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ON THE LOMAX-UNIT TEISSIER DISTRIBUTION: PROPERTIES AND ITS APPLICATIONS

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ABSTRACT

In this paper, a new modified continuous asymmetric probability distribution Lomax-Unit Teissier distribution (LxUTD) which extends the baseline Unit Teissier (UTD) distribution was developed. However, the new distribution is capable of handling an asymmetric data sets. Some statistical properties like moments, moment generating function, renyi entropy, quantile function, and order statistics was derived and presented theoretically. We also, proceed to test the validity of the new constructed continuous asymmetric probability distribution. A simulation was conducted to determine the efficiency of the estimated value by increasing the sample size. The result shows that when simulated or real life datasets are being used the new asymmetric probability distribution will give better fit than the competitive models.

Keywords: Lomax, Teissier distribution, Asymmetric, Simulation

INTRODUCTION

The asymmetric statistical distributions are commonly applied to describe real-world phenomena. Due to the usefulness of the asymmetric continuous probability distributions, their theory is widely studied and new modified distributions are developed. The idea behind the development of more flexible continuous probability distributions remains high in the statistics field. More generalized categories of continuous distributions have been developed and applied to describe various field.

The common properties of these asymmetric continuous probability distribution is that, they contain more statistical parameters. The quality of any good continuous probability distribution is basically really on fitting the assumed probability distribution to the random variable or data. Moreover, there are situation where any of these continuous probability distributions do not fit some complex data appropriately, especially in finance, medicine, engineering, and environmental situation. The use of appropriate probability distributions to be applied on real-life situation plays a vital role in order to the improving the quality, efficiency, power, and sensitivity of statistical tests. The distribution that will fit the data appropriately is term as effective model. Therefore, good knowledge of the suitable distribution to be used for an identified data set is absolutely important. Asymmetrical continuous probability distributions are very useful in data analysis. They can be apply to analyze a great range of data shapes in applied areas.

In the statistical literature of continuous probability distributions is important with various continuous distributions and still improving rapidly. Various extensions of some well-defined continuous distributions have been developed during the last three decades for modeling of types of real life situation that form different nature. The improvement is attain to examine various techniques for generating/modifying new lifetime or existing distributions due to baselines once have different distributions and hence gives the probabilityin analyzing a complex data. The idea used to introduce new asymmetrical distributions are basically term as as connectors in the statistical review and are they can be able give better fit than the baseline models. Some well-known family generators include Lomax-G by Cordeiro et al.(2013), Kumaraswamy Marshall-Olkin by Alizadeh et al.(2015), Generalized transmuted family by Alizadeh et al.(2016), Another generalized transmuted family by Merovci et al.(2016), Transmuted geometric G family by Afify et al.(2016), Beta transmuted-H family by Afify et al.,(2017), Kumaraswamy transmuted-G family by Afify et al.(2016), Topp–Leone Family of Distributions by Al-Shomrani et al.,(2016), The extended Weibull-G family of distributions by Kokmaz (2018). The aim of this research paper is to propose new modified probability distributions that are flexible and adaptive in modeling datasets that are asymmetric in nature

MATERIALS AND METHODS

The Unit Teissier distribution

Krishna *et al*., (2022) proposed the one parameter namely unit Teissier distribution. The UTD is dcomputed from the transformation $Z = e^{k}$using Teissier distribution as the baseline distribution. The UTD serves as the extension of Teissier distribution by the French biologist Georges Teissier (1934). A random variable X is said to follow the UTD with one scale parameter$\theta > 0$, if its CDF is of the following form:

$$F(x;\theta) = x^{-\theta}\, exp\left(-x^{-\theta} + 1\right), \qquad x \in (0,1) \quad (1)$$

The corresponding pdf is given by

$$f(x;\theta) = \theta\left(x^{-\theta} - 1\right)x^{-(\theta+1)}\, exp\left(-x^{-\theta} + 1\right) \quad (2)$$

From equations (1) and (2), the survival function $S(x;\theta)$, hazard rate function $h(x;\theta)$ and reversed hazard rate function $\tau(x;\theta)$ of UTD are obtained as follows.

$$S(x;\theta) = 1 - x^{-\theta}\, exp\left(-x^{-\theta} + 1\right) \quad (3)$$

$$h(x;\theta) = \frac{\theta(x^{-\theta}-1)x^{-(\theta+1)}\, exp(-x^{-\theta}+1)}{1-x^{-\theta}\, exp(-x^{-\theta}+1)} \quad (4)$$

$$\tau(x;\theta) = \frac{\theta(x^{-\theta}-1)x^{-(\theta+1)}\, exp(-x^{-\theta}+1)}{x^{-\theta}\, exp(-x^{-\theta}+1)} \quad (5)$$

New Lomax -G Family

Sapkota *et al*., (2023) proposed a family of distribution called the New Lomax-G family of distribution which serves as the extension of Pareto type II (Lomax) distribution so that its support begins at zero by Lomax (1954).

The CDF and PDF of the proposed new family New Lomax-G Family distribution are given in (6) and (7) respectively.

$$G(x;\alpha,\beta) = \beta^{\alpha}[\beta - log(F(x))]^{-\alpha} \quad \text{and,} \quad (6)$$

$$g(x;\alpha,\beta) = \alpha\beta^{\alpha} f(x)F(x)^{-1}[\beta - log(F(x))]^{-(\alpha+1)} \quad (7)$$

Where $\alpha > 0$and$\beta > 0$are shape and scale parameters respectively.From equations (3) and (4), the survival function

$S(x;\alpha,\beta)$, hazard rate function $h(x;\alpha,\beta)$ and reversed hazard rate function $\tau(x;\alpha,\beta)$ of the LG family are obtained as follows.

$$S(x;\alpha,\beta) = 1-\beta^{\alpha}[\beta - log(F(x))]^{-\alpha} \quad (8)$$

$$h(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}f(x)F(x)^{-1}[\beta-log(F(x))]^{-(\alpha+1)}}{1-\beta^{\alpha}[\beta-log(F(x))]^{-\alpha}} \quad (9)$$

$$\tau(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}f(x)F(x)^{-1}[\beta-log(F(x))]^{-(\alpha+1)}}{\beta^{\alpha}[\beta-log(F(x))]^{-\alpha}} \quad (10)$$

Where$x > 0\alpha > 0$ and $\beta > 0$ are the Shape and Scale parameters respectively.

The New Modified Distribution

Lomax-Unit Teissier Distribution

To obtain the CDF and PDF of the Lx-UT distribution, let the CDF and PDFof the Unit Teissier distribution given in (1) and (2) are then substitute in (6) and (7) respectively.

Then the CDF of Lx-UTD is obtained as in (11).

$$G(x;\alpha,\beta,\theta) = \beta^{\alpha}\left[\beta - log\left(x^{-\theta}\,exp\left(-x^{-\theta}+1\right)\right)\right]^{-\alpha}$$

and, (11)

The corresponding PDF is given by (12)

$$g(x;\alpha,\beta,\theta) = \alpha\beta^{\alpha}\frac{\theta(x^{-\theta}-1)x^{-(\theta+1)}\,exp(-x^{-\theta}+1)}{x^{-\theta}\,exp(-x^{-\theta}+1)}\left[\beta - log\left(x^{-\theta}\,exp\left(-x^{-\theta}+1\right)\right)\right]^{-(\alpha+1)} \quad (12)$$

Where $x > 0$ and$\theta > 0$ is the scale parameter and $\alpha,\beta > 0$are the shape parameters respectively.

Proof:

To prove that, the CDF for new Lx-UTD we take the integral of the PDF (12) with respect to x.

$$G_{Lx-UT}(x;\alpha,\beta,\theta) = \int_0^{G(x)} g(x;\alpha,\beta,\theta)dx \quad (13)$$

$$G(x;\alpha,\beta,\theta) = \int_0^{T(x)} \alpha\beta^{\alpha}\frac{\theta(x^{-\theta}-1)x^{-(\theta+1)}\,exp(-x^{-\theta}+1)}{x^{-\theta}\,exp(-x^{-\theta}+1)}\left[\beta - log\left(x^{-\theta}\,exp\left(-x^{-\theta}+1\right)\right)\right]^{-(\alpha+1)}dx \quad (14)$$

Let

$$y = x^{-\theta}\,exp\left(-x^{-\theta}+1\right) = uv$$

We differentiate y using the product rule given by;

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad u = x^{-\theta}, \quad v = e^{-x^{-\theta}+1}$$

$$u\frac{dv}{dx} = \theta x^{-\theta-1}x^{-\theta}e^{-x^{-\theta}+1}, \quad v\frac{du}{dx} = -\theta x^{-\theta-1}e^{-x^{-\theta}+1}$$

$$\frac{dy}{dx} = \theta x^{-2\theta-1}e^{-x^{-\theta}+1} - \theta x^{-\theta-1}e^{-x^{-\theta}+1}$$

$$= \theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}$$

$$dx = \frac{dy}{\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}}$$

$$\int_0^{F(x)} \alpha\beta^{\alpha}y^{-1}(\beta - log(y))^{-(\alpha+1}\,dy \quad (15)$$

Let

$k = \beta - log(y)$, $\frac{dk}{dy} = -\frac{1}{y}$, Therefore, $dy = -\frac{1}{y^{-1}}dk$

We then substitute for k and dyin (15) which implies (16)

$$\int_0^{\beta-log(k)} \alpha\beta^{\alpha}(k)^{-(\alpha+1}\,dk \quad (16)$$

When $y = x^{-\theta}\,exp\left(-x^{-\theta}+1\right)$ then

$$k = \beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)$$

$$= -\frac{\alpha\beta^{\alpha}\left[k^{-(\alpha+1)+1}\right]_0^{\beta-log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)}}{-(\alpha+1)+1}$$

$$G(x;\alpha,\beta,\theta) = \beta^{\alpha}\left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-\alpha} \quad (17)$$

Where $x > 0$ and$\theta > 0$ is the scale parameter and $\alpha,\beta > 0$are the shape parameters respectively.

To obtain the corresponding PDF we differentiate the CDF (17) with respect to x.

$$y = \left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-\alpha},$$

$$u = \beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right), \quad k = \left(x^{-\theta}e^{-x^{-\theta}+1}\right)$$

$y = u^{-\alpha}, u = \beta - log\,k,\quad k = \left(x^{-\theta}e^{-x^{-\theta}+1}\right) = uv$, $\frac{dy}{du} = -\alpha u^{-\alpha-1}$, $\frac{du}{dk} = -\frac{1}{k}$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \theta x^{-\theta-1}x^{-\theta}e^{-x^{-\theta}+1} - \theta x^{-\theta-1}e^{-x^{-\theta}+1}$$

That is, $\frac{dk}{dx} = \theta\left(x^{-\theta}-1\right)x^{-\theta-1}e^{-x^{-\theta}+1}$

$$\frac{dy}{dx} = \frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}}{\left(x^{-\theta}e^{-x^{-\theta}+1}\right)}\left(\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right)^{-(\alpha+1)}$$

Therefore,

$$f_{L-UTD}(x;\alpha,\beta,\theta) = \frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}}{\left(x^{-\theta}e^{-x^{-\theta}+1}\right)}\left(\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right)^{-(\alpha+1)} \quad (18)$$

Where $x > 0$ and$\theta > 0$ is the scale parameter and $\alpha,\beta > 0$are the shape parameters respectively.

Validity Check for Lomax-Unit TeissierDistribution

To ensure the proposed probability distribution function (PDF) of the new L-UTD is valid, it must satisfy the fact that;

$$\int_0^{\infty} f_{L-UTD}(x;\alpha,\beta,\theta) = 1 \quad (19)$$

$$\int_0^{\infty}\frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}}{\left(x^{-\theta}e^{-x^{-\theta}+1}\right)}\left(\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right)^{-(\alpha+1)}dx = 1 \quad (20)$$

Let

$y = \beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)$, and $k = x^{-\theta}e^{-x^{-\theta}+1}$, $\Rightarrow y = \beta - log\,k$, $k = uv$

$\frac{dy}{dk} = -\frac{1}{k}$, $\frac{dk}{dx} = \theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}$, and $dx = \frac{dk}{\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}}$

$$\int_0^{\infty}\frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}\left(x^{-\theta}e^{-x^{-\theta}+1}\right)^{-1}(\beta-log\,k)^{-(\alpha+1)}}{\left(x^{-\theta}e^{-x^{-\theta}+1}\right)}\frac{dk}{\theta(x^{-\theta}-1)x^{-\theta-1}e^{-x^{-\theta}+1}} \quad (21)$$

$$\int_0^{\infty} -\alpha\beta^{\alpha}(\beta - log\,k)^{-(\alpha+1)}dk \quad (22)$$

$$= \frac{-\alpha\beta^{\alpha}(\beta - log\,k)_0^{\infty}}{-\alpha}$$

$$\int_0^{\infty} f(x;\alpha,\beta,\theta) = (1-(0)) - (1-(1)) = 1$$

Hence, the model in equation (12) is a valid probability density function

Survival Function, Hazard, and Reverse Hazard Rate Functions for Lomax-Unit Teissier Distribution.

To obtain the survival function we substitute (1) into (8) which will give us (23).

$$S(x;\alpha,\beta) = 1 - \beta^{\alpha}\left[\beta - log\left(x^{-\theta}\, exp\left(-x^{-\theta}+1\right)\right)\right]^{-\alpha} \quad (23)$$

Then to obtain the corresponding hazard rate function is by substitute (1) and (2) into (9) respectively.

$$h(x;\alpha,\beta,\lambda,\theta) = \frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-(\theta+1)}\, exp(-x^{-\theta}+1)\left(x^{-\theta}e^{-x^{-\theta}+1}\right)^{-1}\left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-(\alpha+1)}}{1-\beta^{\alpha}\left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-\alpha}} \quad (24)$$

Also, the reverse hazard rate function is by substituting (1) and (2) into (10) respectively.

$$\tau(x;\alpha,\beta,\lambda,\theta) = \frac{\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-(\theta+1)}\, exp(-x^{-\theta}+1)\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-(\alpha+1)}}{\beta^{\alpha}\left[\beta - log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right]^{-\alpha}} \quad (25)$$

Where $x > 0$and $\theta > 0$ is the scale parameter and $\alpha, \beta > 0$are the shape parameters

Linear Representation

Moments for Lomax-Unit Teissier distribution

Let X be a random variable that have a Lomax-Dagum-X family distribution, then the r^{th} moment about the origin, $E(X^r)$is given by

$$M_x(t) = E(X^r) = \int_0^{\infty} X^r f(x)dx \quad (26)$$

$$E(X^r) = \int_0^{\infty} X^r \alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-1}\left\{\beta - \log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1} dx \quad (27)$$

Let $A = \left\{\beta - \log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}$

$$A = \left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$A = \sum_{i=0}^{\infty}(-1)^i \binom{\alpha+i-1}{i}\beta^{-i}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^{-\theta j}e^{-y}$$

$$E(X^r) = \int_0^{\infty} X^r \alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-1}e^{-x^{-\theta}+1}\frac{\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i-1}{i}\beta^{-i}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^{-\theta j}e^{-y}}{-\theta x^{-\theta-1}j(x^{-\theta}+1)}dy \quad (28)$$

$$E(X^r) = \int_0^{\infty}\alpha\beta^{\alpha-i}\sum_{i=0}^{\infty}\binom{\alpha+i-1}{i}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!j}\left(\left(\frac{y-j}{j}\right)^{-\frac{1}{\theta}}\right)^{r+\theta+\theta j} e^{-y}\, dy \quad (29)$$

$$E(X^r) = \alpha\beta^{\alpha-i}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j}{j!j}\binom{\alpha+i-1}{i}\Gamma\left(1 - \frac{r+\theta+\theta j}{\theta}\right) \quad (30)$$

Moment Generating Function for Lomax-Unit Teissier distribution

Let X be a random variable that have a Lomax-Unit Teissier distribution, then the r^{th} moment about the origin, $E(e^{tx})$is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx \quad (31)$$

$$E(e^{tx}) = \int_0^{\infty} e^{tx}\alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-1}\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-(\alpha+1)} dx \quad (32)$$

Let

$$A = \left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-(\alpha+1)}$$

$$(1-\beta^{-1})^{-\alpha-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i}{i}\beta^{-i}$$

$$\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(x^{-\theta}e^{-x^{-\theta}+1}\right)^k$$

$$A = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i}{i}\beta^{-i}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^{-\theta k}e^{-y}$$

Where $e^{tx} = \sum_{v=0}^{\infty}\frac{(tx)^v}{v!}$

Therefore, the moment generating functioin (33) is obtained by applying gamma function

$$E(e^{tx}) = \sum_{v=0}^{\infty}\frac{(t)^v(-1)^{i+k}\alpha\beta^{\alpha-i}}{k!kv!}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+i}{i}\Gamma\left(1+\frac{v+\theta k}{\theta}\right) \quad (33)$$

Quantile Function for Lomax-Unit Teissier distribution

Let X be a random variable that has the CDF given in (11). The quantile function, $Q(u)$ of X can be derived as follows:

$$Q(u) = F^{-1}(u)$$

$$x = Q(u) = F^{-1}(u)$$

Let $F(x;\alpha,\beta,\theta) = \beta^{\alpha}\left\{\beta - \log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha}$

$$u = \beta^{\alpha}\left\{\beta - \log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha} \quad (34)$$

Divide both sides by β^{α}

$$\frac{u}{\beta^{\alpha}} = \left\{\beta - \log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha}$$

$$\left(\frac{u}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}} = 1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)$$

$$\left(\frac{u}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}} - 1 = -\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)$$

$$\frac{1-\left(\frac{u}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}} = logx^{-\theta}$$

$$log\left[\frac{1-\left(\frac{u}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}}\right] = x^{-\theta}$$

After simplification, the quantile function is express in (35) as

$$\text{Therefore,} Q(u) = x = log\left[\frac{1-\left(\frac{u}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}}\right]^{-\frac{1}{\theta}} \tag{35}$$

Where u is a uniform random number on the interval (0,1) and $G^{-1}(.)$ is the inverse function of $G(.)$. in particular, $Q(0.5)$ is the median of the family and defined by substituting $u = 0.5$ in equation (35):

$$Q(0.5) = x = log\left[\frac{1-\left(\frac{0.5}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}}\right]^{-\frac{1}{\theta}} \tag{36}$$

The first and third quartile can be obtained also by substituting $u = 0.25$ and $u = 0.75$, respectively in equation (35), as follows in (37) and (38).

$$Q(0.25) = x = log\left[\frac{1-\left(\frac{0.25}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}}\right]^{-\frac{1}{\theta}} \tag{37}$$

and

$$Q(0.75) = x = log\left[\frac{1-\left(\frac{0.75}{\beta^{\alpha}}\right)^{-\frac{1}{\alpha}}}{\beta^{-1}}\right]^{-\frac{1}{\theta}} \tag{38}$$

Order Statistics for Lomax-Unit Teissier distribution

The pdf $f_{i,n}(x)$ of the i^{th} order statistics for a random sample ($x_1 \dots \dots \dots . . x_n$) from the LUTD can be obtain by:

$$f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{39}$$

$$[1-F(x)]^{n-i} = \sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F(x)]^m$$

$$\binom{n-i}{m} = \frac{(n-i)!}{m!\,(n-i-m)!}$$

$$f_{i,n}(x) = \sum_{m=0}^{n-i}\frac{n!}{(i-1)!m!(n-i-m)!} f(x)[F(x)]^{i+m-1}$$

Therefore,

$$f_{i,n}(x) = \sum_{m=0}^{n-i}\frac{n!}{(i-1)!m!(n-i-m)!} \alpha\beta^{\alpha}\theta(x^{-\theta}-1)x^{-1}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$\left[\beta^{\alpha}\left\{\beta-\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha}\right]^{i+m-1} \tag{40}$$

Let $\quad U = \left[\beta^{\alpha}\left\{\beta-\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha}\right]^{i+m-1}$

Therefore,

$$U = \sum_{i=0}^{\infty}\sum_{w=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{i+r+l+w}}{r!w!}\beta^{\alpha i}x^{\theta\alpha w-\theta l}(\alpha w)^r\binom{i+m-1}{i}\binom{r}{l} \tag{41}$$

Let

$$A = \left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$(1-\beta^{-1})^{-\alpha-1} = \sum_{=0}^{\infty}(-1)^i\binom{\alpha+i}{i}\beta^{-i}$$

$$\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(x^{-\theta}e^{-x^{-\theta}+1}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^{-\theta k}e^{-k(x^{-\theta}+1)}$$

$$e^{-k(x^{-\theta}+1)} = \sum_{h=0}^{\infty}\frac{(-1)^h}{h!}(k)^h\sum_{t=0}^{\infty}(-1)^t\binom{h}{t}x^{-\theta t}$$

Also, $\left(x^{-\theta}-1\right) = \sum_{n=0}^{\infty}(-1)^n x^{-\theta n}$

Let $z = i+r+l+w+n+h+k+t,$ d$= \alpha i+\alpha-i$ and $c = \theta\alpha w-\theta k-\theta t-\theta n-\theta l-1$

$$f(x) = \alpha^r\theta w^r\frac{x^c(k)^h}{k!h!}\frac{\beta^d(-1)^z}{r!w!}\sum_{i=0}^{\infty}\sum_{w=0}^{\infty}\sum_{r=0}^{\infty}\sum_{h=0}^{\infty}\sum_{t=0}^{\infty}\binom{i+m-1}{i}(k)^h$$

$$\sum_{=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+i}{i}\binom{r}{l}\binom{h}{t} \tag{42}$$

$$f_{i,n}(x) = \sum_{m=0}^{n-i}\frac{n!}{(i-1)!m!(n-i-m)!}\alpha^r\theta w^r\frac{x^c(k)^h}{k!h!}\frac{\beta^d(-1)^z}{r!w!}$$

$$\sum_{i=0}^{\infty}\sum_{w=0}^{\infty}\sum_{r=0}^{\infty}\sum_{h=0}^{\infty}\sum_{t=0}^{\infty}\sum_{=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+i}{i}\binom{r}{l}\binom{h}{t}(k)^h\binom{i+m-1}{i} \tag{43}$$

We then substitute at $i = 1$ and $i = n$; that is minimum and maximum order of X_1 and X_n.

Renyi Entropyfor Lomax-Unit Teissier distribution

The entropy of a random variable $X$is one of the major properties of probability distribution. It's basically used to represents a measure of uncertainties obtained in a distribution. Let X be a random variable that has the PDF given in equation (12), then the Renyi entropy of the random variable X is defined as:

$$R_\lambda(x) = \frac{1}{1-\lambda} log\left[\int_0^\infty (f(x))^\lambda dx\right] \quad (44)$$

By inserting equation (12) into (44) which gives

$$R_\lambda(x) = \frac{1}{1-\lambda} log\left[\int_0^\infty \left(\alpha\beta^\alpha\theta(x^{-\theta}-1)x^{-1}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}\right)^\lambda dx\right]$$

$$R_\lambda(x) = \frac{1}{1-\lambda} log\left[\int_0^\infty \alpha^\lambda\beta^{\lambda\alpha}\theta^\lambda(x^{-\theta}-1)^\lambda x^{-\lambda}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\lambda(\alpha+1)} dx\right] \quad (45)$$

Let $V = \left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\lambda(\alpha+1)}$

$$=\sum_{i=0}^\infty(-1)^i\binom{\lambda(\alpha+i)}{i}\beta^{-i}\sum_{j=0}^\infty\frac{(-1)^j}{j!}\ x^{-\theta ij}\sum_{k=0}^\infty\frac{(-1)^k}{k!}\sum_{m=0}^\infty(-1)^m\,(ij)^k\binom{k}{m}x^{-\theta m}$$

$$\left(x^{-\theta}-1\right)^\lambda = \sum_{n=0}^\infty(-1)^n\binom{\lambda}{n}x^{-\theta n}$$

$$R_\lambda(x) = \frac{1}{1-\lambda} log\int_0^\infty \alpha^\lambda\beta^{\lambda\alpha}\theta^\lambda\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{i=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(-1)^{i+j+k+m+n}}{j!k!}x^{-(\theta ij+\theta m+\theta n-\lambda)}(ij)^k$$

$$\binom{\lambda(\alpha+i)}{i}\beta^{\lambda\alpha-i}\binom{\lambda}{n}\binom{k}{m}dx \quad (46)$$

$$C_1 = \alpha^\lambda\beta^{\lambda\alpha}\theta^\lambda\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{i=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(-1)^{i+j+k+m+n}}{j!k!}(ij)^k\binom{\lambda(\alpha+i)}{i}\beta^{\lambda\alpha-i}\binom{\lambda}{n}\binom{k}{m}$$

$$R_\lambda(x) = \frac{1}{1-\lambda} log\left[C_1\int_0^\infty x^{-(\theta ij+\theta m+\theta n-\lambda)}dx\right] \quad (47)$$

$$R_\lambda(x) = \frac{1}{1-\lambda} log\left[C_1\frac{1}{\theta ij+\theta m+\theta n-\lambda+1}\right]$$

$$R_\lambda(x) = \frac{1}{1-\lambda}\left[\frac{\theta ij+\theta m+\theta n-\lambda+1}{C_1}\right]$$

$$R_\lambda(x) = \frac{\theta ij+\theta m+\theta n-\lambda+1}{C_1(1-\lambda)} \quad (48)$$

Asymptotic Behavior/Limiting Function for Lomax-Unit Teissier distribution

The limit of the pdf of LUTD $f(x;\alpha,\beta,\theta)$ as $x\to\infty$ and as $x\to\theta$ is zero 0. This can be proved by taking the limit of $f(x;\alpha,\beta,\theta)$ as $x\to\infty\ and\ x\to\theta$.

That is; $\lim_{x\to\infty}f(x;\alpha,\beta,\theta) = \lim_{x\to 0}f(x;\alpha,\beta,\theta) = 0$

The proof

$$\lim_{x\to\infty}f(x;\alpha,\beta,\theta) = \lim_{x\to\infty}\alpha\beta^\alpha\theta(x^{-\theta}-1)x^{-1}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1} \quad (49)$$

$$= \alpha\beta^\alpha\theta(\infty^{-\theta}-1)\infty^{-1}\left\{1-\beta^{-1}\log\left(\infty^{-\theta}e^{-\infty^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$= 0$$

$$\lim_{x\to 0}f(x;\alpha,\beta,\theta) = \lim_{x\to 0}\alpha\beta^\alpha\theta(0^{-\theta}-1)0^{-1}\left\{1-\beta^{-1}\log\left(0^{-\theta}e^{-0^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$= \lim_{x\to 0}\alpha\beta^\alpha\theta(0^{-\theta}-1)0^{-1}\left\{1-\beta^{-1}\log\left(0^{-\theta}e^{-0^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$= 0$$

When the limit of the *pdf* for the distribution follow $x\to\infty$ and $x\to 0$. It proved that the *pdf* of LUTD have a mode (uni-modal).

Lemma II:

The limit of the CDF of LUTD $F(x;\alpha,\beta,\theta)$, as $x\to\infty$ is 1 and also as $x\to 0$ is 0.

That is,

$$\lim_{x\to\infty}f(x;\alpha,\beta,\theta) = 1$$

$$\lim_{x\to 0}f(x;\alpha,\beta,\theta) = 0$$

The proof

$$\lim_{x\to\infty}f(x;\alpha,\beta,\theta) = \lim_{x\to\infty}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1} \quad (50)$$

$$= 1-\beta^\alpha\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)^{-\alpha}$$

$$= 1-0$$

$$= 1$$

$$\lim_{x\to 0}f(x;\alpha,\beta,\theta) = \lim_{x\to 0}\left\{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1}$$

$$= 1-\beta^\alpha\log(0)$$

$$= 1-1$$

$$= 0$$

Parameter Estimation for Lomax-Unit Teissier distribution

Let $x_1 \ldots \ldots \ldots . . x_n$ be a random sample from the LUT-X family distributions with parameter $\alpha, \beta, \delta, \theta, \lambda$. The log-likelihood function for $\psi, says\ \mathcal{L} = \mathcal{L}(\psi)$.

$$\mathcal{L}(\psi) = \prod_{i=1}^{n} f(x; \alpha, \beta, \theta,) \tag{51}$$

$$\mathcal{L}(\psi) = \prod_{i=1}^{n} \alpha\beta^{\alpha}\theta(x^{-\theta} - 1)x^{-1}\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-\alpha-1} \tag{52}$$

$$= \alpha^{n}\beta^{n\alpha}\theta^{n}\sum_{i=1}^{n}(x^{-\theta} - 1)\sum_{i=1}^{n} x^{-1}\sum_{i=1}^{n}\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\}^{-(\alpha+1)}$$

By taking the log of the likelihood function

$$log\mathcal{L}(\psi) = nlog\alpha + \alpha nlog\beta + nlog\theta - \sum_{i=1}^{n} logx + \sum_{i=1}^{n} log\left(x^{-\theta} - 1\right)$$

$$-(\alpha + 1)\sum_{i=1}^{n} log\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\} \tag{53}$$

In order to maximize the log likelihood, we solve the nonlinear likelihood equation simultaneously obtained from the differentiation of (53) with respect to α, β, θ, as shown below.

$$\frac{\partial log\mathcal{L}(\psi)}{\partial\alpha} = \frac{n}{\alpha} + \frac{n}{\beta} - \sum_{i=1}^{n} log\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\} \tag{54}$$

$$\frac{\partial log\mathcal{L}(\psi)}{\partial\beta} = \frac{n\alpha}{\beta} - (\alpha + 1)\sum_{i=1}^{n} log\left\{1 - \beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)\right\} \tag{55}$$

$$\frac{\partial log\mathcal{L}(\psi)}{\partial\theta} = \frac{n}{\theta} - \frac{\theta\sum_{i=1}^{n} x^{-\theta-1}}{x^{-\theta}-1} - \frac{(\alpha+1)\sum_{i=1}^{n}\left(x^{-\theta-1}\theta(x^{-\theta}-1)e^{-x^{-\theta}+1}\right)}{1-\beta^{-1}\log\left(x^{-\theta}e^{-x^{-\theta}+1}\right)} \tag{56}$$

That is $\frac{\partial log\mathcal{L}(\psi)}{\partial\alpha} = 0, \frac{\partial log\mathcal{L}(\psi)}{\partial\beta} = 0, \frac{\partial log\mathcal{L}(\psi)}{\partial\theta} = 0$ which gives the estimate of the maximum likelihood estimate of the parameters.

RESULTS AND DISCUSSION

Plots CDF and PDF of Lomax-Unit Teissier distribution

Plots of cumulative distribution function and probability density function of Lomax-Unit Teissier distribution for selected/varying parameter values are given in Figures 3 and 4 and furthermore, for a similarity, $a = \alpha, b = \beta, and\ c = \theta$.

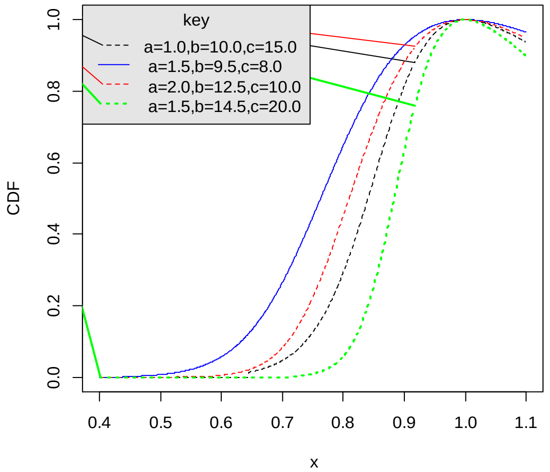

Figure 1: CDF plot of LUTD

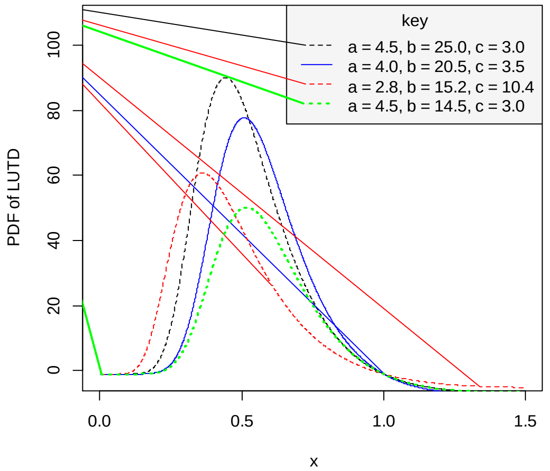

Figure 2: PDF plot of LUDT

The above Figure (1) displays the cumulative distribution function of L-UTD for different values of the shape and scale parameters. Figure (2) displays the density function of L-UTD for different values of the shape and scale parameters. It is both left and right skewed distribution and has different level kurtosis which shows the flexibility of the distribution for modeling asymmetric datasets.

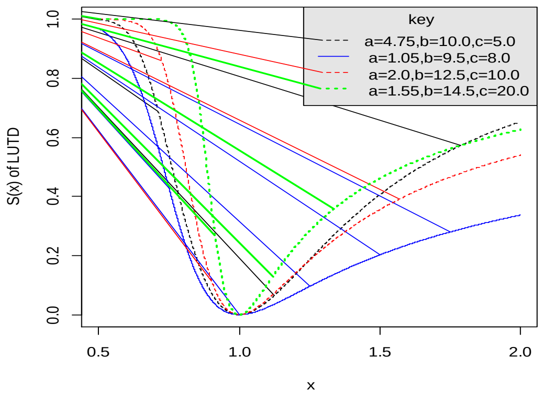

Figure 3: Survival plot of LUTD

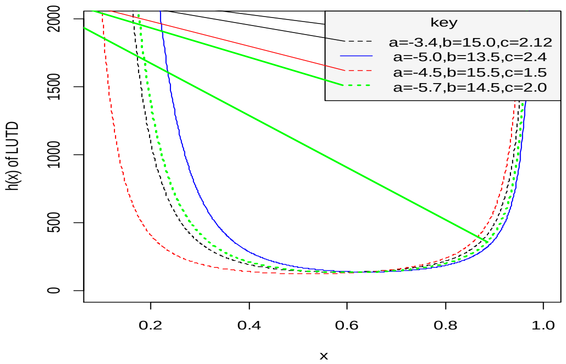

Figure 4: Hazard plot of LUTD

Also, figure (3) of survival function shows varying shapes decreasing, constant and reversed-J shapes. Whereas figure (4) of hazard rate plot show varying shapes like bathtub, reversed-J shape and constant failure rate.

Numerical Application of LxUTD and Goodness of Fit Statistics

In this part, the goodness of fit of LxUTD has been computed by using numerical example (datasets) which represents times between failures of secondary reactor pumps by Suprawhardana and Prayoto (1999).

We demonstrate the application of the developed models by the use of different criteria to compare all our developed distribution with other releted model were discussed.

Information criteria are used for model selection, particularly in the context of comparing different statistical models. They provide a means to evaluate the trade-off between model complexity and goodness of fit. In a situation where models are not nested, the likelihood ratio test become inappropate. To ovaercome this challenge, information criteria such as Akaike Information criteria (AIC), Bayesian Information criteria (BIC), Hannan Queinn Information criteria (HQIC), Consistent Akaike Information criteria (CAIC) are employed. The mathematical expression for AIC, BIC, HQIC and CAIC are given as:

Akaike Information Criteria

$$AIC = 2k - 2\ln(L) \quad (57)$$

Where k is the number of parameters and L is the likelihood of the model.

Bayesian Information Criteria (BIC)

$$BIC = \ln(n)\, k - 2\ln(L) \quad (58)$$

Where n is the number of observations. It also penalizes complexity but does not so more strongly than AIC

Both goodness of fit test and information criteria are essential tools in statistical modeling, helping researchers evaluate how well their models perform and choose among competing models effectively.In comparing the vaues of the information criteria AIC, BIC, HQIC and CAIC for each model or distribution, the model with least values of these criteria is regarded as the best model among the competing once.

Data 1:This data was previously used by Musa et.al (2021)
0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392.

Table 1: The MLEs of the LUTD parameters

Models	Parameter	Parameter	Parameter	P-Value
LUTD	a = 0.629398	b = 7.841393	c = 2.300553	5.16e-09
UTD	a = 0.3625	-	-	0.7341
LLD	-	-	c = 2.12	0.5573
TLEx	-	b = 0.0070	c = 1.2220	-
UBD	a = 0.1639	b= 2.4273	-	0.1688
ETLD	a = 0.6567	b = 1.6566	-	0.6536

Table 2: Goodness of Fit Test for LUTD compared with other models for Datasets I

Models	AIC	CAIC	BIC	HQIC
LUTD	-390.3034	-389.7181	-384.8835	-388.2829
UTD	-250.3635	-230.3579	-240.1678	-243.889
LLD	-244.3731	-222.7812	-7.3517	-
TLEx	-274.4007	-	-	-
UBD	-31.0588	-30.4588	-28.7878	-
ETLD	-36.3418	-26.7878	-34.0708	

Data sets II: The datasets consist of the annual flood discharge of the North Saskatchew in units of 1000 f/second of river at Edmonton over 48 years which is reported by Van Montfort (1970).
19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39..200, 40.000, 40..400, 40.400, 42.250, 44.020, 44.730, 44.900, 46..300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75..800, 84.100, 106.600, 109.700, 121.970, 185.560

Table 3: Goodness of Fit Test for LUTD compared with other models for Datasets II

Models	AIC	AICc	BIC	HQIC
LUTD	-431.3917	-429.8917	-428.4045	-32.81144
UTD	-25.1635	-23.1679	-24.1678	-
LLD	-9.3431	-	-7.3517	-
TLEx	174.4007	-	-	-
UBD	-31.0588	-30.4588	-28.7878	-
ETLD	-36.3418	-26.7878	-34.0708	

Simulation Study of the Models

The simulation studies conducted for LUTD has the samples 5000 that was generated of varying sizes, $n = 10, 50, and\ 100$, for the arbitrary choice of parameters for $\alpha = 0.0004$, $\beta = 0.0003$ and $\theta = 0.0002$ while another sizes $n = 200,\ 300, and\ 700$, for the true parameter values of $\alpha = 0.00002$, $\beta = 0.00004$ and $\theta = 0.00003$ as shown in Table 4 and 5 respectively.

Table 4: Estimate of parameter, bias, MSE, and Variance

	$\alpha = 0.0004$	$\beta = 0.0003$	$\theta = 0.0002$		
Sample	**Parameters**	**Estimate**	**Bias**	**MSE**	**Variance**
	α	1601705	1601725	5.305008e + 16	2.73954e + 16
10	β	2622820	2622820	2.764835e + 17	2.07691e + 17
	θ	1.042554	1.04233	2.247208	1.160746
	α	2549051	2549052	1.192739e + 17	5.4297e + 16
50	β	2896627	2896627	3.415831e + 19	2.576786e + 19
	θ	0.898483	0.898286	0.9392732	0.1323616
	α	24502276	24502273	2.068785e+19	1.468424e+19
100	β	4.968378	4.968378e +12	1.48095e + 26	1.234111e+26
	θ	0.790053	0.790053	0.9560469	0.3321792

Table 5: Estimate of parameter, bias, MSE, and Variance

	$\alpha = 0.00002$	$\beta = 0.00004$	$\theta = 0.00003$		
Sample	**Parameters**	**Estimate**	**Bias**	**MSE**	**Variance**
	α	0.06841382	0.06839282	0.00474858	7.086507e - 05
200	β	508.4523	508.4523	901383	642859.3
	θ	1.222264	1.222234	1.509834	0.01597703
	α	0.0523367	0.05231666	0.0028837	0.00014670
300	β	116.5617	116.5617	24582.72	10996.1
	θ	1.218703	1.218673	1.500924	0.0157594
	α	837813	837813	4.211589e+18	3.509657e+18
700	β	157135.7	157135.7	13486466	1.10173e+11
	θ	0.805852	0.805852	0.7232361	0.0738868

Discussion

Figure (1) displays the cumulative distribution function of LxUTD for different values of the shape and scale parameters. Figure (2) displays the density function of LxUTD for different values of the shape and scale parameters. It is both left and right skewed distribution and has different level kurtosis which shows the flexibility of the distribution for modeling asymmetric datasets.

Table 1 represents the MLEs of the computed parameters. Also, table 2 shows the different models selection criteria, it can be observed that LxUTD has the lowest values which makes best fit for the above data.

CONCLUSION

The Lomax-Unit Teissier LxUTD distribution has been introduced in this study. Moment, Moment generating function, order statistics, reliability analysis and quantile functions was derived. Also, method of parameter estimations (MLEs) and its application was computed using real life numerical data sets. Hence, looking at the output presented, Lomax-Unit Teissier LxUTD distribution perform better than compared models with it.

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