FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 8 No. 6, December, (Special Issue) 2024, pp 461 - 465
DOI: https://doi.org/10.33003/fjs-2024-0806-3036

EULER POLYNOMIALS COLLOCATION METHOD FOR SOLVING LANE-EMDEN EQUATIONS

***Kyato, D. D. and Aboiyar, T.**

Joseph Sarwuan Tarka University Makurdi, P.M.B. 2373 Makurdi, Benue state, Nigeria.

*Corresponding authors' email: Kyatodickson@gmail.com; aboiyar@kiu.ac.ug

ABSTRACT

In this paper, Euler polynomials were utilized to formulate a collocation method for approximating Lane-Emden equations. The approximation formulations were initialized producing truncation function. The residual functions were constructed by using the truncation function. The collocation points were substituted into the residual function to form system of equations and were solved by Matlab application fsolve or Newton-Raphson methods. The results were tabulated and compared with the Herodt (2004) results for absolute error. It was observed that the Euler polynomial is very accurate and converges faster producing zero error as compared. The study recommended in solving Lane-Emden and higher order equations.

Keywords: Lane-Emden equation, Euler Polynomials, Collocation method, Truncation function, Residual function, System of equations

INTRODUCTION

Lane –Emden equation is nonlinear second order differential equation. The equation has many applications in sciences and classified as singular initial value problem. The components of the equation are dimensionless radius, normalized density and the polytrophic index of the circular stars. The variables are related to temperature, pressure, mass and specific heat of the stars. The singular point distorts the solution at origin. Presently series and collocation techniques are used for solving the equation. Here Euler polynomials are utilized in developing a collocation method for approximating Lane-Emden equation.

The Lane-Emden equations are described by a second order ordinary differential equation. The equations include theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, radioactivity cooling, self-gravitating gas clouds and clusters of galaxies (Taghavi, 2013). Recently, many methods for solving the Lane-Emden equations are considered by various authors. However, the major problem comes from the singularity of the equation at the origin. The methods used for solving the equations were series solutions commonly referred to as semi-analytical methods. Collocation methods are effective for approximating the equations because of the highly accurate solutions provided as stated by (Gholanmeza and Parand , 2011). This method simplifies the equations into solvable system of equations. The existence of singularities among the equations makes the solutions lose its accuracy at $x = 0$. Therefore, suitable treatment of singularities is necessary for making the solutions accurate. The work is aimed at developing a collocation method for approximating the various equations using Euler Polynomials as basis function.

Various attempts have been made to solve the equations. Horedt (2004) presented the solutions of all Lane-Emden equations using Bessel functions. Pandey *et al*. (2016) introduced an approximating schemes using Legendre operational matrix of derivatives for computing Lane-Emden equation. The result was fairly accurate. Parand and Hashemi (2018) presented a solution of non-homogeneous Lane-Emden equations by approximating the derivatives. The results when compared with the other numerical methods show that the method is fairly efficient and accurate while Ben-Romdhane and Temimi (2018) modeled an iterative method to solve standard Lane-Emden equation. The result was fairly accurate. Scadatmundi *et al*. (2019) presented collocation methods using Berstein polynomials for solving fractional Lane-Emden equations. The result was fairly accurate. Sahu and Mallick (2019) solved Lane-Emden equation using Bernoulli polynomials. Examples were illustrated and the present method was fairly accurate. Morteza (2020) presented a numerical method to solve Lane-Emden equations. The equation has been reduced to system of equations using the presented method and the results were accurate. Waheed and Zulgurnain (2020) designed a Bernoulli collocation method for approximating standard Lane-Emden equation. Some examples were illustrated and results compared favorably with analytical solution.

In this work, the Euler polynomials are used to develop a collocation method for approximating the Lane-Emden equations with a view of eliminating singularities of these equations

MATERIALS AND METHODS

Euler Numbers and Polynomials

Euler numbers are used to generate Euler polynomials. Catma (2013) defined Euler numbers E_n as stated in Equation (1)

$$\frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n \, . \tag{1}$$

Cosh= hyperbolic cosine. The odd Euler numbers are zeroes while the even numbers have positive or negative signs. The Euler numbers are also stated in the Taylor series in Equation(2).

Using idea that,

$$\cosh(t).\operatorname{sech}(t) = \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right)\left(\frac{\sum_{n=0}^{\infty}(-1)^{2n} E_{2n} t^{2n}}{(2n)!}\right) = 1 \tag{2}$$

Hence the first twelve Euler numbers are stated in Equation (3).

$$\left.\begin{aligned} E_0 &= 1 \\ E_2 &= -1 \\ E_4 &= 5 \\ E_6 &= -61 \\ E_8 &= 1385 \\ E_{10} &= -50521 \\ E_{12} &= 2702765 \end{aligned}\right\} \tag{3}$$

Euler polynomial $E_n(x)$, according to Grospava and Gradimir (2014) are defined by Equation (4).

$$\frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n. \quad (4)$$

The explicit formula for generating Euler polynomials is given by Equation (5).

$$E_n(x) = \sum_{n=0}^{\infty} \binom{n}{k} \frac{E_k}{2^k} (x - \tfrac{1}{2})^{n-k}. \quad (5)$$

where, E_k= Euler numbers defined by Catma (2013) as stated in Equation (3). Some Euler polynomials are provided in Equation (6).

$$\left.\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - \tfrac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{4} \\
E_4(x) &= x^4 - 2x^3 + x \\
E_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{2}x^2 - \tfrac{1}{2} \\
E_6(x) &= x^6 + 3x^5 + 5x^3 - 3x \\
E_7(x) &= x^7 + \tfrac{7}{2}x^6 + \tfrac{35}{4}x^4 - \tfrac{21}{2}x^2 + \tfrac{17}{8} \\
E_8(x) &= x^8 - 4x^7 + 14x^5 - 28x^3 + 17x \\
E_9(x) &= x^9 - \tfrac{9}{2}x^8 + 21x^6 - 63x^4 + \tfrac{153}{2}x^2 - \tfrac{31}{2} \\
E_{10}(x) &= x^{10} - 5x^9 + 30x^7 - 126x^5 + 255x^3 - 155x
\end{aligned}\right\} \quad (6)$$

Lane-Emden Equations

Morteza (2020) described Lane-Emden equations as stated in Equations (7) and (8).

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x)g(y) = h(x), \ \ x \geq 0, \quad (7)$$

and the initial conditions,

$$y(0) = C, \ y'(0) = D \quad (8)$$

α, C and D are constants while f, g and h are variables. There are many types of Equations (7) and (8). Parand and Taghavi (2008) described the standard Lane-Emden as stated in Equations (9) and (10).

$$xy''(x) + 2y'(x) + xy^m(x) = 0, \ 0 \leq m \leq 5 \quad (9)$$

$$y(0) = 1, \ y'(0) = 0 \quad (10)$$

Taghavi (2013) described isothermal gas sphere Lane-Emden equation homogeneous differential equation provided by Equations (11) and (12).

$$xy''(x) + 2y'(x) + xe^{y(x)} = 0, \ x \geq 0, \quad (11)$$

$$y(0) = 0, \ y'(0) = 0. \quad (12)$$

The non-homogenous equation is described in form of Equation (7) which $h(x) \neq 0$ and is given by the Equations (13) and (14).

$$xy''(x) + 2y'(x) + xy(x) = 6x + 12x^2 + x^3 + x^4, \quad x \geq 0. \quad (13)$$

$$y(0) = 0, \ y'(0) = 0 \quad (14)$$

Euler polynomial collocation method for approximating Lane-Emden equation is assumed by the Equation (15).

$$y_E(x) = \sum_{j=0}^{N} a_j E_j(x) \quad (15)$$

Where x is collocation points defined in Equation (16).

$$\left.\begin{aligned} x_j &= a + \frac{b-a}{N} j \\ a &\leq x \leq b \end{aligned}\right\}, \ \mathbf{j} = 0,1,..N. \quad (16)$$

Initializing and simplifying Equation (15) gives the truncation function. The truncation function, first and second derivatives are substituted into the residual function. The collocation points stated in Equation (16) is substituted into residual function to form system of equations which are solved. The constants values are replaced in the truncation function to obtain the approximate solution.

Residual function of standard Lane-Emden equation is written in Euler polynomial collocation form as in Equation (17).

$$R_{ESTm}(x) = x\sum_{j=0}^{N} a_j E_J''(x) + 2\sum_{j=0}^{N} a_j E_j'(x) + x[\sum_{j=0}^{N} a_j E_n(x)]^m \ , \ 0 \leq m \leq 5 \quad (17)$$

After simplifying the residual function, collocation points are substituted into the function.

The Euler polynomial collocation method to approximate isothermal gas sphere Lane-Emden equation is given by the residual function in Equation (18).

$$R_{EIT}(x) = x\sum_{j=0}^{N} a_j E_J''(x) + 2\sum_{j=0}^{N} a_j E_j'(x) + xe^{\sum_{j=0}^{N} a_j E_J(x)} \quad (18($$

The collocation points are substituted in the simplified Equation (18) in order to obtain system of equations.

To obtain a collocation method, using Euler polynomials for approximating the non-homogeneous equation, stated in Equation (13) is written in residual function stated in Equation (19).

$$R_{EN}(x) = x\sum_{j=0}^{N} a_j E_j''(x) + 2\sum_{j=0}^{N} a_j E_j'(x) + x\sum_{j=0}^{N} a_j E_j(x) - 6x - 12x^2 - x^3 - x^4 \quad (19)$$

The simplified form of Equation (19) is subjected by the collocation points to get system of equations.

RESULTS AND DISCUSSION

The Results of Lane-Emden Equations

Approximate Solution of the Standard Lane-Emden Equation.

Approximate solution for Value m=0.

$$y_{EST0}(x) = 1 - \frac{1}{6}x^2 \quad (20)$$

Table 1: Comparison of $y_{EST0}(x)$ and Herodt (2004) obtained from Standard Lane-Emden Equation for $m = 0$

x	Present method $y_{EST0}(x)$	Horedt method y(x)	Errors Euler
0	1.000000S	1.000000	0.000000
0.1	0.998333	0.998333	0.000000
0.2	0.993333	0.993333	0.000000
0.3	0.985000	0.985000	0.000000
0.4	0.973333	0.973333	0.000000
0.5	0.958333	0.958333	0.000000
0.6	0.940000	0.940000	0.000000
0.7	0.918333	0.918333	0.000000
0.8	0.893333	0.893333	0.000000
0.9	0.865000	0.865000	0.000000
1.0	0.833333	0.8333333	0.000000

Approximate solution for Value m=1.

$$\begin{aligned} y_{EST1}(x) = 1 &- \frac{92402060309106083578843212546747}{55448574467315742692245567333990 4}x^2 - \frac{1428849524127459756635533}{1362120406579193334205723443 2}x^3 \\ &+ \frac{4487857144173236495212564769}{5396714301158835234021237063 68}x^4 + \frac{33382857788352244632 3}{67845648963577834079518 72}x^5 \\ &- \frac{220811705170506033}{118059162071741130342 4}x^6 - \frac{11324635629598709}{590295810358705651712}x^7 \\ &+ \frac{4033094500377183}{590295810358705651712}x^8 \end{aligned} \tag{21}$$

Table 2: Comparison of $y_{EST1}(x)$ and Herodt (2004) obtained from the Standard Lane-Emden equation for $m = 1$

x	Present method $y_{EST1}(x)$	Horedt method y(x)	Errors
0	1.000000	1.000000	0.000000
0.1	0.998333	0.998334	0.000001
0.2	0.993347	0.993347	0.000000
0.3	0.985066	0.985067	0.000001
0.4	0.973543	0.973546	0.000003
0.5	0.958844	0. 958851	0.000007
0.6	0.941057	0. 941071	0.000014
0.7	0.920290	0.920311	0.000021
0.8	0. 896664	0.896695	0.000031
0.9	0.870321	0.870363	0.000042
1.0	0. 841415	0.841487	0.000072

Approximate solution for Value m=2.

$$\begin{aligned} y_{EST2}(x) = 1 &- \frac{91310467314173746212434694107 9}{54786248132881469277164955238 40}x^2 \\ &+ \frac{163296344503}{763375180415120}x^3 + \frac{30710882626756649465687803 99}{18424673592200421524193987788 8}x^4 \\ &- \frac{725112417}{6434962517776}x^5 - \frac{569572611168973103161}{396383636655870845124608}x^6 \\ &- \frac{9166541646605413237}{147745507309561911758028 8}x^7 + \frac{723019127397715028225 5}{47278562339059811762569216}x^8 \\ &- \frac{8392465499495363}{295147905179352825856}x^9 \end{aligned} \tag{22}$$

Table 3: Comparison of $y_{EST2}(x)$ and Herodt (2004) obtained from the Standard Lane-Emden Equation for $m = 2$

x	Present method $y_{EST2}(x)$	Horedt method y(x)	Errors
0	1.000000	1.000000	0.000000
0.1	0.998336	0.998335	0.000001
0.2	0.993361		
0.3	0.985139		
0.4	0.973767		
0.5	0.959376	0.959353	0.000023
0.6	0.942133		
0.7	0.922228		
0.8	0.899876		
0.9	0.875314		
1.0	0.848784		

Approximate solution for Value m=5.

$$\begin{aligned} y_{EST5}(x) = 1 &- \frac{3247874251026265780 3}{1948735116518106808 32}x^2 - \frac{3079438683328073}{230359365689757757440}x^3 \\ &+ \frac{219225293825299}{5248869003586065}x^4 - \frac{151887307390781}{333195942781655070}x^5 - \frac{547769995054}{53737337802291}x^6 \\ &- \frac{198632945783}{70029603504718}x^7 + \frac{5217871}{715189033}x^8 - \frac{1730665}{502170708}x^9 + \frac{43}{74828}x^{10} \end{aligned} \tag{23}$$

Table 4: Comparison of $y_{EST5}(x)$ and Herodt (2004) obtained from the Standard Lane-Emden Equation for $m = 5$

x	Present method $y_{EST5}(x)$	Horedt method y(x)	Errors
0	1.000000	1.000000	0.000000
0.1	0.998337	0.998337	0.000000
0.2	0.993399	0. 993399	0.000000
0.3	0.985330	0.985329	0.000001
0.4	0.974355	0.974355	0.000000
0.5	0.960769	0.960769	0.000000
0.6	0.944911	0. 944911	0.000000
0.7	0.927146	0.927146	0.000000
0.8	0.907840	0. 907841	0.000001
0.9	0.887356	0.887357	0.000001
1.0	0.866025	0.866025	0.000000

Results for Isothermal Gas Sphere Lane-Emden Equation

Approximate Solution for Isothermal Gas Sphere Lane-Emden Equation

$$y_{EIT}(x) = \frac{3725815853106109648542709638111141}{2235488883205766189874582095134720 0}x^2 + \frac{1925517155563265727899 11}{354601303849665682717487923 2}x^3 + \frac{4440464270992657416517752688 7}{5328221917243239606609749475328}x^4 - \frac{8105390212414865879}{27950506620484712608563 2}x^5 - \frac{3195587347681375781 81}{61043965488719648207667 2}x^6 - \frac{28650429953824211291}{8546155168420750749073408}x^7 + \frac{28548637194156969}{5902958103587056517 12}x^8 - \frac{637703541937747}{73786976294838206464}x^9 \quad (24)$$

Table 5: Comparison of, $y_{EIT}(x)$ and Herodt (2004) obtained from the Isothermal Gas Sphere Lane-Emden Equation

x	Present method $y_{EIT}(x)$	Horedt method y(x)	Errors
0	0.000000	0.000000	0.000000
0.1	-0.001666	-0.001666	0.000000
0.2	-0.0066653	-0.006653	0.000000
0.3	-0.014931	-0.014933	0.000002
0.4	-0.026452	-0.026455	0.000003
0.5	-0.041147	-0.041154	0.000007
0.6	-0.058932	-0.058944	0.000012
0.7	-0.079708	-0.079726	0.000018
0.8	-0.103365	-0.103386	0.000021
0.9	-0.129766	-0.129798	0.000032
1.0	-0.158788	-0.158827	0.000039

The Results for Non-Homogeneous Lane-Emden Equation

Approximate Solution of Non-Homogeneous Lane-Emden Equation.

$$y_{ETN}(x) = x^2 + x^3. \quad (25)$$

Table 6: Comparison of $y_{ENT}(x)$ and Herodt (2004) obtain from the Non-Homogeneous Lane-Emden Equation

x	Present methods $y_{ENT}(x)$	Horedt method y(x)	Errors
0	0.000000	0.000000	0.000000
0.1	0.011000	0.011000	0.000000
0.2	0.048000	0. 048000	0.000000
0.3	0.0117000	0.0117000	0.000000
0.4	0.224000	0.224000	0.000000
0.5	0.375000	0. 375000	0.000000
0.6	0. 576000	0. 576000	0.000000
0.7	0.833000	0.833000	0.000000

0.8	1.152000	1.152000	0.000000
0.9	1.539000	1.539000	0.000000
1.0	2.000000	2.000000	0.000000

Discussion of Results

Tables 1 to 6 give the dimensionless radius, normalized density and the absolute error. The polytropic index m=0, 1, 5 produces a finite series or closed form. For m=0, the present results are the same as the Herodt (2004) results, therefore producing zero error. For m=1, the present result has small error. For polytropic index m=5, the result has zero error at the long run. For index m=2 the solution yielded infinite series. The fsolve approximates the solutions to the machine accuracy. The normalized density of standard Lane-Emden equation decreases as the dimensionless radius increases. Table 5 shows the compilation result of the isothermal gas sphere Lane-Emden equation. The result has the small error. The dimensionless radius isothermal gas sphere Lane-Emden equation of increases with decrease in normalized density. Table 6 shows the result of the Non-Homogeneous Lane-Emden Equation. The dimensionless radius increases with normalized density. The present result is equal to the Herodt (2004) solution thereby producing zero error.

CONCLUSION

The aim of this work is to develop a collocation method for solving equations by utilizing Euler polynomials. The fsolve or Newton Raphson MATLAB programs are adopted for the computation of results. The solution for the case m=0, 1, 5 comes out in a closed form, whereas m= 2, solution is an infinite series. The isothermal gas sphere is solved and the result is stable, reliable and accurate. The non-homogeneous equation is also solved and the result is the same as the exact solution. The dimensionless radius increases as normalized density decreases as in Homogeneous equations while in Non-homogeneous Lane-Emden equation the dimensionless radius increases as normalized density increases. The results confirmed the presented method is reliable, stable, and accurate. Based on the finding of the work, Euler polynomial is recommended for solving of Lane-Emden equations, higher order equations and the use of some polynomials as a basis function.

REFERENCES

Ben-Romdhance, M. and Temimi, H. (2018). An iterative numerical method for solving the Lane–Emden initial and boundary value problems. *International Journal of Computational Methods*, 15(4): 1-16.

Catma, G. (2013). Bernoulli and Euler Polynomials. M.Sc. thesis. Institute of Graduate Studies and Research, Eastern Mediterranean University, Gazimagusa, North Cyprus.86pp.

Gholamreza, H. and Parand, K. (2011). An efficient computational algorithm for solving the nonlinear Lane-Emden types equations. *International Journal of Mathematical and Computational Sciences,* 5(8):1396-1401.

Gospava, B. O. and Gradimir, V. M. (2014). Special classes of polynomial. New edition University of Nis, Faculty of Technology, Leskovac. Pp 15-16.

Horedt, G. P. (2004).Polytropes: Applications in Astrophysics and Related Field. Astrophysics and Space Science Library: a series of books on the recent development of space science and of general geophysics and astrophysics.*Springer*,Pp46-78.

Morteza, B. N. (2020). A computational method for solving the Lane-Emden initial value problems. *Journal of Computational Methods for Differential Equation*, 8(4): 673-684.

Pandey, R .K., Kumar, N., Bhardwaj, A. and Dutta, G. (2016). Solution of Lane Emden type equations using legendre operational matrix of differentiation. *Applied mathematics and computation*, 218(14):7629-7637.

Parand, K. and Hashemi, S. (2018). Radial basis function differential quadrature method for solvingnonlinear differential equations of Lane-Emden type. *Ain Shams Engineering Journal*, 9(4): 615-629.

Parand, K. and Taghavi, A. (2008). Generalized Laguerre polynomials collocation method for solving Lane-Emden Equation. *Journal of Applied Mathematical Sciences,* 2(60):2955-2961.

Saadatmandi, A., Ghasemi, N. and Effekhari, A. (2019). Numerical study of singular fractional Lane-Emden type equation arising in astrophysics. *Journal of Astrophysics and Astronomy*, 40(3):1-27.

Sahu, P. K. and mallick, B. (2019). Approximate solution of fractional order Lane-Emden type differential equation by orthonormal Bernoulli's Polynomials. *International Journal of Applied and Computational Mathematics,* 5:89.

Taghavi, A. (2013). Spectral methods on the semi-infinite line. MSc thesis. Mathematics Department Simon Fraser University Burmaby, British Columbia, Canada. 83pp.

Waheed, A. and Zulgurnain, S. (2020). Solving a new design of non-linear second order Lane-Emden Pantograph delay differential model via Bernoulli collocation method. *The European Physical Science Journal,135,427.*