0}+1-\mu)^{2}}{r_{20}^{5}}-\frac{\mu q_{2}}{r_{20}^{3}},$$
$$\Omega_{yy}^{0}=n^{2}+\frac{15A_{1}(1-\mu)q_{1}z_{0}^{2}}{2r_{10}^{7}}-\frac{3A_{1}(1-\mu)q_{1}}{2r_{10}^{5}}-\frac{(1-\mu)q_{1}}{r_{10}^{3}}+\frac{15A_{2}\mu q_{2}z_{0}^{2}}{2r_{20}^{7}}-\frac{3A_{2}\mu q_{2}}{2r_{20}^{5}}+-\frac{\mu q_{2}}{r_{20}^{3}},$$
$$\Omega_{zz}^{0}=-\frac{105A_{1}(1-\mu)q_{1}z_{0}^{4}}{2r_{10}^{9}}+\frac{45A_{1}(1-\mu)q_{1}z_{0}^{2}}{r_{10}^{7}}-\frac{9A_{1}(1-\mu)q_{1}}{2r_{10}^{5}}+\frac{3(1-\mu)q_{1}z_{0}^{2}}{r_{10}^{5}}-\frac{(1-\mu)q_{1}}{r_{10}^{3}}$$
$$-\frac{105A_{2}\mu q_{2}z_{0}^{4}}{2r_{20}^{9}}+\frac{45A_{2}\mu q_{2}z_{0}^{2}}{r_{20}^{7}}-\frac{9A_{2}\mu q_{2}}{2r_{20}^{5}}+\frac{3\mu q_{2}z_{0}^{2}}{r_{20}^{5}}-\frac{\mu q_{2}}{r_{20}^{3}},$$
$$\Omega_{xz}^{0}=-\frac{105A_{1}(1-\mu)q_{1}(x_{0}-\mu)z_{0}^{3}}{2r_{10}^{9}}+\frac{45A_{1}(1-\mu)q_{1}(x_{0}-\mu)z_{0}}{2r_{10}^{7}}+\frac{3(1-\mu)q_{1}(x_{0}-\mu)z_{0}}{r_{10}^{5}} \tag{11}$$
$$-\frac{105A_{2}\mu q_{2}(x_{0}+1-\mu)z_{0}^{3}}{2r_{10}^{9}}+\frac{45A_{2}\mu q_{2}(x_{0}+1-\mu)z_{0}}{2r_{10}^{7}}+\frac{3\mu q_{2}(x_{0}+1-\mu)z_{0}}{r_{10}^{5}}.$$

Equations (10) are known as the variational equations of motion and we have $\Omega_{xz}^{0}=\Omega_{zx}^{0}$, while the superscript 0 indicates that the derivatives are to be evaluated at the OEPs $L_{6,7}(x_0,0,z_0)$ and $L_{8,9}(x_0,0,z_0)$.

3.2 Characteristic equations

Let the solution of system (9) be

$\xi=\varpi e^{\lambda t}$, $\eta=\vartheta e^{\lambda t}$ and $\zeta=\upsilon e^{\lambda t}$

where $\varpi,\vartheta,\upsilon$ and λ are constants. Then equations (10) can be written as

$$\begin{aligned}&\lambda^{2}\varpi e^{\lambda t}-2n\lambda\vartheta e^{\lambda t}=\Omega_{xx}^{0}\varpi e^{\lambda t}+\Omega_{xz}^{0}\upsilon e^{\lambda t},\\&\lambda^{2}\vartheta e^{\lambda t}+2n\lambda\varpi e^{\lambda t}=\Omega_{yy}^{0}\vartheta e^{\lambda t},\\&\lambda^{2}\upsilon e^{\lambda t}=\Omega_{zx}^{0}\varpi e^{\lambda t}+\Omega_{zz}^{0}\upsilon e^{\lambda t}.\end{aligned} \tag{12}$$

By representing system (12) in matrix notation, we obtain the non-trivial solution if

$$\begin{vmatrix}\lambda^{2}-\Omega_{xx}^{0} & -2n\lambda & -\Omega_{xz}^{0}\\ 2n\lambda & \lambda^{2}-\Omega_{yy}^{0} & 0\\ -\Omega_{zx}^{0} & 0 & \lambda^{2}-\Omega_{zz}^{0}\end{vmatrix}=0. \tag{13}$$

Expanding the system (13), we obtain

$$\lambda^6+N_1\lambda^4+N_2\lambda^2+\mathrm{N}_3=0, \tag{14}$$

where

$$N_1=4n^2-\Omega_{xx}^{0}-\Omega_{yy}^{0}-\Omega_{zz}^{0},$$
$$N_2=\Omega_{xx}^{0}\Omega_{yy}^{0}+\Omega_{yy}^{0}\Omega_{zz}^{0}+\Omega_{zz}^{0}\Omega_{xx}^{0}-4n^2\Omega_{zz}^{0}-(\Omega_{xz}^{0})^2,$$
$$N_3=(\Omega_{xz}^{0})^2\Omega_{yy}^{0}-\Omega_{xx}^{0}\Omega_{yy}^{0}\Omega_{zz}^{0}.$$

The last equation is the characteristic equation to system (10) and its roots are

$$\lambda_1=-[-\frac{N_1}{3}+\frac{(2^{\frac{1}{3}}N_1^2)}{3(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}$$
$$-\frac{(2^{\frac{1}{3}}N_2)}{(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27_3^2)})^{\frac{1}{3}}}$$
$$+\frac{1}{(3\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}},$$
$$\lambda_2=[-\frac{N_1}{3}+\frac{(2^{\frac{1}{3}}N_1^2)}{3(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}$$
$$-\frac{(2^{\frac{1}{3}}N_2)}{(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}$$
$$+\frac{1}{(3\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}},$$

$$
\begin{aligned}
\lambda_3 = -[&-\frac{N_1}{3}-\frac{N_1^2}{3\times 2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{i\aleph_1^2}{2^{\frac{2}{3}}\sqrt{3}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{i\sqrt{3}N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(6\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(2\times 2^{\frac{1}{3}}\sqrt{3})i(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_4 = [&-\frac{N_1}{3}-\frac{N_1^2}{3\times 2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{i\aleph_1^2}{2^{\frac{2}{3}}\sqrt{3}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{i\sqrt{3}N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(6\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(2\times 2^{\frac{1}{3}}\sqrt{3})i(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_5 = -[&-\frac{N_1}{3}-\frac{N_1^2}{3\times 2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{iN_1^2}{2^{\frac{2}{3}}\sqrt{3}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27_3^2)})^{\frac{1}{3}}}\\
&+\frac{N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{i\sqrt{3}N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(6\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{1}{(2\times 2^{\frac{1}{3}}\sqrt{3})i(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_6 = [&-\frac{N_1}{3}-\frac{N_1^2}{3\times 2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{iN_1^2}{2^{\frac{2}{3}}\sqrt{3}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{i\sqrt{3}N_2}{2^{\frac{2}{3}}(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&-\frac{1}{(6\times 2^{\frac{1}{3}})(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}\\
&+\frac{1}{(2\times 2^{\frac{1}{3}}\sqrt{3})i(-2N_1^3+9N_1N_2-27N_3+3\sqrt{3}\sqrt{(-N_1^2N_2^2+4N_2^3+4N_1^3N_3-18N_1N_2N_3+27N_3^2)})^{\frac{1}{3}}}]^{\frac{1}{2}}.
\end{aligned}
$$

Lyapunov's theorem posits that equilibrium points exhibit stability when their characteristic roots are either complex numbers with negative real parts, purely imaginary values, or solely negative real numbers. Conversely, equilibrium points are deemed unstable in all other cases. The physical parameters listed in Table 1 are applied to calculate the numerical values of these characteristic roots, as derived from Equation (14). This calculation is subsequently used to assess the linear stability of the out-of-plane equilibrium points across five binary systems. The results of this analysis are summarized in Tables 2 and 3.

Table 2: The roots of the characteristic equations of the five binary systems for the OEPs $L_{6,7}(x, 0, \pm z)$

Binary Systems	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
Lalande 21258	$\pm 3.01304480i$	$-2.31033959 \pm 1.45377079i$	$2.31033959 \pm 1.45377079i$
BD+19 5116	$\pm 2.60982987i$	$-2.03710645 \pm 1.45574610i$	$2.03710645 \pm 1.45574610i$
Ross 614	$\pm 2.41633066i$	$-1.89624695 \pm 1.45307452i$	$1.89624695 \pm 1.45307452i$
70 Ophiuchi	$\pm 1.68621643i$	$-1.24750498 \pm 1.27656013i$	$1.24750498 \pm 1.27656013i$
61 Cygni	$\pm 2.23708008i$	$-1.66593272 \pm 1.35206072i$	$1.66593272 \pm 1.35206072i$

Table 3: The roots of the characteristic equations of the five binary systems for the OEPs $L_{8,9}(x, 0, \pm z)$

Binary Systems	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
Lalande 21258	$\pm 2.80977142i$	$-2.07755178 \pm 1.29761847i$	$2.07755178 \pm 1.29761847i$
BD+19 5116	$\pm 2.70325019i$	$-2.01658734 \pm 1.33710274i$	$2.01658734 \pm 1.33710274i$
Ross 614	$\pm 2.61590679i$	$-1.94665609 \pm 1.34275306i$	$1.94665609 \pm 1.34275306i$
70 Ophiuchi	$\pm 2.76378657i$	$-1.75939160 \pm 0.87818034i$	$1.75939160 \pm 0.87818034i$
61 Cygni	$\pm 2.34477314i$	$-1.69369949 \pm 1.29407782i$	$1.69369949 \pm 1.29407782i$

It is observed from Tables 2 and 3 that the out-of-plane equilibria are unstable since we found no case in which the roots are pure imaginary.

Discussion

The stability of out-of-plane equilibrium points (OEPs) was examined through simulations of the primaries as radiating and oblate bodies within the context of the Circular Restricted Three-Body Problem (CRTBP). The analysis encompassed five binary systems: 21258, BD+19 5116, Ross 614, 70 Ophiuchi, and 61 Cygni. We established plausible values for the radiation components and made assumptions regarding the oblateness parameters based on the physical characteristics of these systems. As part of the stability analysis, we derived the characteristic equation, and a numerical assessment of its roots revealed four complex roots alongside two distinct pure imaginary roots. The presence of positive real parts in the complex roots indicates that all binary systems demonstrate instability at the OEPs, corroborating findings from previous studies.

These results are in agreement with:

Perezhogin (1976) when $A_1 = 0 = A_2, q_2 = 1, r_1 \to r, r_2 \to r$, and $q_1 \to q$ and

Singh and Umar (2012) when $e \neq 0, a \neq 1, q_2 = 1, A_1 = 0, A_1 \to A$ and $q_2 \to q$.

And contradict those of:

Ragos and Zagouras (1988) when $A_1 = 0 = A_2$, and

Das et al. (2009) in the absence of PR-drag and when $A_1 = 0 = A_2$.

CONCLUSION

This study demonstrates that the inclusion of oblateness and radiation significantly impacts the stability of OEPs in the CRTBP. For the five binary systems analyzed, the numerical results show that the OEPs are unstable due to the presence of complex roots with positive real parts. This instability implies that any perturbations will amplify over time, resulting in significant deviations from the equilibrium states. These results underline the importance of considering both oblateness and radiative effects in the stability analysis of binary systems.

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