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MODIFIED INVERTED KUMARASWAMY DISTRIBUTION USING INVERSE POWER FUNCTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In the area of distribution theory, statisticians have proposed and developed new models for generalizing the existing ones to make them more flexible and to aid their application in a variety of fields. In this article, we present a new distribution called the Modified Inverted Kumaraswamy Distribution Using Inverse Power Function with three positive parameters, which extends the Inverted Kumaraswamy distribution with two parameters. Some statistical properties of the MIK distribution, such as explicit expressions for the quantile function, probability-weighted moments, moments, generating function, Reliability function, hazard function, and order statistics are discussed. A maximum likelihood estimation technique is employed to estimate the model parameters and the simulation study is presented. The superiority of the new distribution is illustrated with an application to a real data set. The results showed that the new distribution fits better in the real data set amongst the range of distributions considered.

Keywords: Kumaraswamy distribution, Inverted Kumaraswamy distribution, Quantile function, Reliability function, Maximum likelihood, Order Statistics

INTRODUCTION

All parametric statistical techniques, such as inference, modelling, survival analysis, and reliability, are based on statistical distributions. Fitting the data to a statistical model is a critical step when analyzing lifetime data. For this reason, several lifespan distributions have been established in the literature. The majority of lifespan models have a limited set of behaviours. Such distributions are unable to provide a better fit for all real scenarios. As a result, a variety of distribution classes have been created by expanding common continuous distributions. The generated family of continuous distributions is a new enhancement for developing and expanding classic distributions. The newly generated distributions have been extensively researched in a variety of fields, and they provide greater application flexibility.

One of the most well-known lifetime distributions is the inverted Kumaraswamy distribution (Abd AL-Fattah *et.al.*,2017) have a wide range of applications in problems related to econometrics, biological sciences, survey sampling, engineering sciences, medical research and life testing problems. In addition, it is employed in financial literature, environmental studies, survival and reliability theory. Many researchers focused on the inverted distributions and their applications;for example, Calabria and Pulcini,(1990) studied the inverse Weibull distribution, AL-Dayian (1999) introduced the inverted Burr Type XII distribution, Abd EL-Kader *et al.* (2003) also described the inverted Pareto Type I distribution, AL-Dayian (2004) discussed inverted Pareto Type II distribution and Aljuaid (2013) presented exponentiated inverted Weibull distribution.

Kumaraswamy (1980) presented a distribution, which has many similarities to the beta distribution. This distribution applies to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures and hydrological data such as daily rainfall and daily stream flow (see Kumaraswamy, 1980; Jones, 2009; Sindhu, 2013; and Sharaf EL-Deen *et al.*,2014).Other notable contributions to distributions theory include the development of the NOF-G family of distributions (Sadiq *et al.*, 2022), the NGOF-G family of distributions (Sadiq *et al.*, 2023a), the NGOF-Et-G family of distributions (Sadiq *et al.*, 2023b), and the NGOF-OE-G family of distributions (Sadiq *et al.*, 2023c). Additional advancements include the NETD model utilizing a generalized logarithmic function (Obafemi *et al.*, 2024) and the extension of the T-L distribution (Habu *et al.*, 2024).

MATERIALS AND METHODS

The inverted Kumaraswamy distribution

The inverted Kumaraswamy distribution developed by AL-Fattah *et.al.* (2017) can be derived from Kumaraswamy (Kum)distribution using the transformation $\frac{1}{x} - 1$when X has a Kum distribution with probability density function (pdf) and cumulative distribution function (cdf) respectively given by;

$$f(t,\alpha,\beta) = \alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}, 0 < t < \infty, \alpha,\beta > 0 \quad (1)$$

$$F(t;\alpha,\beta) = (1-(1+t)^{-\alpha})^{\beta}, 0 < t < \infty, \alpha,\beta > 0 \quad (2)$$

Modified Inverted Kumaraswamy distribution

Modified Inverted Kumaraswamy (MIK) is an extension of inverted Kumaraswamy using the power function, the cumulative distribution function (cdf) and the probability density function (pdf) of the proposed distribution will be derived by transforming $t = x^{\frac{1}{\lambda}}$ as follows:

$$F(x;\alpha,\beta,\lambda) = (1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta}; 0 < x < \infty, \alpha,\beta,\lambda > 0 \quad (3)$$

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta}{\lambda} x^{\frac{1}{\lambda}-1}(1+x^{\frac{1}{\lambda}})^{-(\alpha+1)}(1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta-1}; 0 < x < \infty, \alpha,\beta,\lambda > 0 \quad (4)$$

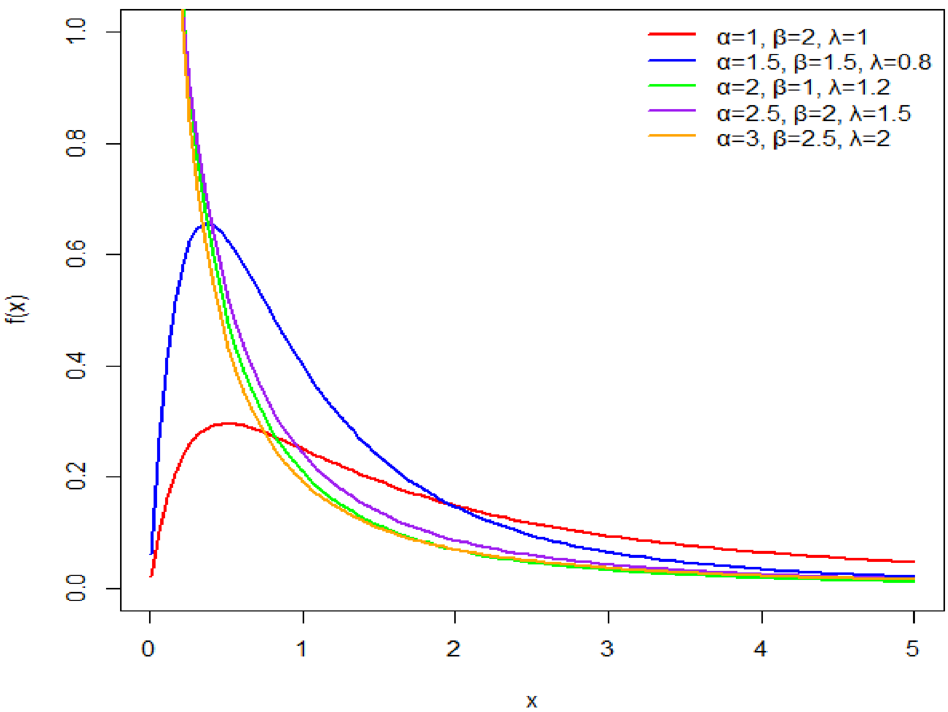

Figure 1: PDF Plot of the Modified Inverted Kumaraswamy Distribution

Figure 1 represents the probability density functions (PDFs) for different parameter values of the Modified Inverted Kumaraswamy Distribution. This distribution is a generalization of the Kumaraswamy distribution, which is known for its flexibility in modelling various types of data.Each curve shows how the shape of the PDF changes with different parameter values, which is useful for understanding the behaviour of the distribution under various conditions.The Modified Inverted Kumaraswamy Distribution is particularly useful in fields such as reliability engineering, survival analysis, and environmental studies due to its ability to model data with varying shapes and tail behaviours.

The goal of this paper is to develop a more flexible model by extending the two-parameter invertedKumaraswamy distribution with α>0 and β>0 while the proposed distribution will havethree parameters$\alpha, \beta, \lambda > 0$.

The rest ofthe paper is organized as follows: useful expansion and representations of the MIK distribution are presented in Section 3. Section 4, providesthe statistical properties such as moments, moments generating function, quantile function, reliability function, hazard function and order statistics. The parameters of the new model were estimated using the maximum likelihood estimation (MLE) approach in Section 5. The applications of the new model to the real dataset were shown in Section 6 to demonstrate its flexibilityagainst the competitors. Finally, Section 7 concludes the paper.

Important Representation

In this section, the simplest and most useful representation of the MIKis provided based on Generalized Binomial and power series expansion as follows:

Using generalized Binomial expansion equation (3) becomes:

$$F(x;\alpha,\beta,\lambda) = (1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta} = \sum_{i=1}^{\infty}(-1)^{i}\binom{\beta}{i}(1+x^{1/\lambda})^{-\alpha i} \quad (5)$$

While the probability density function (pdf) is obtainedusing generalized binomial expansionas follows:

From equation (4)

$(1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta-1} = \sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}(1+x^{\frac{1}{\lambda}})^{-\alpha j}$, then equation (4) becomes:

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta}{\lambda} x^{\frac{1}{\lambda}-1}\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}(1+x^{\frac{1}{\lambda}})^{-\alpha(1+j)-1} \quad (6)$$

Apply $(1+z)^{-b} = \sum_{k=0}^{\infty}\binom{-b}{k}z^{k} for|b| < 1$ to equation (6) and simplify gives:

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta}{\lambda}\sum_{j,k=0}^{\infty}(-1)^{j+k}\binom{\beta-1}{j}\binom{-\alpha(1+j)-1}{k}x^{\frac{1}{\lambda}(1+k)-1} \quad (7)$$

Equation (7) can be rewritten as:

$f(x;\alpha,\beta,\lambda) = \sum_{k=0}^{\infty} \psi_j \, x^{\frac{1}{\lambda}(1+k)-1}$ (8)

Where,

$\psi_j = \frac{\alpha\beta}{\lambda} \sum_{j=0}^{\infty} (-1)^{j+k} \binom{\beta-1}{j} \binom{-\alpha(1+j)-1}{k}$

Statistical Properties

In this section, some of the statistical properties of the new distribution are derived as follows.

Moments

Suppose a random variable X follows MIK- distribution, then the r^{th} moment is obtained as:

$E(x^r) = \int_0^{\infty} x^r \, f(x,\alpha,\beta,\lambda) dx$ (9)

Substituting equation (4) in equation (9), gives:

$E(x^r) = \frac{\alpha\beta}{\lambda} x^{\frac{1}{\lambda}-1} \sum_{j=0}^{1} (-i)^j \binom{\beta-1}{j} \int_0^1 x^r \, (1 + x^{\frac{1}{\lambda}})^{-\alpha(1+j)-1} dx$

let$y = 1 + x^{\frac{1}{\lambda}}, \frac{dy}{dx} = \frac{1}{\lambda} x^{\frac{1}{\lambda}-1}, dx = \frac{\lambda dy}{x^{\frac{1}{\lambda}-1}}$

$= \alpha\beta \sum_{j=0}^{\infty} (-i)^j \binom{\beta-1}{j} \int_0^{\infty} (1 - y)^{\lambda r} \, (y)^{-\alpha(1+j)-1} dy$

Then, the moments are obtained as follows:

$E(x^r) = \alpha\beta \sum_{j=0}^{\infty} (-i)^j \binom{\beta-1}{j} B(\lambda r - 1, -\alpha(1+j))$ (10)

Equation (10) is the moments of the MIK distribution.

Moment generating function (mgf)

The moment-generating function can be obtained using equation (4) as follows;

$M_X(t) = \int_0^{\infty} e^{tx} \, f(x;\alpha,\beta,\lambda) \, dx, \qquad x > 0$ (11)

But $e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!}$ (12)

Substituting equation (12) in equation (11), we get:

$$M_X(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^m f(x;\alpha,\beta,\lambda) \, dx \quad (13)$$

Substituting equation (10) in equation (13), we have:

$$M_X(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \alpha\beta \sum_{j=0}^{\infty} (-i)^j \binom{\beta-1}{j} B(\lambda r - 1, -\alpha(1+j))$$

Entropy

This measures the dynamic uncertainty of the probability distribution.

$I_\theta = \frac{1}{1-\theta} log \int_0^{\infty} (f(x,\alpha,\beta,\lambda))^\theta \, dx, \quad for \ \ \theta \neq 1, \theta > 0$ (12)

$I_\theta = \frac{1}{1-\theta} log \int_0^{\infty} (\frac{\alpha\beta}{\lambda} x^{\frac{1}{\lambda}-1} (1 + x^{\frac{1}{\lambda}})^{-\alpha-1} (1 - (1 + x^{\frac{1}{\lambda}})^{-\alpha})^{\beta-1})^\theta \, dx$ (13)

Survival function

The survival function s(x) is a function that gives the probability that a patient, device, or other object of interest will survive after a given time.it is also known as the reliability function. Its mathematical expression is given as:

$S(x) = 1 - F(x,\alpha,\beta,\lambda)$ (14)

$S(x) = 1 - (1 - (1 + x^{\frac{1}{\lambda}})^{-\alpha})^{\beta}$ (15)

Hazard function

It's also known as failure rate and is obtained using the relation.

$h(x) = \frac{f(x,\alpha,\beta,\lambda)}{1-F(x,\alpha,\beta,\lambda)} = \frac{f(x,\alpha,\beta,\lambda)}{s(x)}$

$h(x) = \frac{\frac{\alpha\beta}{\lambda} x^{1/\lambda-1} (1+x^{1/\lambda})^{-\alpha-1} (1-(1+x^{1/\lambda})^{-\alpha})^{\beta-1}}{1-(1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta}}$ (16)

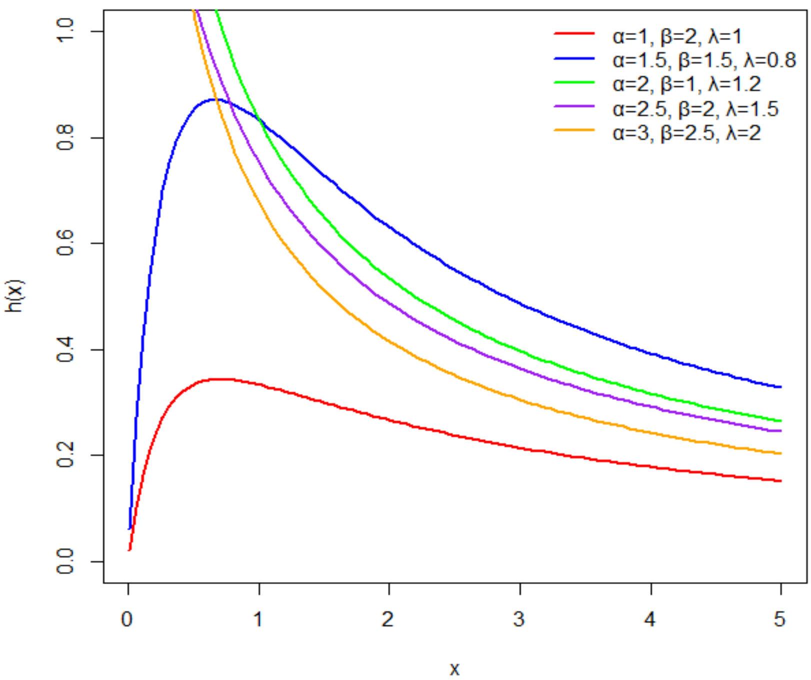

Figure 2: PDF Plot of the Modified Inverted Kumaraswamy Distribution

Figure 2 shows the hazard functions, denoted as $h(x)$, for different parameter values of $\alpha, \beta, and\ \lambda$. Each curve illustrates how the hazard function changes with different parameter values, which is useful in survival analysis and reliability engineering to understand the failure rates of systems or the risk of events over time.

Quantile Function

The Quantile function of the MIKdistribution can be obtained using the CDF. Let F be a CDF of the MIKdistribution, we define the quantile function Q(u) by:

$$Q(u) = F^{-1}(u)$$

$$Q(u) = x = ((1 - u^{\frac{1}{\beta}})^{-\frac{1}{\alpha}} - 1)^{\lambda} \tag{17}$$

Order Statistics

Order statistics help in understanding the position or rank of data points within a sample. $f_{1:n}(x) = \frac{f(x)}{B(r,n-r+1)}\sum_j^{n-r}(-1)^v\binom{n-r}{j}[F(x)]^{n+r-1}$ (18) $f_{n:r}(x) = \frac{\alpha\beta\sum_{j=0}^{\infty}(-i)^j\binom{\beta-1}{j}B(\lambda r-1,-\alpha(1+j))}{B(r,n-r+1)}\sum_j^{n-r}(-1)^v\binom{n-r}{j}\left[\sum_{i=1}^{\infty}(-i)^i\binom{\beta}{i}(1+x^{\frac{1}{\lambda}})^{-\alpha i}\right]^{n+r-1}$ (19)

Parameter Estimation

In this section, the Maximum Likelihood Estimate (MLE) will be used to determine the parameter of the proposed distribution.

Maximum Likelihood Estimation

The maximum likelihood method is the predominant technique for estimating parameters in a model. Suppose that X is a random variable with a probability density function $f(x;\theta)$where θ is a single unknown parameter. Let $X_1, X_2, \ldots, X_n$ be the observed values in a random sample of size n. In such a case, the function is expressed as

$$L(\theta) = \underset{i=1}{\overset{x}{\pi}} f(x;\theta)$$

$$L = n\log(\alpha) + n\log(\beta) - n\log(\lambda) + \left(\frac{1}{\lambda} - 1\right)\sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(1 + x_i^{\frac{1}{\lambda}}\right) + (\beta-1)\sum_{i=1}^{n}\log\left[1 - \left(1 + x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right] \tag{20}$$

Differentiating L with respect to α, β, and λ

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(1 + x_i^{\frac{1}{\lambda}}\right) - (\beta-1)\sum_{i=1}^{n}\frac{\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha}\log\left(1+x_i^{\frac{1}{\lambda}}\right)}{\left[1-\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right]} = 0 \tag{21}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left[1 - \left(1 + x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right] = 0 \tag{22}$$

$$\frac{\partial L}{\partial \lambda} = -\frac{n}{\lambda} - \frac{1}{\lambda^2}\sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\lambda}}\log x_i}{\lambda^2\left(1+x_i^{\frac{1}{\lambda}}\right)} - (\beta+1)\sum_{i=1}^{n}\frac{\alpha\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha-1}x_i^{\frac{1}{\lambda}}\log x_i}{\lambda^2\left[1-\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right]} = 0 \tag{23}$$

Equations (21), (22) and (23) are non-linear, and cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

RESULTS AND DISCUSSION

First Dataset

The first dataset shown below represents 63 observations of the strengths of 1.5cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The data sets are as follows (Wani and Shafi, 2021):

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

Table 1: The Estimates, Log-likelihoods and Goodness of Fits Statistics of the models based on strengths of 1.5cm glass fibres

Model	λ	α	β	LL	AIC
MIK	0.3783	1.8828	7.506	-35.5531	77.1062
TIHLIK	3.8737	0.7367	11.178	-45.4661	96.9322
MOKEIK	1.8737	3.3317	4.512	- 126.1284	258.2568
IK	-	5.6713	110.749	-38.9397	81.8794

Table 1 displays the outcomes of the Maximum Likelihood Estimation for the MIK distribution and three comparator distributions. The MIK demonstrated the lowest AIC value of 77.1062, indicating its superior fit to the failure times of the models based on strengths of 1.5cm glass fibres compared to the other distributions considered.

Second Dataset

The second dataset consists of Survival times (in months) of a sample of 101 patients with Advanced Acute myelogenous leukaemia. The datasets are as follows (Yakubu and Doguwa, 2017):

0.03, 8.882, 41.118, 6.151, 17.303, 0.493, 9.145, 45.033, 6.217, 17.664, 0.855, 11.48, 46.053, 6.447, 18.092, 1.184, 11.513, 46.941, 8.651, 18.092, 1.283, 12.105, 48.289, 8.717, 18.750, 1.48, 12.796 ,57.401, 9.441, 20.625, 1.776, 12.993, 58.322, 10.329, 23.158, 2.138, 13.849, 60.625, 11.48, 27.73, 2.5, 16.612, 0.658, 12.007, 31.184, 2.763, 17.138, 0.822, 12.007, 32.434, 2.993, 20.066, 1.414, 12.237, 35.921, 3.224, 20.329, 2.5, 12.401, 42.237, 3.421, 22.368, 3.322, 13.059, 44.638, 4.178, 26.776, 3.816, 14.474, 46.48, 4.441, 28.717, 4.737, 15, 47.467, 5.691, 28.717, 4.836, 15.461, 48.322, 5.855, 32.928, 4.934, 15.757, 56.086, 6.941, 33.783, 5.033, 16.48, 6.941, 34.211, 5.757, 16.711, 7.993, 34.77, 5.855, 17.204, 8.882, 39.539, 5.987, 17.237.

Table 2: The Estimates, log-likelihoods, and goodness of fit of the models based on the survival time of patients with leukaemia

Model	λ	α	β	LL	AIC
MIK	1.8961	1.8682	10.0338	-413.0059	826.0118
TIHLIK	44.5965	0.1018	2.4515	-499.201	1004.402
MOKEIK	2.0782	6.6123	1.6743	-415.7044	831.4088
IK	-	0.8056	3.9796	-470.8729	947.7458

Table 2 reveals the outcomes of the maximum likelihood estimation for the MIK distribution and three comparator distributions. The MIK distribution exhibited the lowest AIC value at 826.0118, signifying its superior fit to the survival times of leukaemia patients. This implies that the MIK distribution is the most appropriate model among the considered distributions
for capturing the characteristics of the dataset.

CONCLUSION

This paper developed a new distribution and derived its mathematical properties, including moments, moment-generating functions, order statistics and reliability measures. Parameter estimation utilized both Maximum Likelihood Estimation (MLE). Evaluating the proposed model against comparable distributions using the Akaike Information Criterion (AIC) consistentlydemonstrated its superior fit. This indicates the model's proficiency in capturing diverse characteristics of the dataset. Furthermore, this demonstration emphasized the positive impact of introducing additional parameters, enhancing overall distribution fit, and versatility in modelling various datasets of different shapes.

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