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A STUDY OF HYBRID FIXED POINTS ON SEMIGROUPS OF TRANSFORMATION

***[1]Abdulkarim, A., [1]Muhammad, A. A. and [2]Abdullahi, A.**

[1]Ahmadu Bello University, Zaria, Nigeria.
[2]Department of Mathematical Sciences, Kaduna State University Kaduna.

*Corresponding authors' email: abdulkarimabubakar47@gmail.com

ABSTRACT

Fixed point theory plays a fundamental role in mathematical analysis, algebra, and topology, with applications spanning differential equations, game theory, and computer science. This study extends classical fixed point results by exploring hybrid fixed points within semigroups of transformation. Hybrid fixed points generalize standard fixed points by incorporating auxiliary functions, allowing for broader applications in iterative methods and computational mathematics. We establish key results on hybrid fixed points by considering contractive and nonexpansive mappings in semigroups. Using Banach's contraction principle and related fixed point theorems, we prove the existence and uniqueness of hybrid fixed points under suitable conditions. Notable results include hybrid contractions, asymptotic regularity, and their implications in complete and compact metric spaces. Examples illustrate the theoretical findings, demonstrating hybrid fixed points in transformation semigroups.

Keywords: Transformation semigroups, Hybrid fixed points, Existence, Uniqueness

INTRODUCTION

Fixed point theory plays a crucial role in mathematical analysis, algebra, and topology. The study of fixed points on semigroups of transformation extends classical fixed point results to algebraic structures with transformation properties. Fixed points arise naturally in various mathematical disciplines and applications, including differential equations, game theory, and computer science.

The origins of fixed point theory trace back to Brouwer's fixed point theorem, It states that for any continuous function f mapping a nonempty compact convex set to itself, there is a point x_0 such that $f(x_0) = x_0$. That is any continuous function mapping a compact convex set to itself in a Euclidean space has at least one fixed point (Brouwer, 1911). This result laid the foundation for subsequent generalizations, including Banach's contraction principle, which guarantees the existence and uniqueness of a fixed point for contractive mappings in complete metric spaces (Banach, 1922). Banach's theorem has been extensively used in numerical analysis and differential equations due to its constructive nature.

Building upon these early results, Nadler extended Banach's principle to multi-valued mappings in metric spaces, thereby broadening the applicability of fixed point theory (Nadler, 1969). Takahashi investigated non-expansive mappings in Hilbert spaces, proving essential results on the existence of fixed points in broader topological settings (Takahashi,1970). Kirk (Kirk, 2008) explored fixed point results in general metric spaces, emphasizing the role of asymptotic regularity in iterative methods (Kirk, 2008). These developments provided a crucial link between fixed point theory and functional analysis, facilitating applications in operator theory and computational mathematics.

Semigroups of transformation have been a subject of significant study due to their role in algebraic structures and dynamical systems. Howie (Howie, 1995) provided a comprehensive introduction to semigroup theory, outlining their algebraic properties and applications in functional analysis (Howie, 1995). The study of fixed points within transformation semigroups has found applications in Stability Analysis, Differential Inclusions, and Ergodic Theory.

Recent researches have explored contractive and non-expansive mappings in transformation semigroups, leading to results on the existence of fixed points under various conditions. Studies have established that contractive mappings in semigroups ensure the convergence of iterative sequences, which is crucial for proving the existence of hybrid fixed points.

The notion of hybrid fixed points generalizes classical fixed point results by incorporating auxiliary functions, making them suitable for broader applications, including iterative algorithms and computational methods. Hybrid contractions, which involve auxiliary functions modifying the contraction condition, have been explored in metric space and Banach space.

Hybrid fixed point results have found applications in optimization theory, game theory, and nonlinear functional analysis. The extension of these results to semigroups of transformations allows for new insights into the behavior of iterative sequences and their convergence properties. The literature suggests that contractive transformations in semigroups admit hybrid fixed points under suitable conditions, leading to new theoretical advancements in metric space theory and operator analysis.

Preliminaries

We begin with basic definitions and important preliminary results required for our study.

Definition 1 (Howie, 1995) (Semigroup of Transformations)

A non-empty set S equipped with an associative binary operation $*$, is called a semigroup, if for all x, y and z in S, $(x * y) * z = x * (y * z)$. The binary operation $*$ is mostly denoted multiplicatively or in juxtaposition i.e $(xy)z = x(yz)$. If the semigroup S has the property that, for all x, y in $S, xy = yx$, we say that S, is a commutative semigroup. The semigroup S is called a monoid if it has an identity element, that is, there is $1 \in S$ such that for all x in $S, x1 = 1x = x$. An element 0 in S is called zero element of S if $x0 = 0x = 0$ for all x in S and S is called a semigroup with zero. If the semigroup S has no identity or zero element, then it is easy to adjoin an extra identity or zero to $\boldsymbol{S}$, in order to form a monoid or semigroup with zero respectively.

A semigroup S of transformations on a set X is a set of functions $S \subseteq X^X$ closed under composition. That is, for any two transformations $T_1, T_2 \in S$, their composition $T_1 \circ T_1$ is also in S. Semigroups of transformations are fundamental in understanding algebraic structures and dynamic systems.

Definition 2 (Hybrid Fixed Point)

A point $x \in X$ is a hybrid fixed point of a function $T : X \to X$ if there exists a function $g : X \to X$ such that $g(T(x)) = x$. The function g provides additional flexibility compared to classical fixed point definitions, allowing for applications in iterative schemes and generalized contractions.

Definition 3 (Contractive Mapping)

A function $T: X \to X$ is said to be contractive if there exists a constant $c \in [0,1)$ such that for all $x, y \in X$,

$d(T(x), T(y)) \leq c\, d(x,y)$.

This condition ensures the convergence of iterative sequences and plays a key role in proving fixed point existence.

Lemma 4

Let (X, d) be a complete metric space, and let $T : X \to X$ be a transformation satisfying a contractive condition. Then, T has a unique fixed point.

Proof:

Let $x_0 \in X$ be an arbitrary initial point, and define a sequence $\{x_n\}$by

$x_{n+1} = T(x_n)$.

By the contractive condition,

$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq c\, d(x_n, x_{n-1})$.

Since $c \in [0,1)$, the sequence $\{x_n\}$is Cauchy and converges to a limit x^*. Taking the limit on both sides, we conclude that $T(x^*) = x^*$, proving the existence and uniqueness of the fixed point.

Definition 5 (Asymptotic Regularity)

A transformation $T : X \to X$ is said to be asymptotically regular if for every sequence

$\{x_n\}\, in\, X$,

$\lim_{n\to\infty} d(T(x_n), x_n) = 0$.

This property is significant in iterative methods used to approximate fixed points.

Proposition 6

If T is a contractive transformation on a complete metric space, then it is asymptotically regular.

Proof:

Since T is contractive, the sequence $d(T(x_n), x_n)$ forms a decreasing sequence tending to zero, ensuring asymptotic regularity.

Corollary 7

Any contractive transformation in a semigroup S of transformations on a compact metric space admits at least one hybrid fixed point.

Proof:

By compactness, the sequence $T^n(x)\}$ has a convergent subsequence. Using the Banach fixed point theorem, we establish the hybrid fixed point existence.

RESULTS AND DISCUSSION

We present and prove key results on hybrid fixed points within semigroups of transformations.

Theorem 8

Let S be a semigroup of transformations on X such that each $T \in S$ satisfies a contractive condition. Then S has a hybrid fixed point.

Proof:

Let $T \in S$ be a transformation satisfying the contractive condition:

$d(T(x), T(y)) \leq c\, d(x,y)$, for all $x, y \in X$,

where $c \in [0,1)$.

Fix an arbitrary starting point $x_0 \in X$. Define the sequence $\{x_n\}$ iteratively by

$x_{n+1} = T(x_n), for\ n \geq 0$.

By the contractive condition, we have

$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq c\, d(x_n, x_{n-1})$.

Iterating this inequality, we obtain

$d(x_{n+1}, x_n) \leq c^n\, d(x_1, x_0)$.

For any $m > n$, the triangle inequality gives

$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} c^k d(x_1, x_0)$

Since $c \in [0,1)$, the geometric series $\sum_{k=0}^{\infty} c^k$ converges, and thus $\{x_n\}$is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$converges to a limit $x^* \in X$. By continuity of T (implied by the contractive condition), we have

$T(x^*) = T\left(\lim_{n\to\infty} x_n\right)$

$= \lim_{n\to\infty} T(x_n)$

$= \lim_{n\to\infty} x_{n+1}$

$= x^*$

Thus, x^* is a fixed point of T. By Definition 2.2, x^* is a hybrid fixed point of T with respect to the identity function $g(x) = x$. This completes the proof.

Proposition 9

If T is a non-expansive transformation on a compact metric space X, then T admits a hybrid fixed point.

Proof:

A transformation T is non-expansive if

$d(T(x), T(y)) \leq d(x,y), for\ all\ x, y \in X$.

Since X is compact, every sequence in X has a convergent subsequence. Consider the sequence $\{T^n(x_0)\}$ for some $x_0 \in X$. By compactness, there exists a subsequence $\{T^{n_k}(x_0)\}$ converging to a limit $x^* \in X$. Using the non-expansive property, we have

$d(T(x^*), x^*) = \lim_{k\to\infty} d\left(T^{n_{k+1}}(x_0), T^{n_k}(x_0)\right)$

$\leq \lim_{n\to\infty} d(T(x_0), x_0) = 0$

Thus, $T(x^*) = x^*$, and x^* is a fixed point of T. By Definition 2.2, x^* is a hybrid fixed point of T with respect to the identity function $g(x) = x$. This completes the proof.

Corollary 10

If T is a self-mapping on a complete metric space satisfying a hybrid contraction, then it has a unique hybrid fixed point.

Proof:

A hybrid contraction condition is defined as

$d(g(T(x)), g(T(y))) \leq c\, d(x,y), for\ all\ x, y \in X$,

where $c \in [0,1)$ and $g : X \to X$ is a continuous function.

Fix an arbitrary starting point $x_0 \in X$. Define the sequence $\{x_n\}$iteratively by

$x_{n+1} = g(T(x_n)), for\ n \geq 0$.

By the hybrid contraction condition, we have

$d(x_{n+1}, x_n) = d\left(g(T(x_n)), g(T(x_{n-1}))\right) \leq c\, d(x_n, x_{n-1})$.
Iterating this inequality, we obtain:
$d(x_{n+1}, x_n) \leq c^n\, d(x_n, x_0)$.
For any $m > n$, the triangle inequality gives
$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k)$
$\leq \sum_{k=n}^{m-1} c^k d(x_1, x_0)$
Since $c \in [0,1)$, the geometric series converges, and thus $\{x_n\}$is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$converges to a limit $x^* \in X$. By continuity of g and T, we have

$$g(T(x^*)) = g\left(T\left(\lim_{n\to\infty} x_n\right)\right)$$
$$= \lim_{n\to\infty} g(T(x_n))$$
$$= \lim_{n\to\infty} x_{n+1}$$
$$= x^*$$

Thus, x^* is a hybrid fixed point of T with respect to g. Suppose y^* is another hybrid fixed point. Then
$d(x^*, y^*) = d(g(T(x^*)), g(T(y^*))) \leq c\, d(x^*, y^*)$.
Since $c \in [0,1)$, this implies $d(x^*, y^*) = 0$, and thus $x^* = y^*$. This proves uniqueness.

Theorem 11
Let X be a Banach space and $T: X \to X$ a transformation. If T satisfies a weak contractive condition, then T has a hybrid fixed point.
Proof:
A weak contractive condition is defined as;

$$\|T(x) - T(y)\| \leq \|x - y\| - \psi(\|x - y\|), \text{for all } x, y \in X,$$

where $\psi: [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function with $\psi(0) = 0$ $and\ \psi(t) > 0\ for\ t > 0$.
Fix an arbitrary starting point $x_0 \in X$. Define the sequence $\{x_n\}$iteratively by$x_{n+1} = T(x_n), for\ n \geq 0$. By the weak contractive condition, we have
$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\|$
$\leq \|x_n - x_{n-1}\| - \psi(\|x_n - x_{n-1}\|)$.
This implies that $\|x_{n+1} - x_n\|$ is a decreasing sequence. Since $\|x_{n+1} - x_n\|$ is bounded below by 0, it converges to a limit $L \geq 0$. Taking the limit as $n \to \infty$in the weak contraction inequality, we obtain:
$L \leq L - \psi(L)$.
This implies $\psi(L) = 0$, and thus $L = 0$. Therefore, $\{x_n\}$is a Cauchy sequence. Since X is a Banach space (and hence complete), the sequence $\{x_n\}$converges to a limit $x^* \in X$. By continuity of T, we have

$$T(x^*) = T\left(\lim_{n\to\infty} x_n\right)$$
$$= \lim_{n\to\infty} T(x_n)$$
$$= \lim_{n\to\infty} x_{n+1}$$
$$= x^*$$

Thus, x^* is a fixed point of T. By Definition 2.2, x^* is a hybrid fixed point of T with respect to the identity function $g(x) = x$. This completes the proof.

Example 1
Consider the set $X = \mathbb{R}$ (real numbers) and the transformation semigroup $S = \{T_a(x) = ax \mid a \in (0,1]\}$, where multiplication defines the semigroup operation.

Let $T(x) = \frac{1}{2}x$ be a transformation in S. A hybrid fixed point satisfies
$g(T(x)) = x$
For $g(x) = 2x$, we get
$g(T(x)) = 2 \times \frac{1}{2}x = x$
Thus, every $x \in \mathbb{R}$ is a hybrid fixed point with respect to $g(x) = 2x$.

Example 2
Let $X = [0,1]$ with the standard metric $d(x,y) = |x - y|$. Define the transformation
$T(x) = \frac{x}{2} + \frac{1}{4}$
This is contractive because:
$|T(x) - T(y)| = \left|\frac{x}{2} + \frac{1}{4} - \frac{y}{2} - \frac{1}{4}\right|$
$= \frac{|x-y|}{2}$
which satisfies $d(T(x), T(y)) \leq \frac{1}{2} d(x,y)$.
for $g(x) = x + \frac{1}{4}$, we solve
$g(T(x)) = x$
Substituting $T(x)$,
$\frac{x}{2} + \frac{1}{4} + \frac{1}{4} = x$
$\frac{x}{2} + \frac{1}{2} = x$
$x + 1 = 2x$
$1 = 2x - x$
$x = 1$
Thus, $x^* = 1$ is a hybrid fixed point of T with respect to $g(x) = x + \frac{1}{4}$

CONCLUSION
This study extends fixed point theory by investigating hybrid fixed points in semigroups of transformation. The key findings focus on the existence and uniqueness of hybrid fixed points under various conditions, with proofs supported by theorems, lemmas, propositions, and corollaries. This work extends classical fixed point results by incorporating hybrid fixed points in transformation semigroups, offering new insights into metric space analysis and iterative algorithms. Future research could explore extensions to stochastic semigroups and applications in optimization problems.

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