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THE ODD RAYLEIGH-G FAMILY OF DISTRIBUTION: PROPERTIES, APPLICATIONS, AND PERFORMANCE COMPARISONS

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ABSTRACT

This study introduces the Odd Rayleigh-G (OR-G) family of distribution and explores its mathematical properties, applications, and performance comparisons. The Odd Rayleigh-Weibull distribution (ORWD) is developed by incorporating the "Odd" transformation into the Rayleigh and Weibull distribution, resulting in a flexible model suitable for various real-life and survival data applications. The probability density function (PDF), cumulative distribution function (CDF), hazard function, and survival function of the ORWD are derived and analyzed. Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method, and the performance of the ORWD is assessed through simulation studies. The simulations for parameter estimates at 100 sample sizes were conducted and the plot of the simulated data on the PDF, CDF, survival and hazard function demonstrate a comprehensive view of the characteristics of the Odd Rayleigh Weibull distribution. This information is useful for understanding the behaviour of the distribution and for applications in reliability analysis and survival studies. The results demonstrate the consistency and efficiency of the MLE method for the ORWD. The ORWD is compared with other distributions, including the Weibull, Power Rayleigh, and Rayleigh distributions, using goodness-of-fit measures such as the Akaike Information Criterion (AIC = 111.0238 and 87.4294), Bayesian Information Criterion (BIC = 117.2564 and 96.0320), and Kolmogorov-Smirnov (KS = 0.9559 and 0.9889) test with p-values (p-val = 7.772e-16 and 2.2e-16). The ORWD shows superior performance in fitting the mortality dataset and the Reddit advertisement dataset, highlighting its potential for modelling complex data structures. Overall, this study provides a comprehensive framework for the ORWD, demonstrating its versatility and effectiveness in statistical modelling and data analysis.

Keywords: Odd Link Function, Rayleigh, Weibull, Mortality, Advertisement, MLE, Simulation, Information Criterion, Survival Analysis, Probability

INTRODUCTION

The development of a generalized form of probability distribution captivates academics and dedicated statisticians due to its remarkable flexibility and potential applications. (sadiq *et al*., 2023a). Statistics deal with variability and uncertainty, and from such a vital role of statistical inference and distribution theory usefulness cannot be quantified. The variability and uncertainty in survival and reliability data, simulated data, experimental data, sampling surveys data, and other reliable sources in the field of economics and social sciences, engineering sciences, physical sciences, life sciences, etcetera, little failed to model and characterization by some continuous probability distribution (Thomas 2016).

A bounded, unbounded, semibounded, symmetric or skewed classes of distributions are the regarded types of a continuous probability distribution. These divisions of distributions are according to reasons for their random process generations and applications (PGA I-II), in connection with the quality of being attractive or interested in conformity with the underlying probability model laws (Thomas 2016). The earliest generations of the probability distribution are from an accentuated random process. For instance, the binomial distribution has arrived from the Bernoulli random probability process. Likewise, Poisson distribution is a consequence of a discrete number of the arrival or event happening in a fixed time interval with a well-known constant average rate, which is independent of time after the last event, is a Poisson process (Haight *et al*., 1967). Again, as the number of trials tends to infinity, the probability of failure or success is too small to zero in a binomial random probability process, transformed from a binomial distribution to a normal distribution (De Moivre 1756). Furthermore, the probability distribution of duration between an occurrence in a Poisson random probability process is exponential, and distributions such as exponential families are from the events of gamma, binomial, exponential, Poisson and normal distribution. Many of the mentioned first generations have only a location or scale parameter or both with a limitation of shape parameters. This limitation gives them the outstanding fixed random process to better fit their probability model without flexibility in choice for practical purposes (Thomas 2016).

The next to the first generation of the probability distributions is discovered from the abstraction of the first generation with a smaller aggregate of ability to represent an underlined random process made to the latter. They have the extra outstanding capability to model various real-life datasets. With the availability of data under investigation, the generalization by different scholars with the view of one additional parameter or more to the existing distribution enables goodness of fit to the available data (Thomas 2016). Examples of these include the study by Reda Hafez Osman *et al*. (2025) introduces a new generalization of the Power Rayleigh distribution, focusing on its properties and parameter estimation under Type II censoring. The research provides a comprehensive mathematical framework and demonstrates the flexibility and applicability of the new distribution in modelling various types of data. The study by Yirsaw and Goshu (2024) introduces the Extended Rayleigh Probability Distribution tailored to higher dimensions, enhancing its applicability in multi-dimensional data modelling. This research contributes significantly to the statistical literature by generalizing the classic Rayleigh distribution, widely used in reliability and survival analysis, to address higher-dimensional data contexts. The study by Ogunde *et al*. (2024) introduces the Half Logistic Generalized Rayleigh Distribution (HLGRD), a novel extension of the Rayleigh distribution designed to model hydrological data. This innovative distribution addresses the limitations of traditional Rayleigh distributions by incorporating additional flexibility, making it well-suited for complex datasets often encountered in hydrology and related fields. The study by Obiet al. (2024) introduces a Novel Extension of the Rayleigh Distribution (NERD) that enhances the adaptability of the traditional Rayleigh distribution for more diverse and complex datasets. This innovative approach aims to address the limitations of classical models in handling skewed, heavy-tailed, or multi-modal data patterns. The study by Barranco-Chamorro *et al*. (2021) introduces a generalized Rayleigh family of distributions using the modified slash model. It explores the properties, parameter estimation, and applications of the proposed distribution, providing insights into its flexibility and potential use in modelling skewed and heavy-tailed data. The study by Agu, Eghwerido, and Nziku (2022) presents the Alpha Power Rayleigh-G Family of Distributions, a novel generalization aimed at enhancing flexibility in modelling diverse datasets. The research focuses on the mathematical properties of the new family, including its probability density function (PDF), cumulative distribution function (CDF), moments, and reliability measures. The authors employ parameter estimation techniques, primarily using the Maximum Likelihood Estimation (MLE) method.

In recent years, statistical modelling of complex data has gained considerable attention in diverse fields, including engineering, finance, biology, and health sciences. The development of generalized probability distributions to accommodate real-life complexities, such as skewness, heavy tails, and multi-modal behaviour, has become crucial for accurate data representation. Among these, the family of generalized Rayleigh distributions has been extensively studied due to its flexibility in modelling diverse phenomena (Sadiq *et al.*, 2023a and Gupta *et al.*, 1998). The Rayleigh distribution, originally developed for applications in physics and engineering, is particularly useful in reliability analysis and survival studies (Rayleigh, 1880). However, its limitations in handling data with heavy tails or skewness have prompted researchers to develop extensions, such as the Odd Rayleigh distribution, to enhance its applicability in real-life data scenarios (Ateeq *et al.*, 2019 and Sadiq *et al.*, 2022). The Odd Rayleigh distribution has been shown to provide a flexible framework for modelling skewed and heavy-tailed data (Nadarajah & Kotz, 2008). Its properties, including shape, scale, and tail behaviour, make it a suitable choice for applications in survival analysis, reliability, and extreme value modelling. Despite these advantages, the parameter estimation of such distributions poses significant challenges, particularly when using methods like Maximum Likelihood Estimation (Coles (2001), Habu *et al.*, 2024). The study by Elgarhy et al. (2024) introduces an Extended Rayleigh-Weibull (ERW) Model, designed to address limitations in traditional survival and reliability models while offering enhanced flexibility in actuarial and applied data analyses. The study by Shala and Merovci (2024) introduces a Three-Parameter Inverse Rayleigh Distribution (3P-IRD), a novel extension of the classical Inverse Rayleigh distribution, aimed at providing greater flexibility and accuracy for modeling real-world datasets.

Statistical simulations play an essential role in evaluating the performance of estimation methods under various conditions. These studies are instrumental in determining the bias, efficiency, and robustness of parameter estimates, especially when applied to data with unique characteristics, such as censored observations or non-normal distributions (Balakrishnan (2019), Obafemi *et al.*, 2024, Sadiq *et al.*, 2023a).

Bhat and Ahmad (2020) formulated a new lifetime probability model, named Power Rayleigh distribution (PRD). They discussed properties of PRD including moments, moment generating function, hazard rate, mean residual life, order statistics and quantiles. In their study, they established a stochastic ordering of random variables. The expression for four different measures of entropy viz., Shannon entropy, Renyi entropy, beta entropy and Mathai and Haubold entropy were also obtained. A maximum likelihood estimation procedure is employed to estimate the unknown parameters. In addition, they illustrated the practical importance of PRD using two real data sets. Yahaya and Doguwa (2021), developed a new variant of T-Exponentiated Odd Generalized-X family of distributions. The new variant titled Rayleigh-Exponentiated Odd Generalized-X family becomes what it is when the variable T follows Rayleigh distribution. Some important functions comprising the cumulative distribution, probability density, survival function and the hazard function of the new sub-family are presented. Other vital derivations include moments, moment generating function, quantile function, entropy and function of order statistics were derived. The proposed sub-family is shown to belong to the Exponentiated-G family of distributions. The method of maximum likelihood was used to derive estimates of the unknown parameters; after which parameter asymptotic confidence bounds were also obtained.

Sadiq *et al.*, (2023b and 2023c) developed an extended Fréchet-G family of distributions and studied their mathematical properties. They used the method of Alzaatreh in developing the new Generalized Odd Fréchet-G Family of Distribution. The developed distribution is flexible for studying positive real-life datasets. The statistical properties related to this family were obtained. The parameters of the family were estimated by using a technique of maximum likelihood. A New Generalized Odd Fréchet-Weibull model was introduced. This distribution was fitted with a set of lifetime data. A Monte Carlo simulation was applied to test the consistency of the estimated parameters of this distribution in terms of their bias and mean squared error with a comparison of M.L.E and the maximum product spacing (MPS). The findings of the Monte Carlo simulation show that the M.L.E method is the best technique for estimating the parameter of New Generalized Odd Frechet-Weibull distribution than the M.PS method. The findings of the application on the data set produce a higher flexibility than some of the competing distributions. In general, their new distributions serve as a viable alternative to other distributions available in the literature for modelling positive data.

This research builds upon these principles, focusing on the Odd Rayleigh distribution's properties and developing robust parameter estimation techniques using simulation studies. By addressing the limitations of existing methods and proposing innovative solutions, this study aims to contribute to the growing body of knowledge on Odd Rayleigh distribution and its applications. The findings have practical implications for fields such as reliability engineering, climate studies, and survival analysis, where accurate modelling of data distributions is critical for decision-making.

MATERIALS AND METHODS

In this section, we present the methodology for extending the Rayleigh distribution. The extended Rayleigh is named the

Odd Rayleigh-G family of distributions. The statistical properties related to these new families are derived and presented. The method of estimating the parameters of this family is presented. In this research, the odd link function will be used for the transformation and generalization of the Rayleigh distribution. The rationale for using an odd link function in generalizing the Rayleigh distribution, or other probability distributions, lies in its ability to introduce greater flexibility and capture a broader range of distributional shapes. This flexibility is crucial in statistical inference and distribution theory, as it allows the model to better represent diverse data patterns and underlying phenomena. The odd link function introduces additional skewness, kurtosis, or heavy-tailed behaviour, depending on the nature of the function and its parameters. This helps the generalized Rayleigh distribution adapt to datasets where the conventional Rayleigh distribution's symmetric or unimodal shape is insufficient. It provides the capacity to model both symmetric and asymmetric distributions, broadening the scope of application. Real-world data, especially in survival analysis, reliability studies, and engineering, often exhibit asymmetry. The odd link function introduces the ability to model such skewed behaviour, which the standard Rayleigh distribution may fail to capture. By incorporating an odd function, the tail behaviour of the distribution can be adjusted, making it suitable for modelling phenomena with heavy tails or outliers. This is particularly valuable in fields like finance or environmental science where extreme values are important. In practice, adding an odd link function allows the generalized Rayleigh distribution to fit a wider range of empirical datasets, leading to better goodness-of-fit metrics and more accurate modelling. Odd link functions, by definition, maintain certain mathematical properties, such as symmetry about the origin when integrated into cumulative distribution functions or survival functions. These properties preserve analytical tractability while intensifying flexibility. Despite the added complexity, odd link functions often retain the interpretability of the parameters, which is essential for the practical application and communication of statistical results. The odd link function provides a systematic way to generalize the Rayleigh distribution, allowing it to address more diverse data scenarios while preserving theoretical and practical usability. It balances increased flexibility with analytical tractability, making it a powerful tool in distribution theory and statistical inference. The r statistical software would be used for the implementation, computations, application and goodness of fit for the proposed model and performance comparison with other existing models.

Bhat and Ahmad (2020) defined a random variable T and is said to have Rayleigh distribution with shape parameter θ if its pdf and CDF are given as,

$$f(t\ ;\ \theta\ ) = \frac{t}{\theta^2} exp\left\{-\left(\frac{1}{2\theta^2}t^2\right)\right\}\ ;\quad 0 < t < \infty\, , \theta > 0 \tag{1}$$

$$F(t\ ;\ \theta) = 1 - exp\left\{-\left(\frac{1}{2\theta^2}t^2\right)\right\};\ 0 < t < \infty\, , \theta > 0 \tag{2}$$

Supposed $C(x;\xi)$ to be our assumed link function for our proposed family. Integrating the density function presented in equation (1) to obtain the CDF of the proposed family as:

$$Y(x;\theta,\xi) = \int_0^{C(x;\xi)} f(t)\ dt$$

$$= \int_0^{C(x;\xi)} \frac{t}{\theta^2} exp\left\{-\left(\frac{1}{2\theta^2}t^2\right)\right\} dt \tag{3}$$

$$Y(x;\theta,\xi) = 1 - exp\left\{-\left(\frac{1}{2\theta^2}\left(C(x;\xi)\right)^2\right)\right\} \tag{4}$$

Odd Rayleigh-G Family of Distribution

We proposed a new family of distribution called the Odd Rayleigh-G (OR-G) family by taking the odd link function, $C(x;\xi) = \frac{M(x;\xi)}{1-M(x;\xi)}$ into equation (4) to obtain the CDF given by:

$$Y(x;\theta,\xi) = 1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\} \tag{5}$$

where $\theta > 0$ is the scale parameter, $x > 0$ and $M(x;\xi)$ is the CDF and ξ is the parameters' vector of the baseline distribution. Differentiating equation (5) with respect to x, its corresponding PDF is given by:

$$y(x;\theta,\xi) = \frac{m(x;\xi)M(x;\xi)}{\theta^2\left(1-M(x;\xi)\right)^3} exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\} \tag{6}$$

where $m(x;\xi)$ is the pdf and ξ is the parameters' vector of the baseline distribution. Therefore, a random variable X with density function and distribution function in equations (6) and (5) is denoted by $X \sim\ ORG\ (\theta,\xi)$.

Test of Validity of the pdf and CDF of OR-G Family

If equations (6) and (5) are valid pdf and CDF respectively, they must satisfy the statistical properties of any continuous distributions:

$\int_{-\infty}^{\infty} y(x;\theta,\xi)\ dx\ = 1$ for x>0

$$\lim_{x\to-\infty} Y(x;\theta,\xi) = 0$$

$$\lim_{x\to\infty} Y(x;\theta,\xi) = 1$$

CASE 1

$$\int_0^{\infty} y(x;\theta,\xi)\ dx\ = \int_0^{\infty} \frac{m(x;\xi)M(x;\xi)}{\theta^2\left(1-M(x;\xi)\right)^3} exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\} dx \tag{7}$$

$$\int_0^{\infty} y(x;\theta,\xi)\ dx = \int_0^{\infty} \frac{m(x;\xi)M(x;\xi)}{\theta^2\left(1-M(x;\xi)\right)^3} exp\{-y\}\, dx \tag{8}$$

where $y = \frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2$

Therefore, it can be easily seen that:

$$\int_0^{\infty} y(x;\theta,\xi)\ dx = \int_0^{\infty} \frac{m(x;\xi)M(x;\xi)}{\theta^2\left(1-M(x;\xi)\right)^3} exp\{-y\} \frac{\theta^2\left(1-M(x;\xi)\right)^3 dy}{m(x;\xi)M(x;\xi)} \tag{9}$$

$$\int_0^{\infty} y(x;\theta,\xi)\ dx = \int_0^{\infty} exp\{-y\}\, dy = [-exp\{-y\}]_0^{\infty} = 1 \tag{10}$$

CASE II

$$\lim_{x\to 0} Y(x;\theta,\xi) = \lim_{x\to 0}\left[1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\}\right] = \left[1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{0}{1-0}\right)^2\right\}\right] = 0 \tag{11}$$

CASE III

$$\lim_{x\to\infty} Y(x;\theta,\xi) = \lim_{x\to\infty}\left[1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\}\right] = \left[1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{1}{1-1}\right)^2\right\}\right] = 1 \tag{12}$$

Survival and Hazard Rate Function of the OR-G Family

The survival function, hazard function, and cumulative hazard function random variable is X which follows the OR-G family are respectively given as,

$$S(x;\theta,\xi)\ =\ 1 - Y(x;\theta,\xi) = exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\} \tag{13}$$

$$h(x;\theta,\xi) = \frac{y(x;\theta,\xi)}{S(x;\theta,\xi)} = \frac{m(x;\xi)M(x;\xi)}{\theta^2(1-M(x;\xi))^3} \quad (14)$$

Quantile Function of OR-G Family

The quantile function of the OR-G family is obtained by inverting the CDF in equation (5). Suppose the variable U is uniformly distributed on (0,1), then.

$$u = 1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right\} \quad (15)$$

Simplifying equation (15) and solving for x, we obtained the quantile function as:

$$x = \Phi(u) = M^{-1}\left(\frac{(-2\theta^2\, log(1-u))^{\frac{1}{2}}}{\left(1+(-2\theta^2\, log(1-u))^{\frac{1}{2}}\right)}\right) \quad (16)$$

where M^{-1} is the quantile function of the baseline distribution $M(x;\xi)$. And $0 < u < 1$.

Linear Representation for the pdf and CDF of OR-G Family

Here, we consider the individual terms in the given CDF of the OR-G family presented in equations (5) via some standard mathematical expansion, which comprises the generalized binomial expansion for negative and positive power, the power series expansion, and so on for instance,

$$Y(x;\theta,\xi) = \sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1}{i}\binom{2ij}{k}\frac{1}{j!(2\theta^2)^{ij}}M^{k+2ij}(x;\xi) \quad (17)$$

Therefore, equation (17) reduces to,

$$Y(x;\theta,\xi) = \sum_{i,j,k=0}^{\infty} A_{i,j,k}M^{k+2ij}(x;\xi) \quad (18)$$

where $A_{i,j,k} = \frac{\binom{1}{i}\binom{2ij}{k}(-1)^{i+j+k}}{j!(2\theta^2)^{ij}}$,

differentiating equation (18) w.r.t. x we have the corresponding pdf as:

$$y(x;\theta,\xi) = \sum_{i,j=0}^{\infty} A_{i,j,k}\ (k+2ij)m(x;\xi)M^{(k+2ij)-1}(x;\xi) \quad (19)$$

further simplification of equation (18) is as:

$$Y(x;\theta,\xi) = \sum_{k=0}^{\infty}\ l_k N_k(x) \quad (20)$$

where $l_k = \sum_{i,j,k=0}^{\infty} A_{i,j,k}$ $\ and\ $ $N_k(x) = M^{(k+2ij)-1}(x;\xi)$,

differentiate equation (20) w.r.t. x

$$y(x;\theta,\xi) = \sum_{k=0}^{\infty}\ l_k n_k(x) \quad (21)$$

where $n_k(x) = km(x;\xi)M^{k-1}(x;\xi)$,

therefore, equations (20) and (21) are the reduced CDF and pdf of the OR-G family.

Moments of OR-G Family

The rth ordinary moment of a random variable X which follows the Odd Rayleigh family by using equation (21) we have:

$$\mu_r^{/} = E(X^r) = \int_0^{\infty} x^r\, y(x;\theta,\xi)dx = \int_0^{\infty} x^r \sum_{k=0}^{\infty}\ l_k n_k(x)\,dx = \sum_{k=0}^{\infty}\ l_k E[Z_k^r] \quad (22)$$

where $E[Z_k^r] = \int_0^{\infty} x^r km(x;\xi)M^{k-1}(x;\xi)\,dx$

Moment Generating Function of OR-G Family

The moment-generating function of a random variable X which follows the Odd Rayleigh family by using equation (21) we have,

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\, y(x;\theta,\xi)dx = \int_0^{\infty} e^{tx}\sum_{k=0}^{\infty}\ l_k n_k(x)\,dx = \sum_{k=0}^{\infty}\ l_k E[e^{tZ_k}] \quad (23)$$

where $E[e^{tZ_k}] = \int_0^{\infty} e^{tx} km(x;\xi)M^{k-1}(x;\xi)\,dx$

Entropies of OR-G Family

The entropy of any random variable X is a measure of indecisiveness, variability, and details innate to the probable results of the variable. It is defined mathematically by (Sadiq *et al.*, 2023a) using equation (21) we have,

$$I_R(\varpi) = \frac{1}{1-\varpi}log\left(\int_0^{\infty} y^{\varpi}(x;\theta,\xi)\ dx\right) = \frac{1}{1-\varpi}log\left(\int_0^{\infty}(\sum_{k=0}^{\infty}\ l_k n_k(x))^{\varpi}\ dx\right) \quad (24)$$

where $\varpi > 0$and $\varpi \neq 1$

The nth entropy is defined (Sadiq *et al.*, 2023b) by

$$I_{nth}(\varpi) = \frac{1}{\varpi-1}log\left(1-\int_0^{\infty} y^{\varpi}(x;\theta,\xi)\ dx\right) = \frac{1}{1-\varpi}log\left(1-\int_0^{\infty}(\sum_{k=0}^{\infty}\ l_k n_k(x))^{\varpi}\ dx\right) \quad (25)$$

where $\varpi > 0$and $\varpi \neq 1$

Order Statistics of OR-G Family

Suppose $X_1, X_2, X_3, \ldots, X_n$ is a random sample from the OR-G distribution and $X_{i\,:\,n}$ represents the ith order statistic defined (Sadiq *et al.*, 2023c), then, using equations (20) and (21) we have

$$f_{i\,:\,n}(x;\theta,\xi) = \frac{n!}{[(i-1)!(n-i)!]}[y(x;\theta,\xi)][Y(x;\theta,\xi)]^{i-1}[1-Y(x;\theta,\xi)]^{n-i} \quad (26)$$

$$f_{i\,:\,n}(x;\theta,\xi) = \frac{n!}{[(i-1)!(n-i)!]}[\sum_{k=0}^{\infty}\ l_k n_k(x)]\ [\sum_{k=0}^{\infty}\ l_k N_k(x)]^{i-1}[1-\sum_{k=0}^{\infty}\ l_k N_k(x)] \quad (27)$$

Estimation of Parameters for OR-G Family

Suppose that $x_1, x_2, x_3, \ldots, x_n$ are the observed values from the proposed OR-G family with parameters θ and ξ. Suppose that $\Phi = [\theta]^T$ is the $[m \times 1]$ vector of the parameter. The log-likelihood function Φ using equation (6) is expressed by

$$\ell_n(\Phi) = -2n\,log(\theta) + \sum_{i=1}^{n} log[m(x;\ \xi)] - 3\sum_{i=1}^{n} log(1-M(x;\xi)) - \frac{1}{2\theta^2}\sum_{i=1}^{n}\left[\left(\frac{M(x;\xi)}{1-M(x;\xi)}\right)^2\right] \quad (28)$$

We find the partial derivatives of the log-likelihood function in equation (28) with respect to each parameter (θ and ξ) in the Odd Rayleigh distribution.

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^2 + (\xi-1)\sum_{i=1}^{n}\frac{-x_i^2\,exp(-\theta x_i^2)}{1-exp(-\theta x_i^2)} \quad (29)$$

$$\frac{\partial \mathscr{L}}{\partial \xi} = \frac{n}{\xi} + \sum_{i=1}^{n\sum[1-exp(-\theta x_i^2)]} ln \quad (30)$$

These derivatives in equations (29) and (30) can be used to solve for the maximum likelihood estimates (MLEs) of θ and ξ by setting them equal to zero and solving numerically.

Weibull Distributions

Sadiq *et al.*, (2023b), defined a random variable X as said to have Weibull distribution with scale parameter ϕ and shape parameter ω if its pdf and CDF are given as,

$$m(x\ ;\ \phi\ ,\ \omega) = \omega\phi^{-\omega}x^{\omega-1}\, exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\ ;\ \ x, \phi, \omega > 0 \quad (31)$$

$$M(x\ ;\ \phi\ ,\ \omega) = 1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\};\ x, \phi, \omega > 0 \quad (32)$$

Suppose that the baseline distribution M has a Weibull distribution with pdf and CDF as in equations (31) and (32)

Odd Rayleigh-Weibull Distributions

Suppose that the baseline distribution M has Weibull distribution with pdf and CDF as in equations (31) and (32), then from (5) and (6), the pdf and CDF of Odd Rayleigh-Weibull distribution (OR-W) are defined by, for all $x; \theta, \phi, \omega > 0$

$$f(x;\theta,\phi,\omega) = \frac{\left(\omega\phi^{-\omega}x^{\omega-1}\, exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)\left(1-exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)}{\theta^2\left(exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^3} exp\left\{-\frac{1}{2\theta^2}\left(\frac{\left(1-exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)}{exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}}\right)^2\right\} \quad (33)$$

$$F(x;\theta,\phi,\omega) = 1 - exp\left\{-\frac{1}{2\theta^2}\left(\frac{\left(1-exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)}{exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}}\right)^2\right\} \quad (34)$$

The hazard and survival functions of the Odd Rayleigh Weibull distribution are derived as:

$$h(x;\theta,\phi,\omega) = \left(\omega\phi^{-\omega}x^{\omega-1}\, exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)\left(1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)\left(\theta^2\left(exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^3\right)^{-1} \quad (35)$$

$$S(x;\theta,\phi,\omega) = exp\left\{-\frac{1}{2\theta^2}\left(\frac{\left(1-exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)}{exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}}\right)^2\right\} \quad (36)$$

The quantile function of the Odd Rayleigh-Weibull distribution (OR-W) is derived as:

$$x = (-\phi)\left(log\left(1-\left(\frac{(-2\theta^2\, log(1-u))^{\frac{1}{2}}}{\left(1+(-2\theta^2\, log(1-u))^{\frac{1}{2}}\right)}\right)\right)\right)^{\frac{1}{\omega}} \quad (37)$$

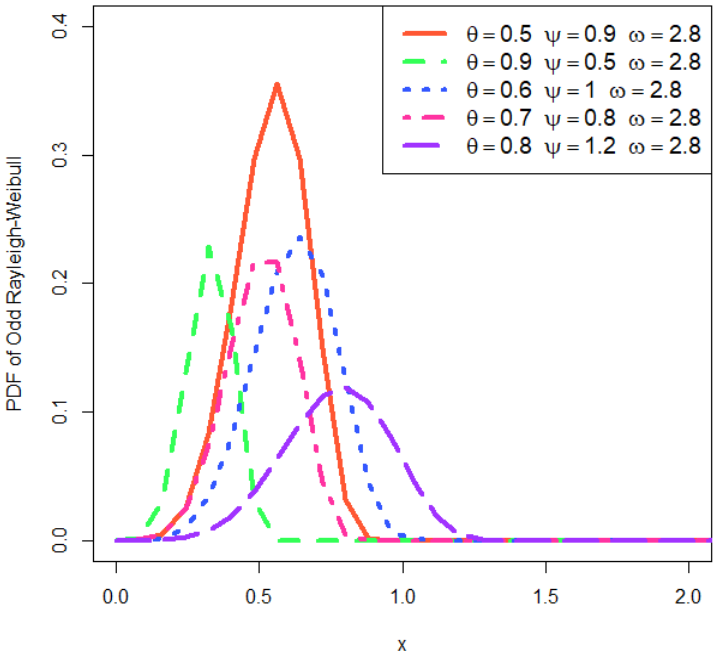

Figure 1: PDF Plot of Odd Rayleigh Weibull Distribution

There are five different curves on the plot in Figure 1, each representing the PDF for different parameter values $\theta,\ \psi,\ and\ \omega$. The legend in the top right corner specifies the parameter values for each curve. The shape of the PDF changes with different parameter values. This is important for understanding how the distribution behaves under various conditions. The peak and spread of each curve indicate the likelihood of different xx values. Higher peaks suggest higher probabilities for those xx values. The parameters $\theta,\ \psi,\ and\ \omega$ influence the shape and spread of the distribution. For example, changes in $\psi\ and\ \theta$shift the peak and alter the spread of the PDF.

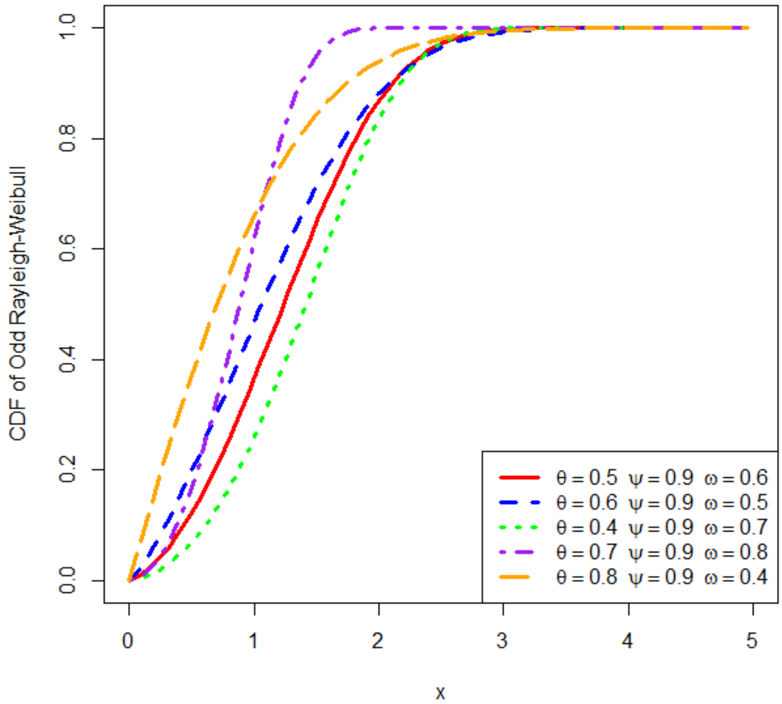

Figure 2: CDF Plot of Odd Rayleigh Weibull Distribution

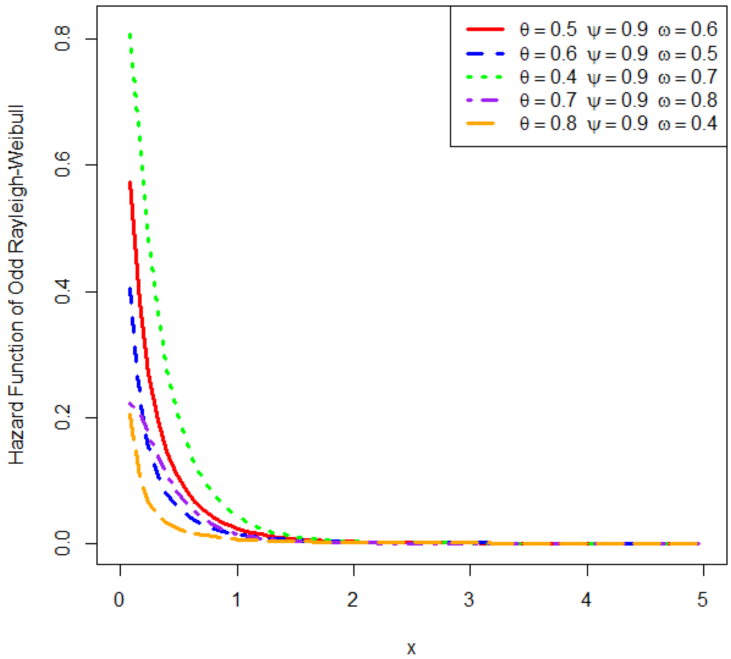

Figure 3: Hazard Function Plot of Odd Rayleigh Weibull Distribution

There are five different curves on the plot in Figure 2, each representing the CDF for different parameter values $\theta,\ \psi,\ and\ \omega$. The legend in the top right corner specifies the parameter values for each curve. The hazard function values start high at $x = 0$and decrease rapidly as $x$increase. This behaviour is typical for hazard functions of distributions that model life data, where the hazard rate decreases over time. The differences in the curves illustrate how changes in the parameters $\theta,\ \psi,\ and\ \omega$ affect the hazard function. For example, higher values $\psi\ and\ \theta$ shift the hazard function downward, indicating a lower initial hazard rate. The hazard function approaches zero as x increases, suggesting that the likelihood of failure or event occurrence decreases over time. Figure 2 is relevant for understanding the reliability and failure rates of systems modelled by the Odd Rayleigh-Weibull distribution. It helps in visualizing how the distribution behaves under different parameter settings, which is crucial for applications in reliability engineering and survival analysis.

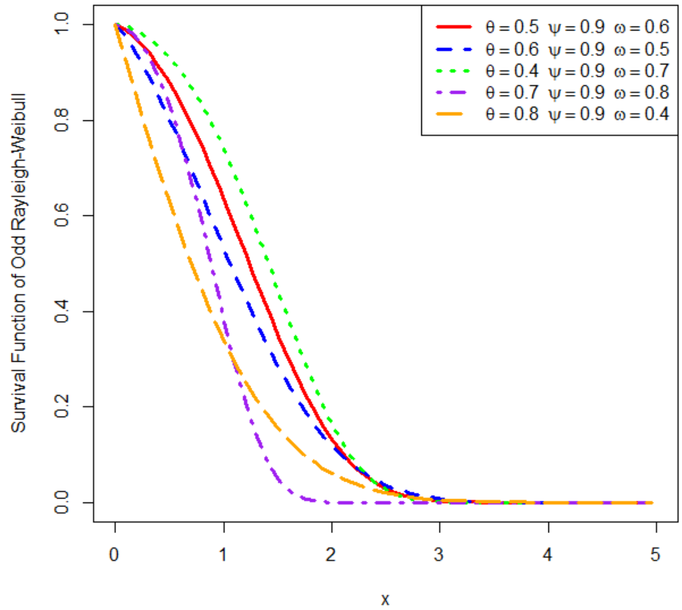

Figure 4: Survival Function Plot of Odd Rayleigh Weibull Distribution

RESULTS AND DISCUSSION

In this section, we present the results and discussion of the Rayleigh model and its simulations and applications to real-life datasets.

Simulation Study

The quantile function (inverse CDF) in equation (37) is used to generate 100 random samples from a given distribution. This randomly generated dataset was used in estimating the parameters of the distribution and the density function, distribution function, hazard and survival function were plotted and shown in Figure 5. This process helps in understanding the behaviour of the Odd Rayleigh-Weibull distribution and its applications in various fields such as reliability analysis and survival studies. PDF (Probability Density Function) shows the likelihood of different values of the random variable. It is derived by differentiating the CDF. The CDF (Cumulative Distribution Function) shows the cumulative probability up to a certain value of the random variable. The Hazard Function represents the instantaneous failure rate at any given time. It is the ratio of the PDF to the survival function. The Survival Function represents the probability of survival beyond a certain time. It is the complement of the CDF.

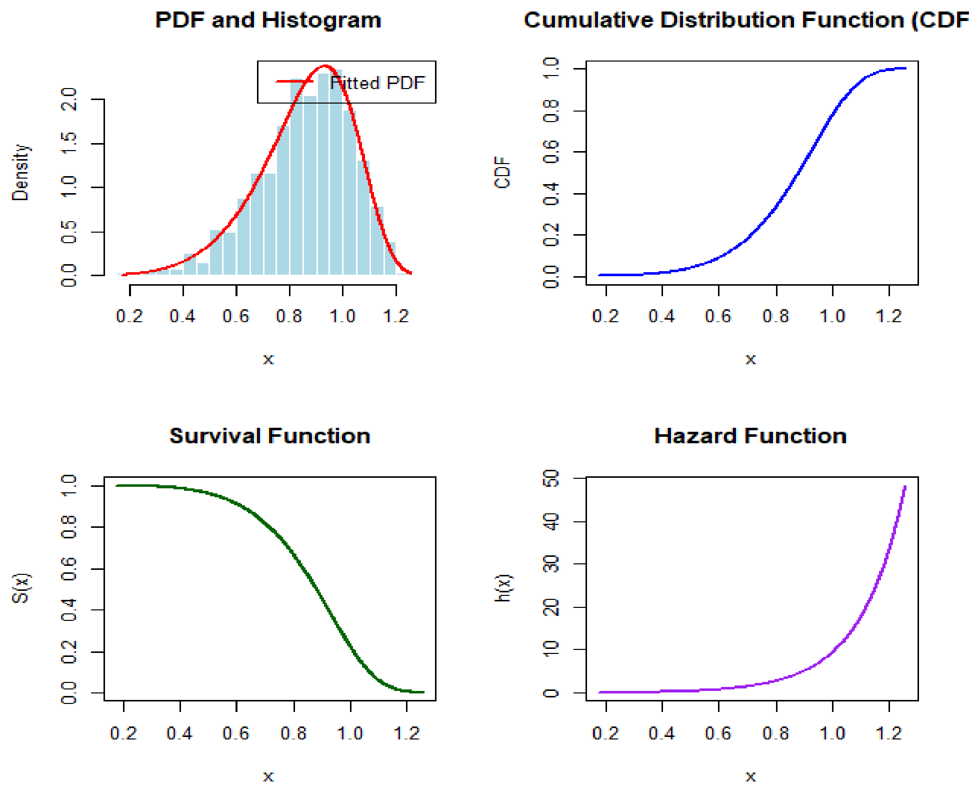

Figure 5: Simulation study for the PDF, CDF, SF, and HF of Odd Rayleigh Weibull Distribution

Figure 5 consists of four plots that represent different functions related to the Odd Rayleigh Weibull distribution. The plot shows the Probability Density Function (PDF) overlaid on a histogram of simulated random data. The histogram bars are light blue, and the fitted PDF is represented by a red curve. The PDF peaks around the value of 0.8 on the x-axis. Figure 5 shows the Cumulative Distribution Function (CDF) of the simulated data. The CDF starts at 0 and increases to 1, indicating the cumulative probability of the distribution. The curve is blue and shows a smooth increase, reaching its maximum value around 1.2 on the x-axis. Figure 5 shows the Survival Function, which is the complement of the CDF. The survival function starts at 1 and decreases to 0, indicating the probability of survival beyond a certain point. The curve is green and shows a smooth decline, reaching its minimum value of around 1.2 on the x-axis. Figure 5 shows the Hazard Function, which represents the instantaneous failure rate at any given time. The hazard function starts near 0 and increases, indicating the rate of failure over time. The curve is purple and shows a steep increase, reaching a high value of around 1.2 on the x-axis. In general, Figure 5 provides a comprehensive view of the characteristics of the Odd Rayleigh Weibull distribution, including its density, cumulative probability, survival probability, and failure rate. This information is useful for understanding the behaviour of the distribution and for applications in reliability analysis and survival studies.

Application to Real-life Dataset

The first data set was originally reported by Almongy *et al.*, (2021) which represents the mortality rate of COVID-19 in Italy for 59 days. This "mortality rate" data set is:

4.571, 7.201, 3.606,8.479, 11.410, 8.961, 10.919, 10.908, 6.503, 18.474, 11.010, 17.337, 16.561, 13.226, 15.137, 8.697, 15.787, 13.333, 11.822,14.242, 11.273, 14.330, 16.046, 11.950, 10.282, 11.775, 10.138, 9.037, 12.396, 10.644, 8.646, 8.905, 8.906, 7.407, 7.445,7.214, 6.194, 4.640, 5.452, 5.073, 4.416, 4.859, 4.408, 4.639, 3.148, 4.040, 4.253, 4.011, 3.564, 3.827, 3.134, 2.780, 2.881,3.341, 2.686, 2.814, 2.508, 2.450, 1.518

The second data set was originally reported by Shen *et al.* (2022), the data set consists of 150 observations and is related to the Reddit advertising data. The Reddit advertising data is given as:

11.340,6.296, 5.136, 7.292, 6.700, 3.648, 7.272, 5.140, 2.980, 6.336,5.128, 7.564, 6.180, 5.588, 8.760, 11.560, 5.192, 10.032, 6.660,7.932, 7.536, 4.436, 17.580, 9.836, 5.580, 6.092, 7.912, 6.760,10.096, 5.948, 11.156, 6.936, 5.424, 8.532, 4.980, 6.536,13.012, 6.668, 5.540, 11.184, 7.052, 9.336, 10.624, 7.148,4.892, 6.584, 4.436, 11.696, 7.868, 4.040, 6.748, 5.336, 9.056,11.496, 11.548, 12.392, 3.636, 7.380, 8.940, 9.068, 4.536,13.060, 8.940, 7.900, 9.972, 5.740, 5.560, 6.136, 10.904, 7.960,9.380, 6.484, 3.512, 4.792, 8.980, 5.512, 2.392, 5.336, 3.440, 4.580, 6.704, 7.296, 4.600, 5.592, 9.292, 7.816, 7.068, 8.492, 7.556, 8.836, 5.960, 3.696, 9.816, 0.908, 5.636, 8.536, 6.260, 7.912, 12.492, 8.880, 6.188, 11.900, 7.692, 7.496, 10.340, 9.636, 3.784, 5.068, 2.940, 9.992, 7.252, 10.956, 7.512, 8.108, 7.796, 6.928, 6.236, 4.924, 8.056, 3.468, 7.904, 3.780, 5.912, 7.756, 9.900, 5.472, 3.956, 5.044, 12.676, 5.376.

Table 1: Performance Comparison of ORWD with 3 other Distributions based on Mortality Rate

Probability Distributions		ORWD	WD	PRD	RD
Parameter Estimates	θ	0.06871	2.0028	4.7954	6.5828
	ω	2.2582	9.5079	0.8774	-
	ψ	14.5529	-	-	-
Log-likelihood		52.51188	167.8221	168.1341	167.7666
AIC		111.0238	339.6442	340.2682	337.5332
CAIC		111.4601	339.8585	340.4825	337.6034
BIC		117.2564	343.7993	344.4233	339.6107
HQIC		113.4567	341.2661	341.8902	338.3444
Cramer von Mises	W^*	0.1002	0.1326	0.1350	0.1327
Anderson Darling	A^*	0.6128	0.8028	0.8072	0.8043
Komogorov Smirnov test	D	0.9559	0.1443	0.0952	0.13597
	p-value	7.772e-16	0.1547	0.6236	0.2058

Table 1 shows the performance comparison of the Odd Rayleigh-Weibull Distribution (ORWD) with three other distributions (WD, PRD, RD) based on the mortality rate dataset. The parameter estimate values indicate the estimated parameters for each distribution. ORWD has three parameters, while the others have fewer. The log-likelihood estimates, the lower values indicate a better fit to the data. The ORWD has the lowest log-likelihood, suggesting it fits the data well. The information criteria (AIC, CAIC, BIC, HQIC), lower values indicate a better model fit. The ORWD has the lowest values across all criteria, suggesting it is the best-fitting model among the four. Overall, the Odd Rayleigh-Weibull Distribution (ORWD) appears to provide the best fit to the mortality rate data based on the information criteria and goodness-of-fit tests. It has the lowest AIC, CAIC, BIC, HQIC, and the lowest values in the Cramer von Mises and Anderson Darling tests.

The results from Table 1 compare the performance of the Odd Rayleigh Weibull Distribution (ORWD) with three other probability distributions (Weibull Distribution (WD), Power Rayleigh Distribution (PRD), and Rayleigh Distribution (RD)) based on mortality rate data.

Parameter Estimates

Each distribution has a distinct set of parameter estimates, reflecting the flexibility and shape of their respective models. ORWD has three parameter estimates (0.06871, 2.2582, 14.5529), indicating a more flexible model compared to others like the RD, which lacks parameter estimates, showing its simplicity and lack of flexibility.

Goodness-of-Fit Measures

Log-Likelihood

The ORWD has a lower log-likelihood (52.51188), indicating it may not fit as well as the other distributions for this dataset.

Akaike Information Criterion (AIC)

The ORWD (111.0238) has the lowest AIC, suggesting that despite its lower log-likelihood, its parameter flexibility balances out the trade-off between fit and complexity.

Other distributions, such as the WD (339.6442), PRD (340.2682), and RD (337.5332), have much higher AIC values, showing poorer performance relative to ORWD.

Bayesian Information Criterion (BIC)

Again, the ORWD (111.4601) has the lowest BIC, outperforming the other models. This indicates its capacity to balance fit and parsimony effectively.

CAIC and HQIC

The ORWD consistently outperforms other distributions in CAIC and HQIC metrics as well, affirming its advantage in model selection criteria.

Goodness-of-Fit Tests

These tests assess how well the model aligns with the observed data.

*Cramér-von Mises (W)**

The ORWD (0.1002) achieves the lowest value compared to WD (0.1326), PRD (0.1350), and RD (0.1327), suggesting it best captures the underlying distribution of the mortality data.

*Anderson-Darling (A)**

The ORWD (0.6128) again outperforms the other distributions, confirming its robustness in capturing extreme values.

Kolmogorov-Smirnov (D)

Lower D values indicate a smaller maximum distance between the empirical and theoretical cumulative distributions.

The PRD (0.0952) achieves the lowest DD-value, followed by RD (0.13597) and WD (0.1443). The ORWD (0.9559) performs poorly here.

Despite this, the p-value for the ORWD (7.772e-16) is extremely small, suggesting that the fit is statistically significant but with noticeable deviations.

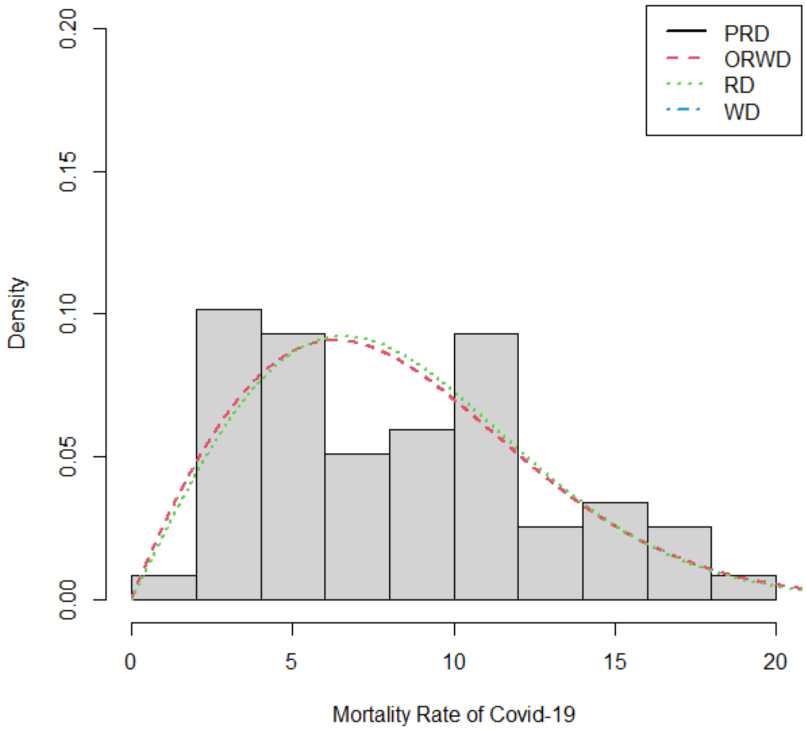

Figure 6: Histogram Plot Mortality Rate Plot of ORW and others Distributions

Figure 6 shows the density plot visually represents the distribution of COVID-19 mortality rates and compares different density estimation methods. The curves provide a smoothed estimate of the distribution, helping to understand the underlying data distribution and make statistical inferences.

Table 2: Performance Comparison of ORWD with 3 other Distributions based on Advertisement

Probability Distributions		ORWD	WD	PRD	RD
Parameter Estimates	θ	0.06871	2.8837	14.3113	5.5013
	ω	2.2582	8.1955	1.4230	-
	ψ	14.5529	-	-	-
Log-likelihood		40.7147	310.9042	311.0430	324.1729
AIC		87.4294	625.8085	626.0860	650.3458
CAIC		87.6199	625.9030	626.1805	650.3771
BIC		96.0320	631.5436	631.8210	653.2134
HQIC		90.9250	628.1388	628.4163	651.5110
Cramer von Mises	W^*	0.0447	0.0899	0.0857	0.0397
Anderson Darling	A^*	0.3380	0.5563	0.5329	0.2813
Komogorov Smirnov test	D	0.9889	0.0757	0.0855	0.1727
	p-value	2.2e-16	0.4455	0.2968	0.0008

Table 2 provides the parameter estimates and goodness of fit measures for the Odd Rayleigh Weibull distribution with competing models (Weibull distribution, Power Rayleigh distribution and Rayleigh distribution) using the advertisement dataset. Akaike's Information Criterion (AIC), Consistent Akaike's Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC) are the performance metrics. Overall, the Odd Rayleigh-Weibull Distribution (ORWD) appears to provide the best fit to the advertisement data based on the information criteria and goodness-of-fit tests. It has the lowest AIC, CAIC, BIC, HQIC, and the lowest values in the Cramer von Mises and Anderson Darling tests.
The results in Table 2 compare the performance of the Odd Rayleigh Weibull Distribution (ORWD) with the Weibull Distribution (WD), Power Rayleigh Distribution (PRD), and Rayleigh Distribution (RD) based on advertisement data.
1. Parameter Estimates
The ORWD has three parameters (0.06871, 2.2582, 14.5529), demonstrating its flexibility to model complex data.
The WD (2.8837, 8.1955) and PRD (14.3113, 1.4230) have fewer parameters, offering less flexibility.
The RD (5.5013) has only one parameter, indicating simplicity but limited adaptability.
2. Goodness-of-Fit Measures
Log-Likelihood
The RD (324.1729) achieves the highest log-likelihood, followed by PRD (311.0430) and WD (310.9042).
The ORWD (40.7147) has the lowest log-likelihood, suggesting a weaker fit compared to the other distributions for this dataset.
Akaike Information Criterion (AIC)
The ORWD (87.4294) achieves the lowest AIC, outperforming the other models, despite its lower log-likelihood. This indicates its effectiveness in balancing fit and flexibility.
Other distributions have much higher AIC values (RD: 650.3458, PRD: 626.0860, WD: 625.8085), showing inferior overall performance.
Bayesian Information Criterion (BIC)
The ORWD (87.6199) still has the lowest BIC, confirming its efficiency.
The RD (650.3771), PRD (626.1805), and WD (625.9030) perform worse.
CAIC and HQIC
The ORWD performs best across CAIC and HQIC metrics, demonstrating its robustness.
3. Goodness-of-Fit Tests
a. *Cramér-von Mises (W)**
The RD (0.0397) achieves the lowest value, followed by the ORWD (0.0447).
The WD (0.0899) and PRD (0.0857) perform slightly worse.
b. *Anderson-Darling (A)**
The RD (0.2813) performs best, followed by the ORWD (0.3380), showing its ability to capture tail behaviour effectively.
The PRD (0.5329) and WD (0.5563) perform worse.
c. Kolmogorov-Smirnov (D)
The WD (0.0757) performs best, followed by the PRD (0.0855).
The RD (0.1727) and ORWD (0.9889) perform poorly. The ORWD's p-value (2.2e-16) is significant but indicates a large deviation from the empirical data.

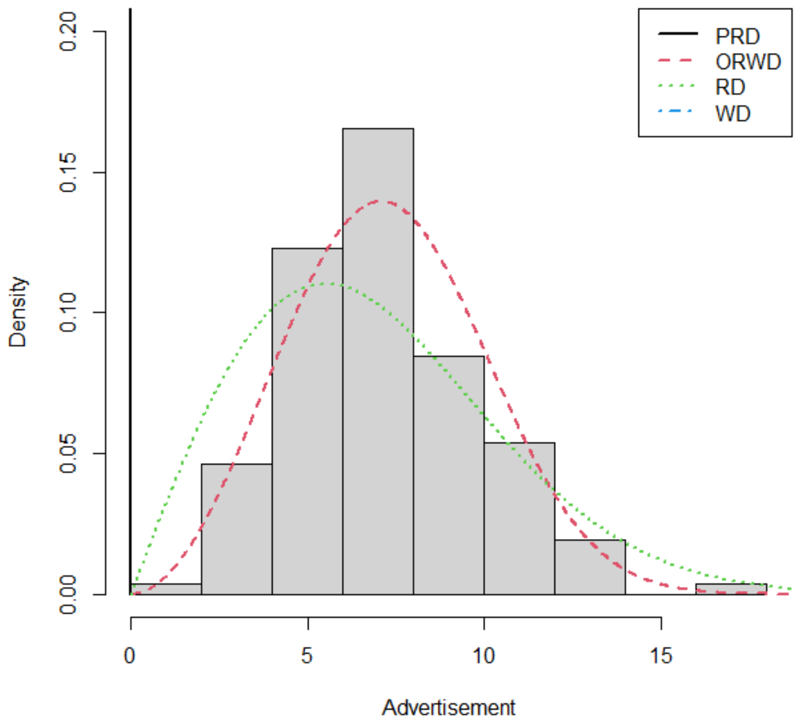

Figure 7: Histogram Plot Advertisement Plot of ORW and others Distributions

Figure 7 shows the density plot visually represents the distribution of advertisement rates and compares different density estimation methods. The curves provide a smoothed estimate of the distribution, helping to understand the underlying data distribution and make statistical inferences.

Table 3: Density Function for the ORWD at $\psi = 1.5$ and $\omega = 2.5$

x	f(x) for $\theta = 1$	f(x) for $\theta = 1.5$	f(x) for $\theta = 2.0$
0.5901	0.0082	0.0036	0.0020
1.2563	0.0314	0.0143	0.0081
2.0135	0.0151	0.0085	0.0052
2.0630	0.0133	0.0077	0.0047
2.3198	0.0054	0.0040	0.0027
2.6221	0.0009	0.0013	0.0010
2.7627	0.0003	0.0006	0.0006
2.8240	0.0001	0.0004	0.0004
3.0943	0.0000	0.0000	0.0001
3.5338	0.0000	0.0000	0.0000
4.0338	0.0000	0.0000	0.0000
4.4065	0.0000	0.0000	0.0000
4.4689	0.0000	0.0000	0.0000
4.6657	0.0000	0.0000	0.0000
4.7758	0.0000	0.0000	0.0000

Table 3 provides a clear comparison of the density function values for the Odd Rayleigh-Weibull Distribution at different x values and parameter sets. The highest density values for all three parameter sets occur around $x = 1.2563$. This suggests that the most likely value of x in the distribution is around this point. As x increases beyond 1.2563, the density values decrease for all parameter sets. This indicates that the likelihood of observing higher values of x diminishes. For $\theta = 1$ consistently has higher density values compared to Sets $\theta = 1.5$ and $\theta = 2.0$. This suggests that the parameter values $\theta = 1$ result in a distribution with higher probabilities for the given x values. For $\theta = 1.5$ has intermediate density values, while $\theta = 2.0$ has the lowest density values across the board.

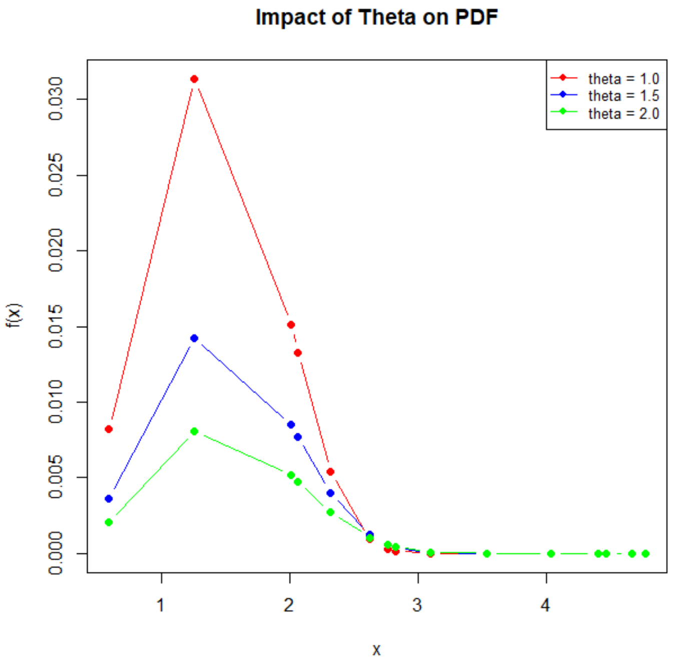

Figure 8: The plot of the influences of parameter theta on the PDF of ORWD

Table 4: Hazard Function for the ORWD at $\theta = 1.5$ and $\omega = 2.5$

x	h(x) for $\psi = 3.5$	h(x) for $\psi = 4.0$	h(x) for $\psi = 6.0$
0.7229	0.0018	0.0011	0.0002
0.8599	0.0029	0.0018	0.0004
1.3011	0.0070	0.0048	0.0012
1.6389	0.0097	0.0072	0.0022
1.7920	0.0105	0.0081	0.0026
2.0359	0.0110	0.0092	0.0034
2.0912	0.0109	0.0093	0.0036
2.1445	0.0108	0.0094	0.0038
2.1490	0.0108	0.0094	0.0038
2.7678	0.0077	0.0086	0.0056

2.9122	0.0066	0.0080	0.0059
3.5885	0.0025	0.0044	0.0064
4.0412	0.0010	0.0024	0.0059
4.7527	0.0002	0.0007	0.0044
4.9682	0.0001	0.0005	0.0039

Table 4 provides a clear comparison of the hazard function values for the Odd Rayleigh-Weibull Distribution (ORWD) at different x values. The table provides the hazard function $h(x)$ for three different parameter sets. The hazard function values increase initially for all three parameter sets, indicating an increasing rate of failure or event occurrence as x increases from 0.7229 to around 2.0359. The highest hazard function values are observed around $x = 2.0359$ for all parameter sets. This suggests that the rate of failure or event occurrence is highest around this point. After reaching the peak, the hazard function values start to decrease. This indicates a decreasing rate of failure or event occurrence as x increases beyond 2.0359. For $\psi = 3.5$ consistently has higher hazard function values compared to Sets $\psi = 4.0$ and $\psi = 6.0$. This suggests that the parameter values $\psi = 3.5$ result in a distribution with a higher rate of failure or event occurrence. For $\psi = 4.0$has intermediate hazard function values, while $\psi = 6.0$ has the lowest hazard function values across the board. For x values greater than 4, the hazard function values approach zero for all parameter sets. This indicates that the probability of failure or event occurrence is extremely low for higher x values.

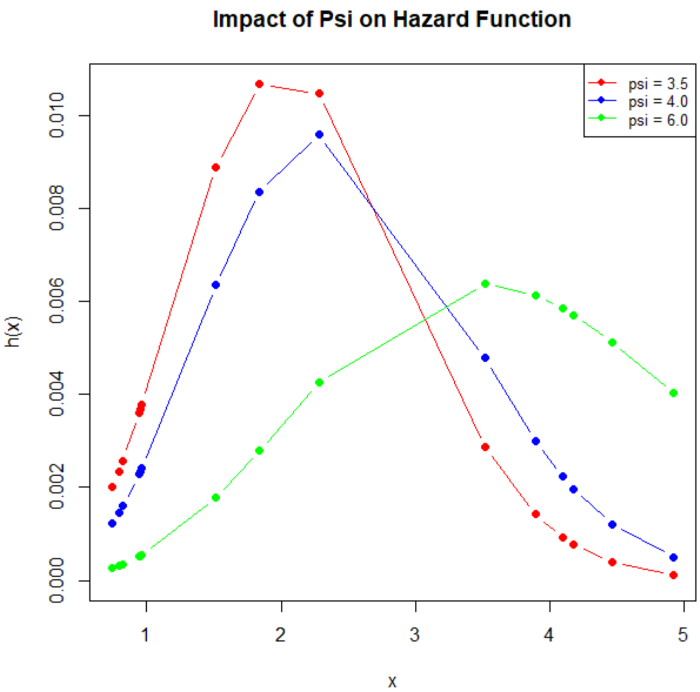

Figure 9: The plot of the influences of parameter psi on the HF of ORWD

Table 5: Survival Function for the ORWD at $\theta = 1.5$ and $\omega = 2.5$

x	S(x) for $\omega = 1$	S(x) for $\omega = 2$	S(x) for $\omega = 3$
0.5990	0.9838	0.9992	1.0000
0.9349	0.9553	0.9950	0.9994
1.1583	0.9257	0.9873	0.9976
1.2829	0.9049	0.9800	0.9954
1.5162	0.8568	0.9570	0.9862
1.5675	0.8445	0.9498	0.9828
1.9332	0.7389	0.8617	0.9261
1.9895	0.7199	0.8407	0.9090
2.1838	0.6488	0.7474	0.8192
3.1277	0.2510	0.0415	0.0003
3.3339	0.1763	0.0046	0.0000
3.5892	0.1024	0.0000	0.0000
4.1064	0.0210	0.0000	0.0000
4.2022	0.0143	0.0000	0.0000
4.9687	0.0001	0.0000	0.0000

Table 5 provides survival function values for the Odd Rayleigh-Weibull Distribution (ORWD) at different x values. The table provides the survival function $S(x)$ for three different parameter sets. For all three parameter sets, the survival function values start high, indicating a high probability of survival at lower x values. For example, at $x = 0.5990$, the survival probabilities are close to 1. As x increases, the survival function values decrease for all parameter sets. This indicates that the probability of survival decreases as x increases. For the $\omega = 1$ generally has lower survival function values compared to $\omega = 2$ and $\omega = 3$, indicating a lower probability of survival. The $\omega = 2$ has intermediate survival function values, suggesting a moderate probability of survival. The $\omega = 3$ has the highest survival function values initially, indicating a higher probability of survival, but it decreases more rapidly than the other sets. For x values greater than 3, the survival function values approach zero for all parameter sets. This indicates that the probability of survival is extremely low for higher x values.

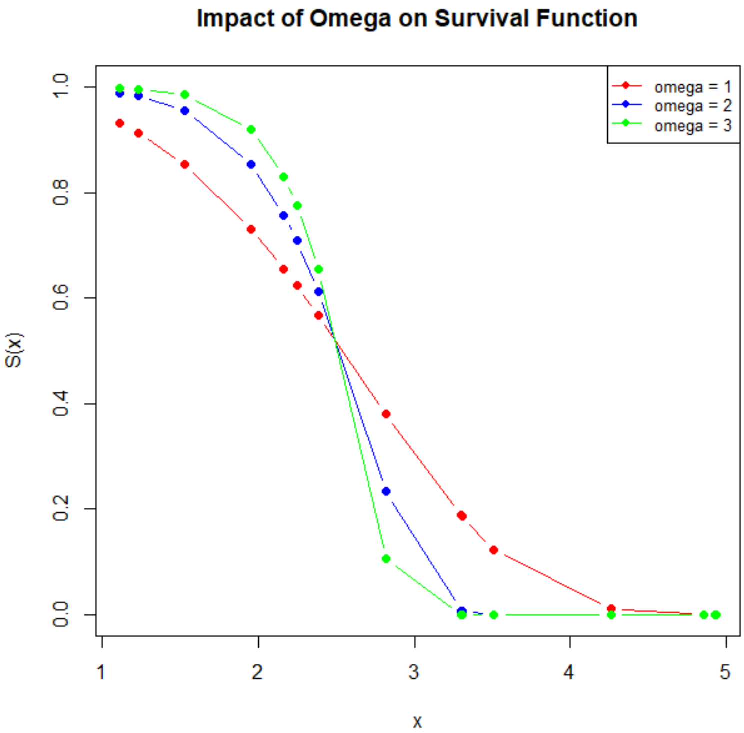

Figure 10: The plot of the influences of parameter theta on the PDF of ORWD

CONCLUSION

This study successfully introduced the Odd Rayleigh-Weibull Distribution (ORWD) and demonstrated its versatility and effectiveness in modelling real-life and survival data. The ORWD was developed by incorporating the "Odd" transformation into the Rayleigh-Weibull distribution, resulting in a flexible model with enhanced fitting capabilities. Through rigorous mathematical derivations, the probability density function (PDF), cumulative distribution function (CDF), hazard function, and survival function of the ORWD were established. The Maximum Likelihood Estimation (MLE) method was employed to estimate the parameters of the ORWD, and its performance was assessed through extensive simulation studies. The simulations revealed that the MLE method provides consistent and efficient parameter estimates across various sample sizes. Comparative analyses with other distributions, such as the Weibull Rayleigh and power Rayleigh distributions, highlighted the superior performance of the ORWD in fitting the mortality dataset. Goodness-of-fit measures, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (KS) test, consistently favoured the ORWD, demonstrating its robustness and applicability. Overall, the ORWD offers a valuable addition to the family of statistical distributions, providing a powerful tool for researchers and practitioners in fields such as reliability engineering, survival analysis, and environmental studies. Future research could explore the application of the ORWD to multivariate data and its integration with copula models to further enhance its utility. The family could also be used to extend other continuous probability distributions and develop their corresponding survival regression models for applications and fitting time-to-event datasets.

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