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ON NONLINEAR BIHARMONIC DISPERSIVE WAVE EQUATIONS

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ABSTRACT

This paper proposes and studies particular nonlinear dispersive biharmonic equation, whose related equations appear in various physical phenomena such as wave propagation in nonlinear media and plasma physics. We chose the power kind of nonlinearity as it is common in these areas. We show that the linear version exhibits strong dispersive behaviour while the nonlinear version reveals possible emergence of singularities for higher degree nonlinearity exponent p. Both versions of the equation, linear and nonlinear, were solved analytically where for the latter we use perturbation approach and Fourier transform for the former. A glimpse towards the symmetry analysis of the underlying equations is provided and somewhat insights into the behaviour of the solution is discussed.

Keywords: Biharmonic, Dispersion, Nonlinearities, Singularities, Perturbation, Numeric

INTRODUCTION

Nonlinear dispersive equations play a fundamental role in theoretical and applied mathematical physics, particularly in modelling wave propagation phenomena where both nonlinear and dispersive effects are significant. Some of the prominent dispersive equations are Airy equation (Airy, 1838 & Stokes, 1847), Korteweg de Vries equation (Korteweg and de Vries 1895), Nonlinear Schrodinger (NLS) equations (Zakharov and Shabat, 1972), Klein-Gordan equation (Klein 1926), Sine-Gordon equation (Lamb 1980), and the likes. These equations possess special properties uniquely defined for their dispersive nature such as local smoothing, regularity, local and global existence.

In our case, we are concern with the equation

$$u_{tt} + \beta u_{xxxx} + \alpha u^p u_{xx} = 0, \quad u:\mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}, \quad \alpha,\beta \in \mathbb{R} \quad p \geq 1 \tag{1}$$

This equation is a complex-valued so that its solution $u(t,x)$ describes certain electromagnetic wave. It is of interest due to its combination of higher-order dispersion (through the u_{xxxx} term) and nonlinearity (through $u^p u_{xx}$), leading to rich and complex dynamics. This equation could share important characteristics with the other well-known equations of mathematical physics. It is similar to the equations: Boussinesq $u_{tt} - u_{xx} + (u^2)_{xx} = 0$ which studies the propagation of the long waves in shallow water bodies (Boussinesq, 1872), KdV equation $u_t + \alpha\, u_{xxx} + \beta\, uu_x = 0$ describing the evolution of waves on surface of shallow water which can be solitons for some α and β (Korteweg and de Vries, 1985). Other closely related work is that of Whitham (1974) for Korteweg-de Vries-Burgers (KdV-Burgers) which merges the KdV equation's dispersive features with Burgers' equation's dissipative characteristics, thereby studying the interplay between the nonlinearity, dispersion and dissipation properties of the equation.

In essence, our equation captures the evolution of dispersive waves in the presence of higher nonlinearities, in a way similar to the NLS equations with power nonlinearity, see Sulem and Sulem (1999) and singularity in the solutions of Zakharov equations (Papanicolaou, 1991).

Dispersive Nature of the Equation

The idea of dispersion is to explain how $u(t,x)$ spreads through space, whereas dispersive equations are characterized by the fact that waves of different wavelengths propagate at different speeds. This behaviour is typically encoded in the dispersion relation of the equation. In our case, for the linearized equation

$$u_{tt} + \beta u_{xxxx} = 0, \tag{2}$$

we perform a Fourier transform to obtain the dispersion relation. This is attained by assuming a plane-wave solution of the kind

$$u(x,t) = e^{i(kx-\omega t)},$$

substitution into the linearized equation (2) gives the dispersion relation

$$\omega^2 = \beta k^4.$$

This fourth-order relationship between ω and k indicates that higher frequency (shorter wavelength) components will propagate at different speeds, thus demonstrating the dispersive nature of the system. The inclusion of the nonlinear term $u^p u_{xx}$ further modifies the dispersion relation, leading to richer dynamics. Nonlinear effects can result in the formation of solitons or other localized structures, depending on the balance between the nonlinear and dispersive effects.

The underlying biharmonic equation (1) poses a number of challenges as a result of the presence of both high order dispersive terms and nonlinearities. Due to its ubiquitous nature in application, such as *waves in elastic rods* or *plasma waves*, where both dispersion and nonlinearity must be accounted for, it is worth studying to address unanswered questions such as the possible existence of *solitonic waves* and *singularities* representing wave collapse or blow-up as can be found in the monograph of self-focusing and wave collapse by Sulem & Sulem (1999).

Our primary objective is to derive exact and approximate solutions, compare them, and analyse the dynamics of the equation. We analytically study the dispersive nature of the equation (1) and determine its solutions via Fourier Transform and perturbation approaches. We further proceed determining the travelling wave forms of the equation and discuss the symmetry analysis for the behaviour of $u(t,x)$.

Related Work

Nonlinear dispersive equations, such as the Korteweg-de Vries (KdV) equation and the nonlinear Schrodinger equation, have been extensively studied due to their ability to model phenomena ranging from shallow water waves to plasma physics. Early work on the KdV equation by Zabusky and Kruskal (1965), led to the discovery of solitons, which are stable, localized waves that result from a balance between dispersion and nonlinearity. Other works that provides intensive studies on equations like nonlinear Schrodinger equations and other dispersive partial differential equation (PDEs) include the work of Klein and Claude-Saut (2022).

More recently, researchers have extended these classical models to higher-order dispersive terms, similar to those found in our equation of interest. For instance, the higher-order nonlinear Schrodinger equation has been used to model ultra-short pulse propagation in optical fibres, Agrawal (2001). Studies by Ablowitz and Clarkson, (1991), have shown that exact solutions to nonlinear dispersive equations can often be found using symmetry methods or inverse scattering techniques...

Other work, which of our central focus for future work is the traveling wave solutions, and have been widely used in understanding nonlinear wave phenomena. In particular, Whitham (2000, 1974 and 2011) in his series of publications on Linear and Nonlinear waves explored the role of traveling waves in dispersive media, highlighting their importance in capturing the balance between nonlinearity and dispersion.

Regarding numerical approaches for dispersive equations, there are several existing techniques consisting of finite differences and other robust techniques like spectral one. In a study by Pan et al. (2023), developed a fourth-order compact finite difference schemes for solving biharmonic equations with Dirichlet boundary conditions. This method achieves high accuracy while maintaining computational efficiency for both two- and three-dimensional problems.

Another recent work by Cheng Ma (2023), explored the solutions of biharmonic and mixed dispersion equations, particularly focusing on variational methods and Sobolev space embeddings. These studies addressed nonlinear effects in high-order dispersion systems, including their stability and energy dynamics. Other important literatures on the numerical approach for such dispersive equations using classical and spectral methods based on the existing symbol and numerical software are available in the following texts, Leveque (2007), Trefethen (2000).

Important works on the geometric structure and advanced techniques for treatment of dispersive equations can be found, respectively, in Bluman and Kunei (1989), on the symmetries of differential equations, Bender and Orszag (1978).

It has been examined that the role of biharmonic dispersive equations can be found in modelling phenomena such as fluid dynamics, elastic plates, and nonlinear optics. This work often incorporates constraints like energy minimization and examines specific boundary conditions.

Dispersive nature, Derivation and Analytical Approaches

Dispersive nature of the nonlinear equation

Dispersion relation for dispersive equations plays important role towards the analysis of the such equations. It tells more about the nature of the underlying unknown functions regarding how it spreads into space as it evolves in time.

To address the dispersive nature of the equation, let us consider the linear part of an equation

$$u_{tt} + \beta u_{xxxx} + \alpha u^p u_{xx} = 0,\ u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C},\ p \geq 1 \quad (3)$$

which is

$$u_{tt} + \beta u_{xxxx} = 0, \qquad u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}, \quad (4)$$

One assumes a plane wave solution $u(x,t) \equiv e^{i(kx-\omega t)}$ where ω is the plane-wave frequency and its wave number k. Upon substituting it into the linear equation, one gets the relation:

$$\omega = \omega(k) = \pm\sqrt{\beta} k^2 \quad (4a)$$

This clearly satisfies the condition that

$$v_g = \nabla\omega = \pm 2\sqrt{\beta} k \neq \frac{\omega}{k} = \pm\sqrt{\beta} k = v_p.$$

Hence, the underlying linear equation is dispersive with dispersive relation (4a).

For the nonlinear equation (3), we need to take deeper analysis of the nonlinear term, for simplicity we will take $p = 1$. The presence of the nonlinear term $\alpha u^p u_{xx}$ complicates the derivation of the dispersion relation. However, we can still attempt to derive a relation for small perturbations around a "ground state" or the so-called base, commonly referred to as linearizing the equation about a steady state.

We assume that the solution $u(x,t)$ consists of a small perturbation $\tilde{u}(x,t)$ around a constant background

$$u_0: \quad u(x,t) = u_0 + \tilde{u}(x,t), \quad \text{where} \quad |\tilde{u}(x,t)| \ll u_0. \quad (5)$$

Substituting (5) into the nonlinear equation (3):

$$(u_0 + \tilde{u})_{tt} + \beta(u_0 + \tilde{u})_{xxxx} + \alpha(u_0 + \tilde{u})(u_0 + \tilde{u})_{xx} = 0, \quad (6)$$

and expanding this (6) to first-order terms in $\tilde{u}$ (neglecting higher-order nonlinear terms), we get:

$$\tilde{u}_{tt} + \beta\tilde{u}_{xxxx} + \alpha u_0 \tilde{u}_{xx} = 0.$$

This is now a linear equation for the perturbation $\tilde{u}$, which can be analysed to find a dispersion relation. We, furthermore, assume a plane wave solution for $\tilde{u}(x,t)$ of the form:

$$\tilde{u}(x,t) = e^{i(kx-\omega t)}, \qquad k, \omega \in \mathbb{R} \quad (7)$$

where k is the wave number and ω is the angular frequency. Substituting this (7) into the linearized equation (4):

$$-\omega^2 e^{i(kx-\omega t)} + \beta k^4 e^{i(kx-\omega t)} + \alpha u_0 k^2 e^{i(kx-\omega t)} = 0, \quad (8)$$

Simplify (8), we get the dispersion relation:

$$-\omega^2 + \beta k^4 + \alpha u_0 k^2 = 0 \quad \Rightarrow \quad \omega^2 = \beta k^4 + \alpha u_0 k^2.$$

Thus, the dispersion relation for the nonlinear version of the equation is:

$$\omega = \pm\sqrt{\beta k^4 + \alpha u_0 k^2} = \pm k\sqrt{\beta k^2 + \alpha u_0}$$

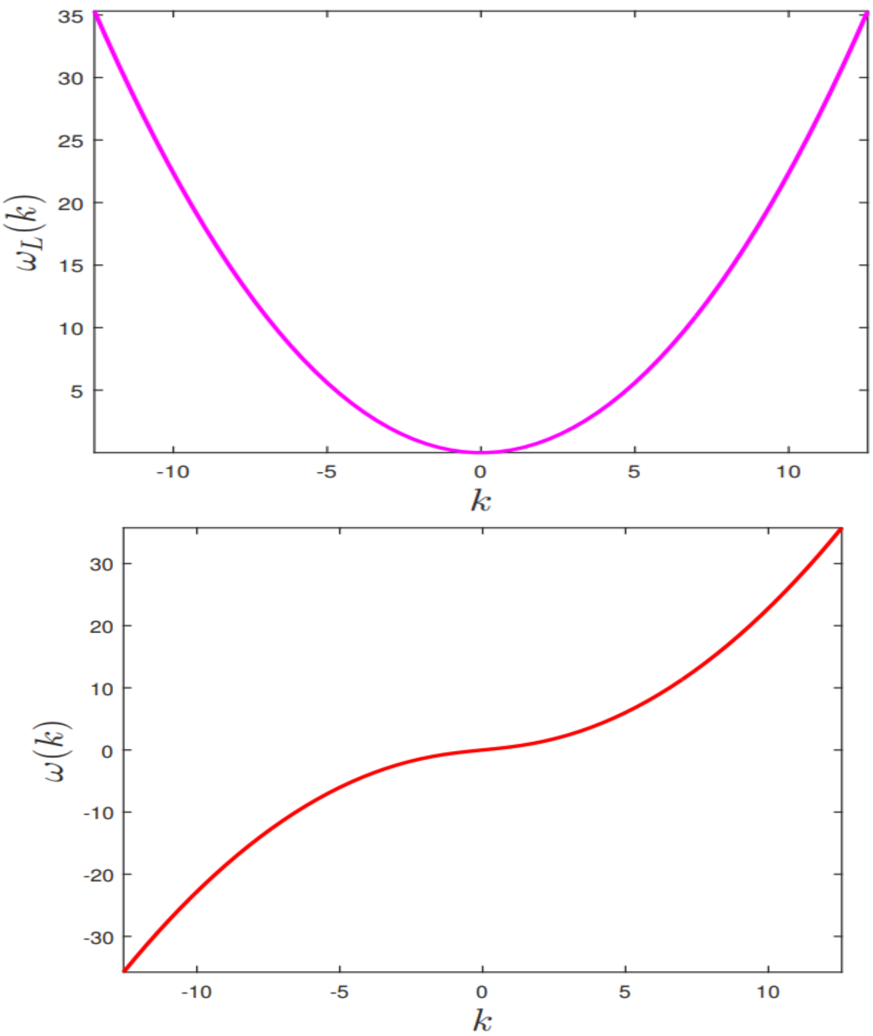


Figure 1: The dispersions relations of (top) the linear-equation (4) and (b) nonlinear-equation (3)

Thus, we will classify the dispersion relation for large and small wave-numbers k. For large wave numbers k, the βk^4 term dominates, and the equation behaves like a purely dispersive equation, similar to the linear dispersive wave equation (4). For smaller wave numbers, the nonlinear term $\alpha u_0 k^2$ contributes, introducing a modification to the frequency ω. This is evident from Fig.1.

This shows how nonlinearity affects the wave dispersion, particularly for lower k. The presence of the nonlinear term adds a quadratic component to the dispersion relation, altering the wave dynamics in comparison to the purely linear case.

Analytical Approaches

Linear problem

Solving the linear problem (2) requires the use of initial data so that it is a potential well-posed problem. This equation is solved via Fourier Transform as suggested by the dispersion relation.

Let us assume $u(t,x)$ has the Fourier transform:

$$\hat{u}(t,k) = \int_{-\infty}^{\infty} u(t,x)\, e^{-ikx} dx$$

where k is the wave number.

Taking the Fourier transform of the linear PDE (3) and the Schwartz kind of initial data $u_0(x) = e^{-x^2}$:

$$\mathcal{F}[u_{tt}] + \beta\mathcal{F}[u_{xxxx}] = 0, \quad \mathcal{F}[u_0] = \mathcal{F}\left[e^{-x^2}\right] = \sqrt{\pi}\, e^{-\frac{k^2}{4}}.$$

Since the Fourier transform of the derivative yields

$$\mathcal{F}[u_{tt}] = \partial_{tt}\hat{u}(t,k), \qquad \mathcal{F}[u_{xxxx}] = (-ik)^4\, \hat{u}(t,k)$$

where $\hat{u}$ denotes the Fourier transform of $u(t,x)$.

Substituting these, we get

$$\frac{\partial^2}{\partial t^2}\hat{u} + \beta\, k^4\, \hat{u} = 0.$$

This is a second order equation whose solution is

$$\hat{u}(t,k) = A(k)\, e^{i\omega t} + B(k) e^{-i\omega t}$$

where $\omega = \sqrt{\beta}\, |k|^2$ is the associated dispersion relation as noticed earlier.

To retrieve $u(t,x)$ take the inverse Fourier transform:

$$u(t,x) = \int_{-\infty}^{\infty} \hat{u}(t,k) e^{ikx} dk.$$

Substituting $\hat{u}(t,k)$ we get

$$u(t,x) = \int_{-\infty}^{\infty} \left[A(k)\, e^{i\omega t} + B(k) e^{-i\omega t}\right] e^{ikx}\, dk.$$

This leads to the final form

$$u(t,x) = \int_{-\infty}^{\infty} \left[A(k)e^{i(kx+\sqrt{\beta}k^2t)} + B(k)e^{i(kx-\sqrt{\beta}k^2t)}\right] dk$$

The energy functionals for linear and nonlinear problem involving both kinetic and potential contributions:

$$E_{\text{linear}}(t) = \frac{1}{2}\int_{-\infty}^{\infty} [u_t^2 - \beta u_{xx}^2]\ dx;\ \ E_{\text{nonl}}(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left[u_t^2 - \beta u_{xx}^2 - \frac{2\alpha}{(p+1)}u^{p+1}u_{xx}\right] dx$$

And the mass functional is:

$$M(t) = \int_{-\infty}^{\infty} u(t,x)\, dx.$$

For simplicity, assume Gaussian initial conditions, say $e^{-a\cdot x^2}$, with $A(k) = B(k) = \sqrt{\frac{\pi}{a}}e^{-\frac{k^2}{4a}}$, the resulting exact solution in real space is:

$$u(t,x) = \frac{1}{\sqrt{1+4i\beta\, t}}e^{-\frac{x^2}{1+4i\beta\, t}}, \quad (9)$$

When this solution is substituted into the linear wave equation, it can be found that $u(t,x)$ is a solution if

$$\beta = 1, \quad \text{or} \quad \beta = \left(\frac{1}{4t} + \frac{1}{6}(-3 \pm x^2\sqrt{6}\,)\right)i.$$

The exact solution (9) is indeed the solution for the linear problem (4) in these cases.
The simulation of the solution for $t = 0 \ldots 2$ and $x \in [-4\pi, 4\,\pi]$ with Gaussian initial data e^{-x^2} is shown in Fig. 2.

Variational Derivation of the Equation

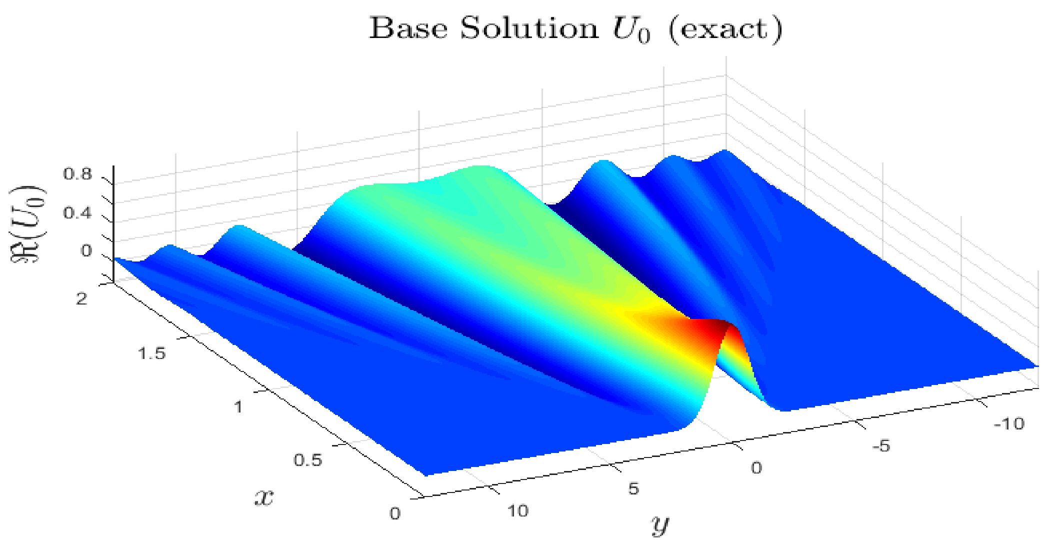
The equation nonlinear problem can be derived from the Lagrangian formulation

$$\mathcal{L}(u,u_t,u_{xx}) = \frac{1}{2}\left[u_t^2 - \beta u_{xx}^2 - \frac{2\alpha}{(p+1)}u^{p+1}u_{xx}\right] \quad (10)$$

so that the equation is derived from the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial u_t}\right] - \frac{\partial^2}{\partial x^2}\left[\frac{\partial\mathcal{L}}{\partial u_{xx}}\right] + \frac{\partial}{\partial x}\left[\frac{\partial\mathcal{L}}{\partial u_x}\right] - \frac{\partial\mathcal{L}}{\partial u} = 0 \quad (11)$$

The Lagrangian (10) when plugged into the equation (11) indeed yields the main equation (3).



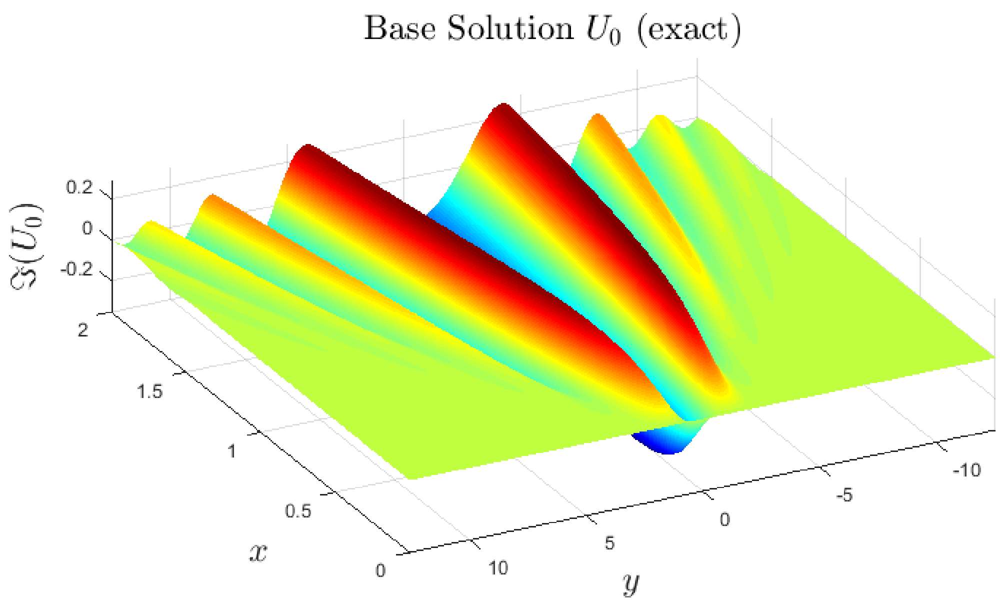


Figure 2: Exact solution to the linear biharmonic equation (2), $\beta = 1$: (a) $\Re(u(t,x))$ (b) $\Im(u(t,x))$

Nonlinear Problem
For the nonlinear problem
$u_{tt} + \beta\, u_{xxxx} + u^p u_{xx} = 0$
one applies, again, the well-known Fourier transforms as in the linear case, which are particularly useful for dispersive equations. We decompose the solution $u(x,t)$ into its Fourier modes:

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx}\, dk.$$

Substituting this into the equation $u_{tt} + \beta u_{xxxx} + \alpha u^p u_{xx} = 0$, we obtain the following ordinary differential equation for the Fourier transform $\tilde{u}(k,t)$:

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} + \beta k^4 \tilde{u}(k,t) + \alpha \mathcal{F}(u^p u_{xx})(k,t) = 0, \quad p \geq 1$$

where $\mathcal{F}$ denotes the Fourier transform and the nonlinear part is treated in a special way.
For time integration, we apply the fourth-order Runge-Kutta scheme, as described in the section on numerical method, to evolve the Fourier-transformed equation in time. The scheme proceeds by computing intermediate steps to approximate the solution at each time step, ensuring stability and accuracy for moderate to long-time simulations. This method is efficient, especially when combined with spectral techniques that handle the spatial derivatives in the Fourier domain. This is treated and properly implemented in the succeeding sections.

Perturbation Approach
In the regime where $\varepsilon \ll 1$, the nonlinearity can be treated as a small perturbation. We expand the solution as a series in powers of ε:
$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$
Here we give complete scheme to be implemented
For the nonlinear equation:
$u_{tt} + \beta\, u_{xxxx} + \alpha\, u^p u_{xx} = 0$
expand $u(t,x)$ in a perturbative series:

$$u(t,x) = u_0(t,x) + \varepsilon u_1(t,x) + \varepsilon^2 u_2(t,x) + \varepsilon^3 u_3(t,x) + \ldots$$

where $u_0(t,x)$ is the solution of the linear equation (2):

Zeroth-Order Equation
$u_0(t,x)$: $\quad u_{0,tt} + \beta u_{0,xxxx} = 0$
whose solution is $u_0(t,x) = \frac{1}{\sqrt{1+4i\beta}} e^{-\frac{x^2}{1+4i\beta}}$.
First-Order Equation: Next we, substitute
$u(t,x) = u_0(t,x) + \varepsilon u_1(t,x)$
into the original equation (1) and collect terms of $O(\varepsilon)$ to get
$u_1(t,x)$: $\quad u_{1,tt} + \beta u_{1,xxxx} = -\alpha u_0^p u_{0,xx}$
where the term $u_0^p u_{0,xx}$ is treated as forcing term.
Second-Order Equation: Substituting again
$u(t,x) = u_0(t,x) + \varepsilon u_1(t,x) + \varepsilon^2 u_2(t,x)$
and collect terms of $O(\varepsilon)$ we get:
$u_2(t,x)$: $\quad u_{2,tt} + \beta u_{2,xxxx} = -\alpha\left(p u_0^{p-1} u_1 u_{0,xx} + u_0^p u_{1,xx}\right)$,
where higher order nonlinear interactions are included.

Third-Order Equation: Similarly, we collect terms of order $O(\varepsilon^3)$ to get:

$$u_3(t,x): \quad u_{3,tt} + \beta u_{3,xxxx} = -\alpha\left(\frac{p(p-1)}{2} u_0^{p-1} u_1^2 u_{0,xx} + p u_0^{p-1} u_2 u_{0,xx} + p u_0^{p-1} u_1 u_{1,xx} + u_0^p u_{2,xx}\right),$$

Integration: Finally, we solve for $u_1, u_2, u_3, \ldots$ using techniques like Fourier transform or numerical methods. Each equation involves the same linear operator applied to different source terms, which are derived from lower-order terms.
Higher order corrections can be computed iteratively, although the complexity increases at each step. This method provides an accurate approximation for weakly nonlinear regimes and offers insight into how the nonlinearity modifies the behaviour of the linear dispersive waves.
Below, we present solutions $u(t,x)$ up to third order corrections with respect to the choice of nonlinearities and dispersion parameters.

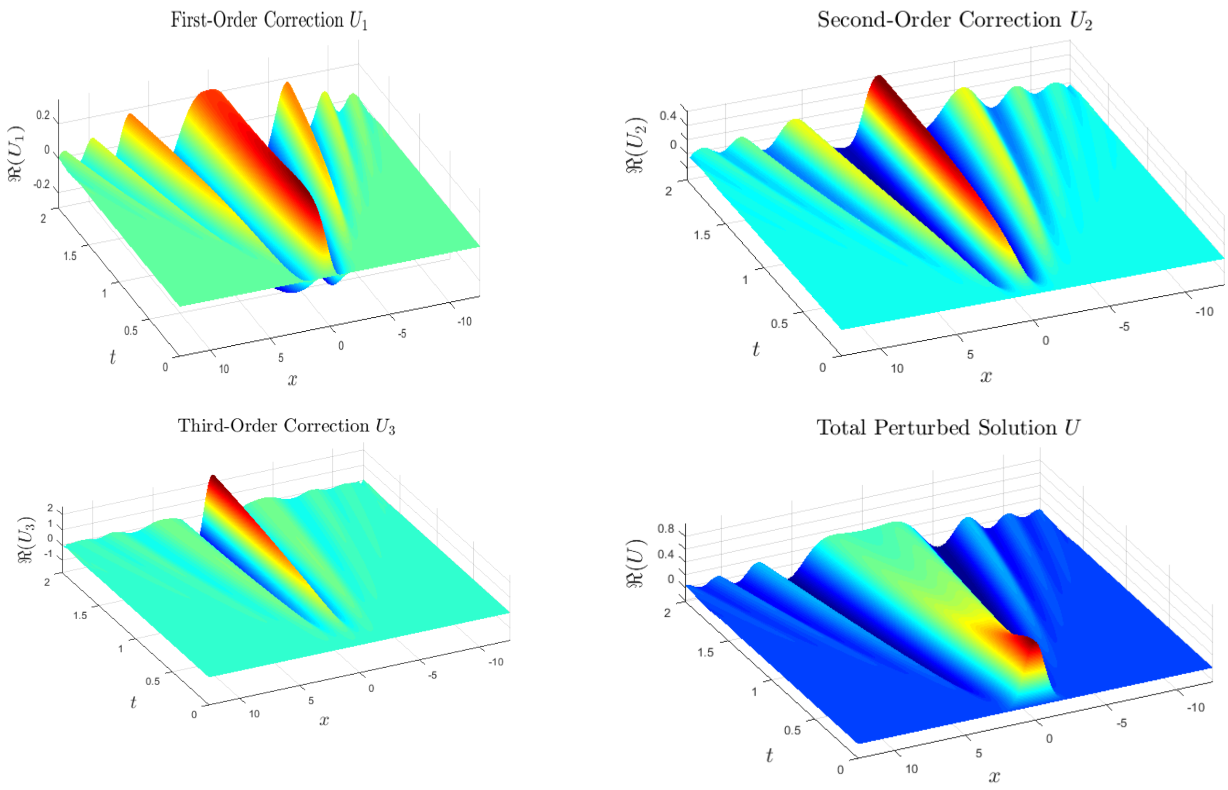


Figure 3: Evolution (for the nonlinear equation (1)) of the real parts of $u_0, u_1, u_2, u_3, \tilde{u}(t,x)$ for $p = 1, \alpha = \beta = 1, \varepsilon = 0.1$; The respective mass and energy conservation of the perturbed solution is presented in Fig 4

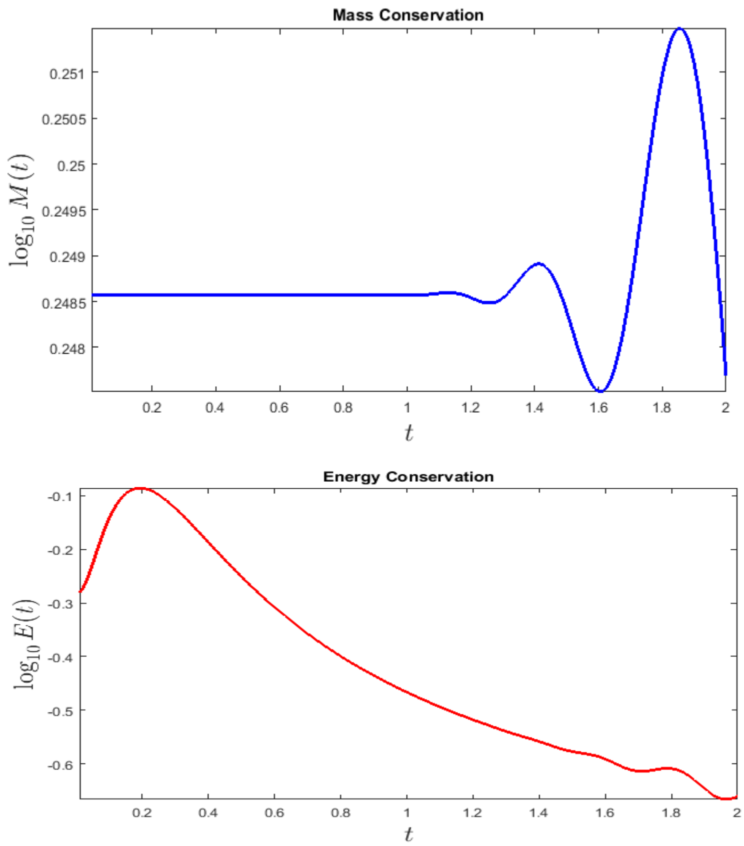


Figure 4: log-plots of the mass $M(t)$ and energy $E(t)$ conservation of the perturbed solution

For (higher nonlinearity) $p = 2, \alpha = \beta = 1, \varepsilon = 0.1$, we have the following results for the real parts of u_1, u_2, u_3 shown the following figures of Fig.6.

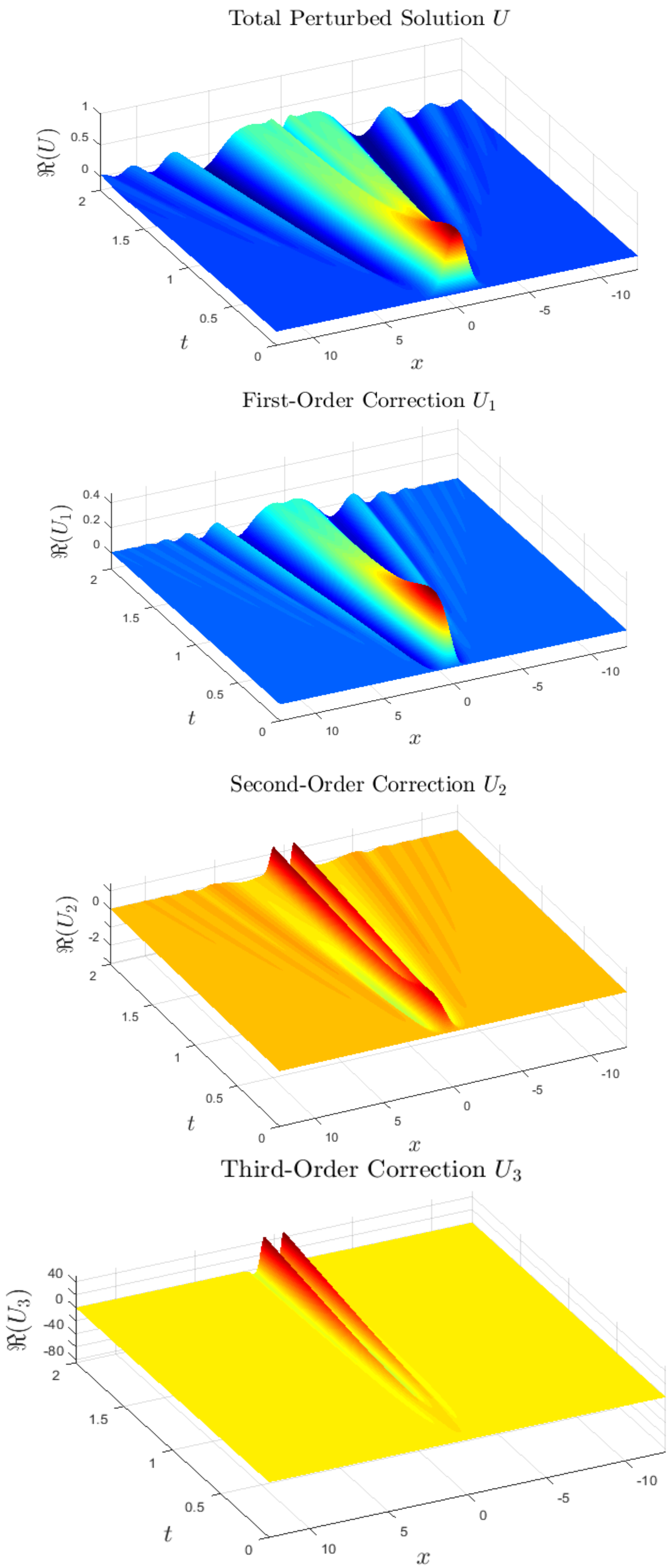


Figure 5: Real parts of the correction terms and perturbed solution for $p = 2, \alpha = \beta = 1, \varepsilon = 0.1$

Imaginary parts of the solutions as described in Fig. 5 is shown as follows, for $p = 2, \alpha = \beta = 1$, and perturbation parameter $\varepsilon = 0.1$ is shown in Fig. 6.

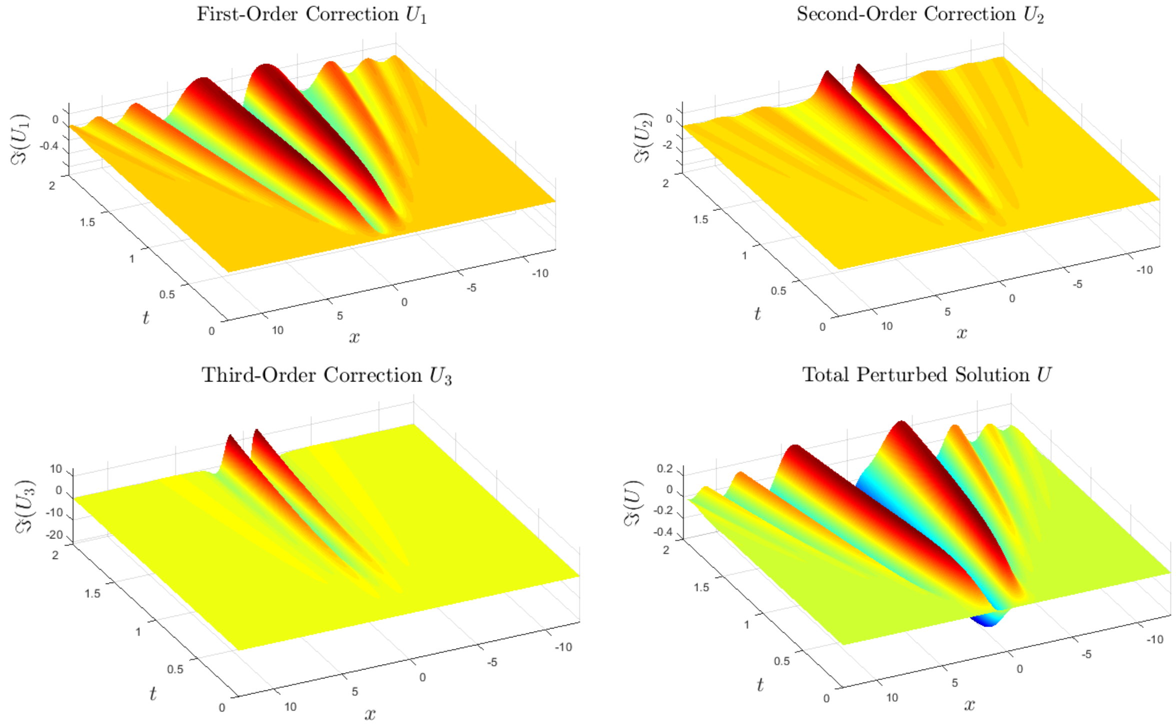


Figure 6: The imaginary part of the perturbed solutions with correction terms as in Fig.5

For $p = 2, \beta = \alpha = 1$, keeping $\varepsilon = 0.1$, we get the imaginary solutions as follows:

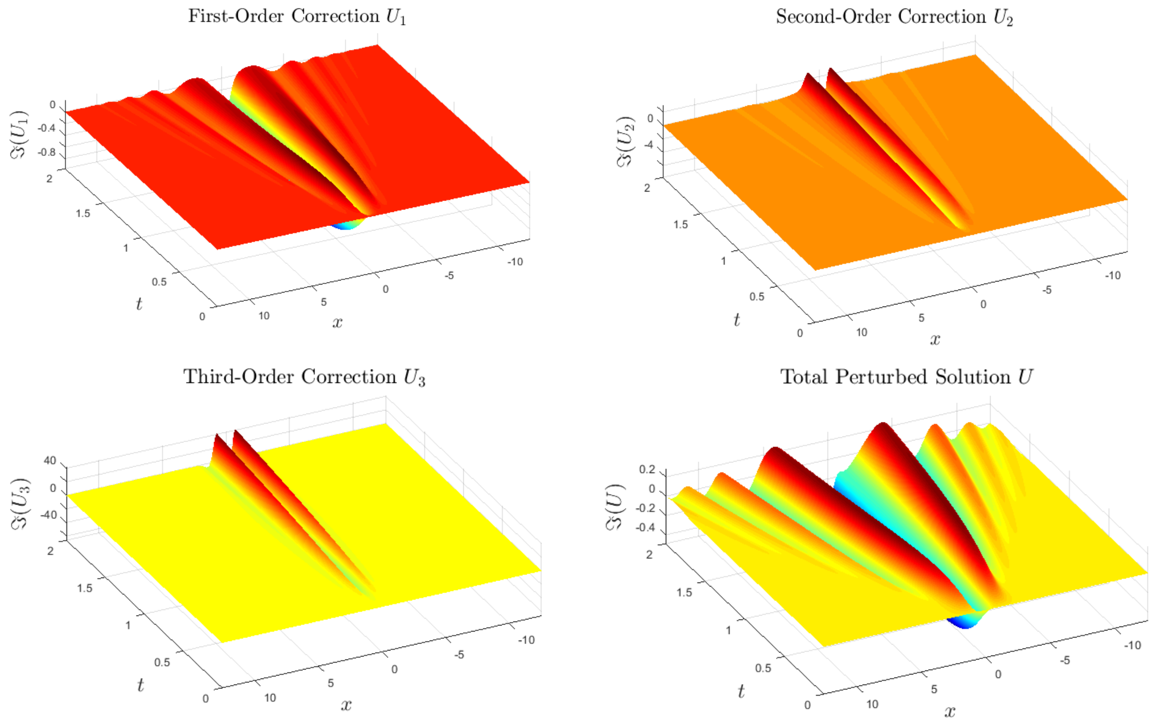


Figure 7: Imaginary parts of the perturbed solutions and its correction terms for $p = 2, \alpha = \beta = 1$

The perturbation approach is understood to be more efficient in terms of numerical implementation than the classical numerical approach like the finite difference and the likes. This is because it allows higher nonlinear parameters $p \geq 2$, unlike the RK4 implemented in the succeeding sections.

Numerical Methods: Spectral Methods

In addition to finite difference methods, spectral methods offer a highly accurate approach for solving partial differential equations, particularly when the solution is smooth. Spectral methods rely on approximating the solution

as a sum of basis functions, typically Fourier series or Chebyshev polynomials.
For the nonlinear equation (2), we use a Fourier spectral method, where the spatial derivatives are computed in the Fourier domain. The nonlinear term $u^p u_{xx}$ is computed in physical space, and then transformed back to Fourier space. This combination of spectral and physical space calculations is known as a pseudo-spectral method.
The advantage of these spectral methods is that they converge exponentially for smooth problems, providing highly accurate solutions with fewer grid points compared to finite difference methods. Thus, we consider, in our settings, only smooth scenarios like Schwarz initial data. However, they require periodic boundary conditions or special treatment of boundaries.

RK4 - Fourier Transform Set-up
This is implemented on the equations

$$\tilde{u}_{tt} = L\tilde{u} + N[\tilde{u}]$$

where L is linear operator and $N(u)$ a nonlinear operator in u.
For an equation of the form:

$$\frac{du}{dt} = f(t,u), \quad u(t_0) = u_0$$

the RK4 update formula is given by:
Computation of Intermediate Steps:

$$K_1 = h\,f(t_n, u_n),$$

$$K_2 = h\,f\left(t_n + \frac{h}{2}, u_n + \frac{K_1}{2}\right),$$

$$K_3 = h\,f\left(t_n + \frac{h}{2}, u_n + \frac{K_2}{2}\right),$$

$$K_4 = h\,f(t_n + h, u_n + K_3),$$

Getting Solution Updating the solution $u(t_n, x)$:

$$u_{n+1} = u_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).$$

Here h is the time step, t_n us the current time and y_n is the solution at the current time t_n.

In our case, the time derivative is of second order, therefore there is need to convert the equation into a system. Letting $v = \partial_t u$, we have the system that

$$\frac{\partial u}{\partial t} = v$$

$$\frac{\partial v}{\partial t} = -\beta\, k^4\, u - \alpha\, \mathcal{F}(u^p u_{xx})$$

so that the solution $u(t_n, x)$ is updated at every instant t_n for all x.

Mathematical Analysis of RK4
The RK4 scheme is chose in order that:
Order of Accuracy: the local truncation error is $O(h^5)$, while the global error is $O(h^4)$. This is because the RK4 is a fourth-order scheme.
Stability: the solution $u(t,x)$ is stable for all t. This is sequel to the fact that RK4 is conditionally stable, and its stability depends on the size of the time step h relative to the eigenvalues of the differential operator in the equation.
Energy Conservation: For biharmonic equations, RK4 does not inherently conserve energy. Special adaptations or symplectic methods might be necessary for long-term simulations.
However, the RK4 method on the nonlinear version may suffer numerical instability especially for higher order power $p \geq 2$. This is due to the possible accumulation of errors when taking the exponents p in generating the solution, except, the when the dispersion coefficient parameterβ is small. For instance, see the solution as shown in the figure below. It, moreover, develops singularity as $t \to \infty$.

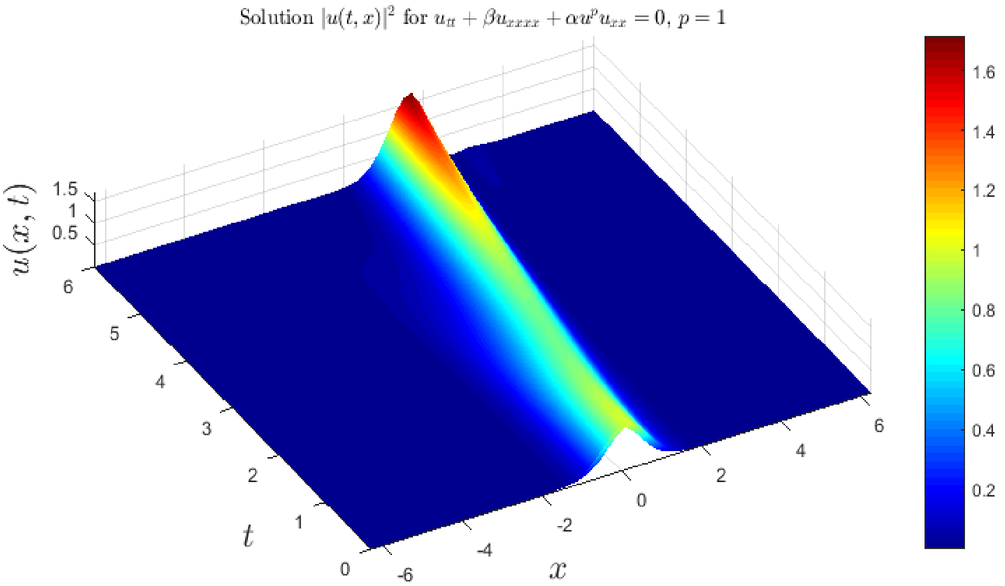


Figure 8: Showing the solution for $p = 1,\ \alpha = 1, \beta = 0.05$

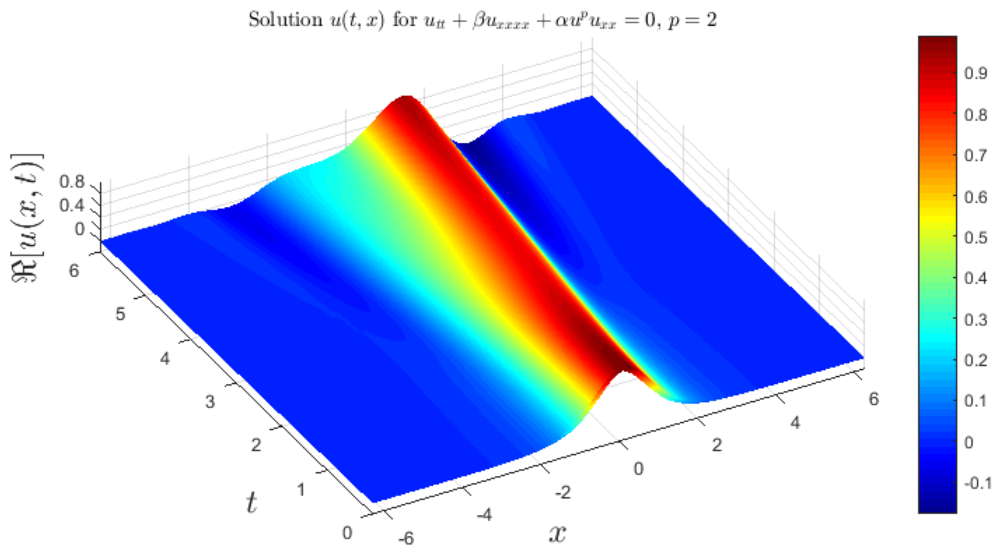


Figure 9: Showing the real part of the solution $u(t,x)$ for $p = 2, \alpha = 1, \beta = 0.05$

Singularity is observed in these cases, just as expected, in a situation where nonlinearity dominates. As one will notice later, the when the dispersion is made stronger and nonlinearity is weakened, the dispersion takes over thereby having the initial lump dispersing away through the space. These are the not surprising since the common property of dispersive equations.

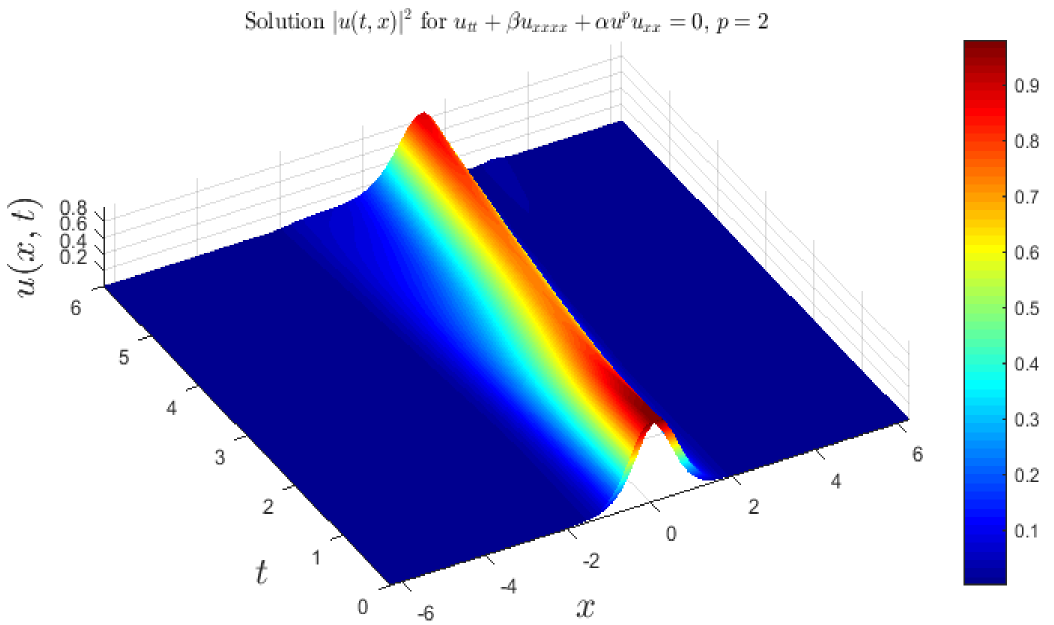
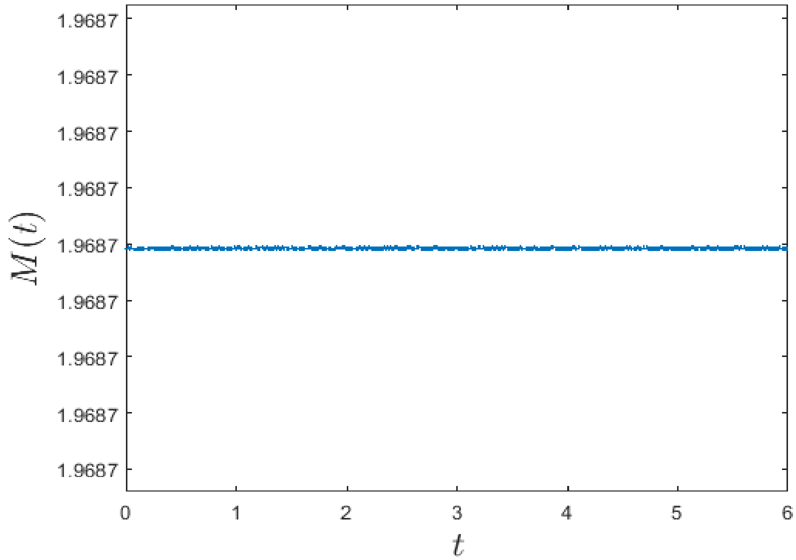


Figure10: Showing the solution for $p = 2, \alpha = 1, \beta = 0.05$

Moreover, we do not get good control of the energy function as it is not preserved for all t. This is illustrated in the second figure in Fig. 1**1** below.



The mass $M(t)$ of the solution of the nonlinear problem with $p = 2$.

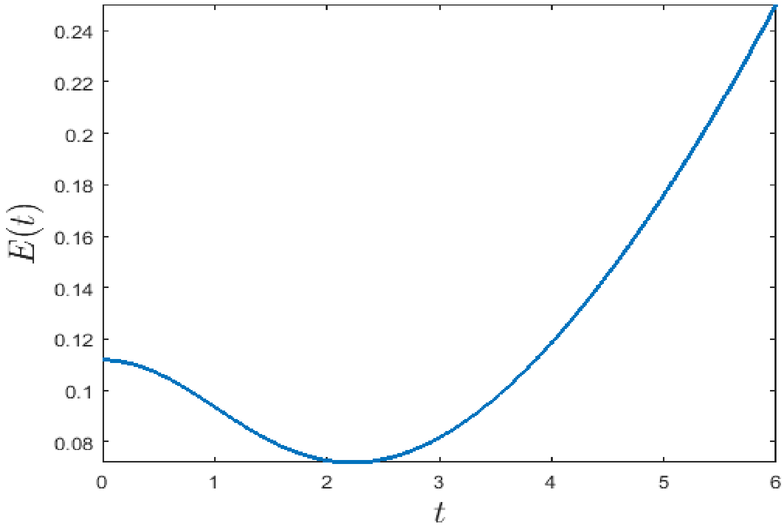


Figure 11: The evolution of mass and energy functions for $p = 2$ $\alpha = 1, \beta = 0.05$

When $\beta = 0.5$, we get the following results. The mass remains conserved as in the case for $\beta = 0.05$, but the energy shows significant change while the solution is as shown.

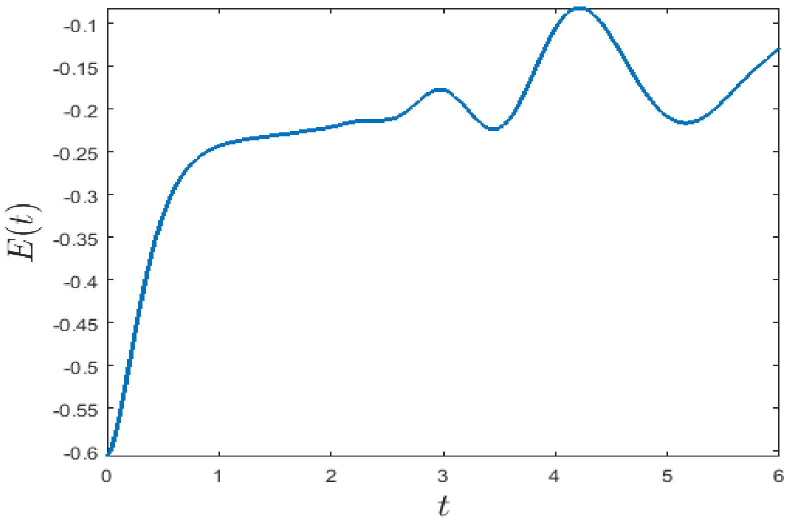


Figure 12: Evolution of energy function with $\beta = 0.5$ and $p = 2$

The solution with these parameters is shown in the Fig.13.

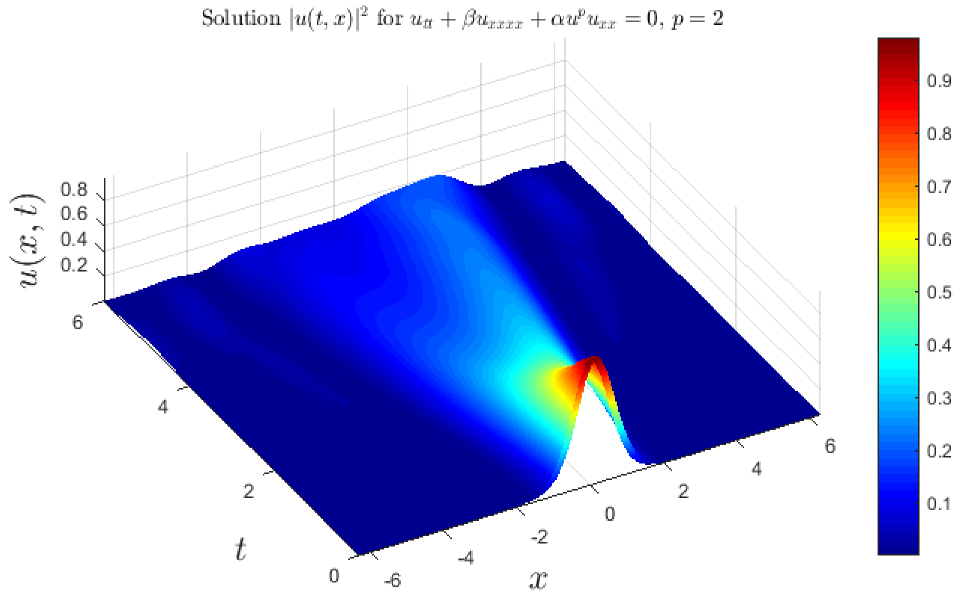


Figure 13: evolution of $u(t,x)$ for $\beta = 0.5, \alpha = 1, p = 2$

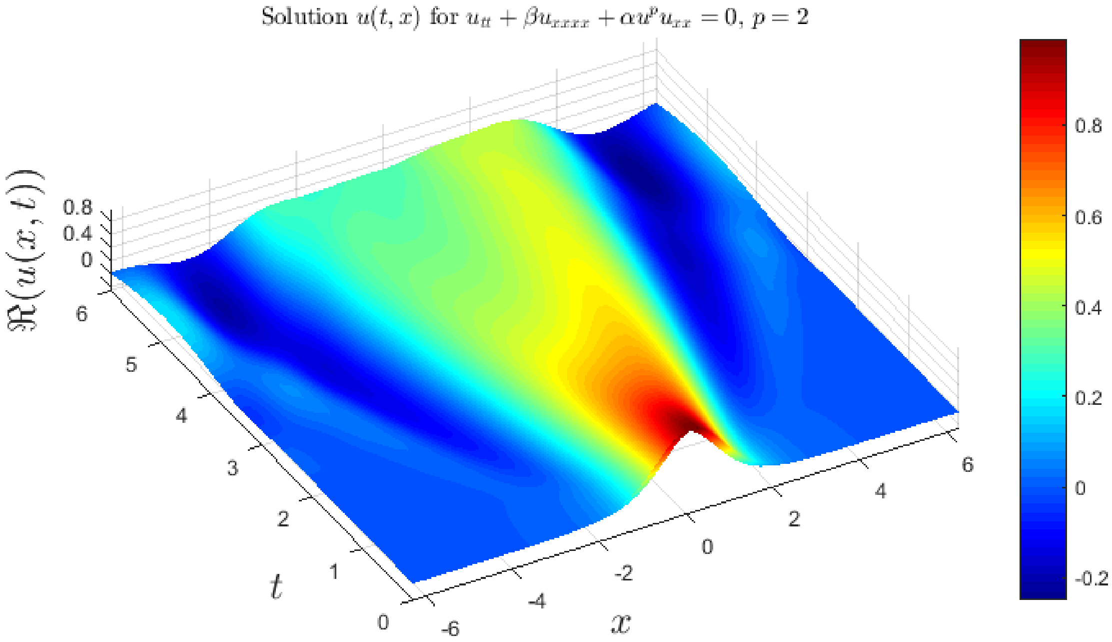


Figure 14: the evolution of the real part of the solution $u(t,x)$ for $\beta = 0.5, \alpha = 1, p = 2$

One notices that in Fig.14 that stronger dispersion causes wave packet to disperse quickly.

Derivation of Travelling Wave Solution

We begin by applying the travelling wave ansatz $u(x,t) = f(\xi),\ \ \xi = x - ct$ to reduce the partial differential equation to an ordinary differential equation. Substituting

$u(x,t) = f(\xi), \qquad \xi = x - ct$

into the nonlinear equation (1), we obtain:

$-c^2 f''(\xi) + \beta f''''(\xi) + \alpha f(\xi) f''(\xi) = 0.$

This equation can be integrated once to give:

$$\beta f'''(\xi) + c^2 f'(\xi) + \frac{\alpha}{2} f(\xi)^2 = A$$

where A is an integration constant. Multiplying through by $f'(\xi)$ and integrating again yields an energy-like equation for $f(\xi)$:

$\frac{\beta}{2}(f''(\xi))^2 + \frac{c^2}{2}(f'(\xi))^2 + \frac{\alpha}{3} f(\xi)^3 = C$, where C is another constant of integration. Solutions to this equation can be studied numerically or approximated using perturbation methods. Results of this equation can be attained by successful implementation as described earlier.

Symmetry Analysis and Conservation Laws

To further explore the structure of the equation, we perform a Lie symmetry analysis. Lie symmetries are transformations that leave the equation invariant and can be used to reduce the number of independent variables, simplifying the problem.

For the nonlinear equation, we seek transformations of the form:
$x' = f(x,t,u), \; t' = g(x,t,u), \quad u' = h(x,t,u)$
that leave the equation invariant (unchanged). Using standard techniques from Lie group theory, we find that the equation admits scaling symmetries of the form:
$x \to \lambda x, \; t \to \lambda^2 t, \; u \to \lambda^{-1} u$
as well as translation symmetries in x and t (at least for $p = 1$). These symmetries allow us to reduce the equation to simpler forms, potentially leading to exact solutions.

The conserved quantities associated with these symmetries can be derived using Noether's theorem. For instance, the time translation symmetry leads to conservation of energy, while
the spatial translation symmetry leads to conservation of momentum. These conserved quantities provide important insights into the long-term behaviour of the solutions.

Lie symmetry analysis is a powerful tool in the study of nonlinear partial differential equations, as it can reveal hidden structures and simplify the problem. In this case, it helps us identify invariant solutions and conserved quantities that govern the evolution of the system.

RESULTS AND DISCUSSION

The linear equation (2) admits an exact solution, especially for $\beta = 1$, and its solution is projected to attain the solution for the nonlinear biharmonic equation (1) via perturbation approach. It has been observed that, in the strongly nonlinear and weakly dispersive regime, the perturbative solution agrees well with the numerical solutions implemented via RK4 scheme. This is evident from the fact that, we cannot maintain stability of the RK4 scheme as we take strongly dispersive terms, unless slightly weaker dispersive, as shown in the Figures Fig. 8-14. It shows employing a hybrid approach helps a lot, particularly in the case analytical one is not easily implementable fully.

The results as found and shown in the Fig. 3 – Fig. 7 show construction can be constructed up to choice of the nonlinearity exponent for the perturbation unlike when the solution is constructed numerically. Overall, the nature of the solutions is a focusing wave-packet or dispersive waves. An initial lump decomposes and propagate through space whereby singularity is formed (blow-up) for larger $p \geq 2$ and $\alpha \geq 1$ and disperses away otherwise. Such behaviour is observed in NLS equation, see Sulem & Sulem (1999), where the equation is classified as critical, subcritical and supercritical depending on the nonlinearity exponent involved in the NLS equation. Our solutions indicate that the equation is subcritical for $p < 2$ and supercritical for $p \geq 2$ for a fixed $\alpha = 1$ and weaker $\beta \sim 1$. It is guaranteed that having $\beta \sim 0$, the solution blows up, as an obvious result. Nevertheless, the equation appears to simulate wave phenomenon, such as shock waves and light pulses, that are strongly sensitive to the dispersion, see Ablowit & Segur (1991), Agrawal (2019).

The trace of conservation of energy and mass, in all the cases, the RK4 scheme does remarkably well for that of mass, thereby implying the conservation nature of the dynamical equation. The wiggly tails we see in the tracing of the mass and energy using perturbation is as a result of truncations of the expansion of the terms. This can somewhat be improved when more correction terms are added.

The symmetry analysis and the travelling wave approach reveals that variant and special classes of solutions, including solitary waves and periodic solutions can be obtained. However, there is need for robust approach towards enhancing the existing methods that could handle well the obtained travelling wave solutions. Moreover, the symmetry approach permits one to scale the solution in some reasonable way so that one studies its behaviour of solutions effectively.

CONCLUSION

The study of the nonlinear dispersive equation $u_{tt} + \beta u_{xxxx} + \alpha u u_{xx} = 0$ using traveling wave solutions, numerical methods, and perturbation techniques has provided a comprehensive understanding of the solution behaviour. Each approach offers unique insights into different aspects of the problem, from exact solutions to numerical approximations. This research demonstrates the power of combining numerical, perturbation, and symmetry methods to tackle complex nonlinear dispersive equations. Future work could focus on extending the analysis to include different forms of nonlinearity or higher-dimensional generalizations of the equation, e.g. for $p \geq 2$.

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