FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 8 No. 6, December, (Special Issue) 2024, pp 234 - 238
DOI: https://doi.org/10.33003/fjs-2024-0806-2857

A TREATMENT OF A MULTISTEP COLLOCATION METHOD FOR THE DIRECT SOLUTION OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

A research has been conducted at the Department of Mathematics, Faculty of Science, Federal University, Gashua, Yobe State to study the treatment of multistep collocation method for the direct solution of second order ordinary differential equations using a class of modified Backward Differentiation Formular (BDF)-type with one super-future point. The research has proposed the construction of a new method of solving second-order initial valued problem of ordinary differential equation. A step-number, $k = 3$, a number of discrete members are obtained and used in block through multi-step collocation approach. The stability properties of the newly constructed methods are investigated using written computer codes and its convergence are established. The numerical efficiency of the method has been tested on some treated second-order initial valued problems, in order to ascertain its suitability. The solutions of the problems are compared with the corresponding exact solutions and the associated absolute errors are presented. Tables and graph have been adopted in the presentation of results, and they have shown the efficiency of the new method for solving solution of second order ordinary differential equations.

Keywords: Modified Backward Differentiation Formula, Block method, Second order Ordinary Differential Equations, Initial Value Problems, Collocation method

INTRODUCTION

Differential equations are mathematical equations that relate some functions of one or more variables with their respective derivatives. A differential equation is used to describe changing quantities and it plays a major role in quantitative studies in many disciplines such as all areas of engineering, physical sciences, life sciences, and economics. According to Butcher (2000), they are widespread in science, and social science. Most realistic differential equations do not have exact solutions, see Abualnaja. (2015). Extensive researches are on the rise to find a close to accurate solution for initial valued problem of second order ordinary differential equation (SODEs) of the form.

$$y'' = f(x, y, y'), \; y(a) = y_0, \; y'(a) = y_1, \; x \in [a, b] \quad (1)$$

Among other numerical methods for solving the problem of the form (1), are as contain in Atsi and Kumleng. (2021), where the interpolation and collocation technique is adopted, but using an extended trapezoidal rule of second kind (ETR_2), for step-number, $k = 3$ as the basis function. Abdelrahim and Omar (2016) proposed a hybrid method using the interpolation and collocation technique with power series approximation as the basis function. In Kuboye et. al., (2018), Awoyemi et. al., (2011), Obarhua (2019), and Ukpebor (2019), numerical methods have been proposed for the solution of (1). Interpolation and collocation technique is mostly adopted in the aforementioned methods.

Motivated with the work presented in Cash (2000), and Atsi and Kumleng (2020), an extension of a family of Modified Backward Differentiation Formula method has been presented in this research. The method is used in block for the solution of second order ordinary differential equation.

MATERIALS AND METHODS

The constructed block method for solving an equation of the form (1) is generated from family of Modified Backward Differentiation Formula method with one superfuture point (as presented in Cash (2000)), denoted by:

$$\bar{y}(x) = \sum_{j=0}^{k-2} \alpha_j(x) y_{n+j} + h[\beta_{k-1}(x) f_{n+k-1} + \beta_k(x) f_{n+k}] + h^2[\gamma_{k-1}(x) g_{n+k-1} + \gamma_k(x) g_{n+k} + \gamma_{k+1}(x) g_{n+k+1}] \quad (2)$$

where,

$\bar{y}(x)$ is the approximation of a continuously differentiable solution $y(x)$ of (1)

$\alpha_j(x)$, $\beta_j(x)$ and $\gamma_j(x)$ are assumed polynomials, called the coefficients of the method

$h = x_{n+1} - x_n$, is a constant step size on $[a, b]$, and a distinct step number is the same as k

y_{n+j} is the approximation to the theoretical solution of (1) at x_{n+j}

$f_{n+j} \equiv f(x_{n+j} y_{n+j})$ and $g_{n+j} \equiv g(x_{n+j}, y_{n+j}, f_{n+j})$.

To ensure a higher order method with smaller error constant is achieved, a superfutre point in the modified BDF method is chosen on $x = x_{n+k+1}$ on g.

Using the technique adopted in Atsi and Kumleng (2020), a D matrix is obtained, for $k = 3$, as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 & (x_n + h)^6 \\ 0 & 1 & 2(x_n + 2h) & 3(x_n + 2h)^2 & 4(x_n + 2h)^3 & 5(x_n + 2h)^4 & 6(x_n + 2h)^5 \\ 0 & 1 & 2(x_n + 3h) & 3(x_n + 3h)^2 & 4(x_n + 3h)^3 & 5(x_n + 3h)^4 & 6(x_n + 3h)^5 \\ 0 & 0 & 2 & 6(x_n + 2h) & 12(x_n + 2h)^2 & 20(x_n + 2h)^3 & 30(x_n + 2h)^4 \\ 0 & 0 & 2 & 6(x_n + 3h) & 12(x_n + 3h)^2 & 20(x_n + 3h)^3 & 30(x_n + 3h)^4 \\ 0 & 0 & 2 & 6(x_n + 4h) & 12(x_n + 4h)^2 & 20(x_n + 4h)^3 & 30(x_n + 4h)^4 \end{bmatrix} \quad (3)$$

The values of the continuous coefficients are obtained from the inverse of D and are presented as follows:

$$\left.\begin{aligned}
\alpha_0 &= \frac{1}{11531}\frac{(h-z)(11531h^5-16549zh^4+10091z^2h^3-3129z^3h^2+486z^4h-30z^5)}{h^6}\\
\alpha_1 &= \frac{1}{11531}\frac{(28080h^5-26640h^4z+13220h^3z^2-3615h^2z^3+516hz^4-30z^5)z}{h^6}\\
h\beta_2 &= \frac{1}{11531}\frac{(h-z)z(102843h^4-130437h^3z+60123h^2z^2-12047hz^3+886z^4)}{h^5}\\
h\beta_3 &= -\frac{1}{11531}\frac{z(h-z)(119392h^4-140528h^3z+63252h^2z^2-12533hz^3+916z^4)}{h^5}\\
h^2\gamma_2 &= \frac{1}{138372}\frac{z(1026324h^4-1096920h^3z+455993h^2z^2-85264hz^3+5975z^4)(h-z)}{h^4}\\
h^2\gamma_3 &= \frac{1}{69186}\frac{z(h-z)(335088h^4-408576h^3z+191500h^2z^2-39683hz^3+3019z^4)}{h^4}\\
h^2\gamma_4 &= -\frac{1}{10644}\frac{z(h-z)(2700h^4+1819h^2z^2-422hz^3+37z^4-3546h^3)}{h^4}
\end{aligned}\right\} \qquad (4)$$

The continuous form of the new method is obtained by substituting the values of the continuous coefficients, (4), into (2), we have:

$$\bar{y}(x) = \left.\begin{aligned}
&\left(\frac{1}{11531}\frac{(h-z)(11531h^5-16549zh^4+10091z^2h^3-3129z^3h^2+486z^4h-30z^5)}{h^6}\right)y_n\\
&+\left(\frac{1}{11531}\frac{(28080h^5-26640h^4z+13220h^3z^2-3615h^2z^3+516hz^4-30z^5)z}{h^6}\right)y_{n+1}\\
&+\left(\frac{1}{11531}\frac{(h-z)z(102843h^4-130437h^3z+60123h^2z^2-12047hz^3+886z^4)}{h^5}\right)f_{n+2}\\
&+\left(-\frac{1}{11531}\frac{z(h-z)(119392h^4-140528h^3z+63252h^2z^2-12533hz^3+916z^4)}{h^5}\right)f_{n+3}\\
&+\left(\frac{1}{138372}\frac{z(1026324h^4-1096920h^3z+455993h^2z^2-85264hz^3+5975z^4)(h-z)}{h^4}\right)g_{n+2}\\
&+\left(\frac{1}{69186}\frac{z(h-z)(335088h^4-408576h^3z+191500h^2z^2-39683hz^3+3019z^4)}{h^4}\right)g_{n+3}\\
&+\left(-\frac{1}{10644}\frac{z(h-z)(2700h^4+1819h^2z^2-422hz^3+37z^4-3546h^3)}{h^4}\right)g_{n+4}
\end{aligned}\right\} \qquad (5)$$

We now interpolate continuous method, (5), at $x_{n+2}, x_{n+3}, x_{n+4}$, and collocating the first and second derivative of (5) at x_n, x_{n+1}, x_{n+4} and x_n, x_{n+1} ,respectively, to get the following system of eight equations:

$$\left.\begin{aligned}
y_{n+4} &= -\frac{693}{11531}y_n+\frac{12224}{11531}y_{n+1}+\frac{13548}{11531}hf_{n+2}+\frac{20352}{11531}hf_{n+3}-\frac{7236}{11531}h^2g_{n+2}+\frac{4128}{11531}h^2g_{n+3}+\frac{84}{887}h^2g_{n+4}\\
y_{n+3} &= -\frac{592}{11531}y_n+\frac{12123}{11531}y_{n+1}+\frac{5184}{11531}hf_{n+2}+\frac{17286}{11531}hf_{n+3}-\frac{10674}{11531}h^2g_{n+2}-\frac{5958}{11531}h^2g_{n+3}+\frac{18}{887}h^2g_{n+4}\\
y_{n+2} &= -\frac{581}{11531}y_n+\frac{12112}{11531}y_{n+1}-\frac{522}{11531}hf_{n+2}+\frac{11472}{11531}hf_{n+3}-\frac{34972}{34593}h^2g_{n+2}-\frac{14776}{34593}h^2g_{n+3}+\frac{50}{2661}h^2g_{n+4}\\
11531hf_{n+4} &= 7827h^2g_{n+2}+19400h^2g_{n+3}+3523h^2g_{n+4}+18619hf_{n+2}-7328hf_{n+3}-240y_n+240y_{n+1}\\
11531hf_{n+1} &= -25509h^2g_{n+2}-13558h^2g_{n+3}+637h^2g_{n+4}-21368hf_{n+2}+30499hf_{n+3}-2400y_n+2400y_{n+1}\\
887hf_n &= 6579h^2g_{n+2}+4296h^2g_{n+3}-225h^2g_{n+4}+7911hf_{n+2}-9184hf_{n+3}-2160y_n+2160y_{n+1}\\
11531h^2g_{n+1} &= 18453h^2g_{n+2}+17067h^2g_{n+3}-949h^2g_{n+4}+42840hf_{n+2}-34920hf_{n+3}+7920y_n-7920y_{n+1}\\
11531h^2g_n &= -353874h^2g_{n+2}-247888h^2g_{n+3}+13533h^2g_{n+4}-466560hf_{n+2}+519840hf_{n+3}+53280y_n-53280y_{n+1}
\end{aligned}\right\} \qquad (6)$$

Stability Analysis of the New Block Scheme

The discrete schemes (6) are represented in block form as:

$$AY_m = ay_m + hBF_m + hbf_m + h^2CG_m + h^2cg_m \qquad (7)$$

where, $Y_m = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}]^T$, $y_m = [y_{n-3}, y_{n-2}, y_{n-1}, y_n]^T$, $f_m = [f_{n-3}, f_{n-2}, f_{n-1}, f_n]^T$, $F_m = [F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}]^T$, $g_m = [g_{n-3}, g_{n-2}, g_{n-1}, g_n]^T$ and $G_m = [G_{n+1}, G_{n+2}, G_{n+3}, G_{n+4}]^T$.
a, B, b, C, c are 4 by 4 matrices and A is a 4 by 4 identity matrix.

Order and Error Constant

As it has been explored in Chollom et. al., (2007) and employed in Atsi and Kumleng (2021), the order and error constant of the new block method are presented thus:
the order is,
$p = [6\ \ 6\ \ 6\ 6]^T$
with error constant,

$$C_{p+1} = \left[-\frac{1600}{242151}\ -\frac{471}{115310}\ -\frac{4727}{1210755}\ -\frac{22688}{315}\ -\frac{181067}{1260}\ \frac{2148}{35}\ \frac{7359}{28}\ -\frac{27982}{7}\right]^T$$

Consistency

The new block scheme is consistent, since $p \geq 1$, see Kumleng et. al., (2013).

Zero Stability

The root of the first characteristic polynomial satisfies $|r| \leq 1$, are, $|r| = 0, 0, 0, 0, 0, 0, 1$ or 1. Hence, the new block method is zero-stable, see Kumleng et. al., (2013).

Convergence

As exploited in Kumleng et. al., (2013), the new block scheme is convergent, since it is both consistent and zero stable.

Numerical Implementation

The proposed method is has been tested in the following examples:

Example 1*:* See Atsi and Kumleng (2021).

$y'' = -1001\, y' - 1000\, y, \qquad y(0) = 1, y'(0) = -1, h = 0.1$

Exact Solution: $y(x) = e^{-x}$

The numerical results of this problem and comparisons are presented in Table 1.

Example 2: See Badmus and Yahaya (2019).

$y'' = x(y')^2, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1$

with the exact solution $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

RESULTS AND DISCUSSION

Table 1: Comparisons of errors in the solutions of Example 1

x	Absolute Error in Atsi and Kumleng (2021)	Absolute Error in (New Method)
0.1	$8.0e-10$	$1.1499e-10$
0.2	$4.1e-9$	$5.0290e-11$
0.3	$9.3e-9$	$2.1631e-10$
0.4	$3.2e-9$	$7.1519e-10$
0.5	$7.0e-10$	$3.5868e-10$
0.6	$2.3e-9$	$1.6595e-10$
0.7	$6.0e-10$	$4.2068e-10$
0.8	$3.5e-9$	$1.2215e-9$
0.9	$6.6e-9$	$4.8237e-10$
1.0	$3.3e-9$	$2.7298e-10$

Table 2: Comparisons of errors in the solutions of Example 2

x	Absolute Error in Badmus and Yahaya (2019)	Absolute Error in (New method)
0.1	$5.891E-06$	$7.51309E-09$
0.2	$8.2399E-05$	$1.8017E-08$
0.3	$3.46421E-04$	$2.8802E-08$
0.4	$7.52101E-04$	$3.6507E-08$
0.5	$1.380283E-03$	$1.3355E-04$

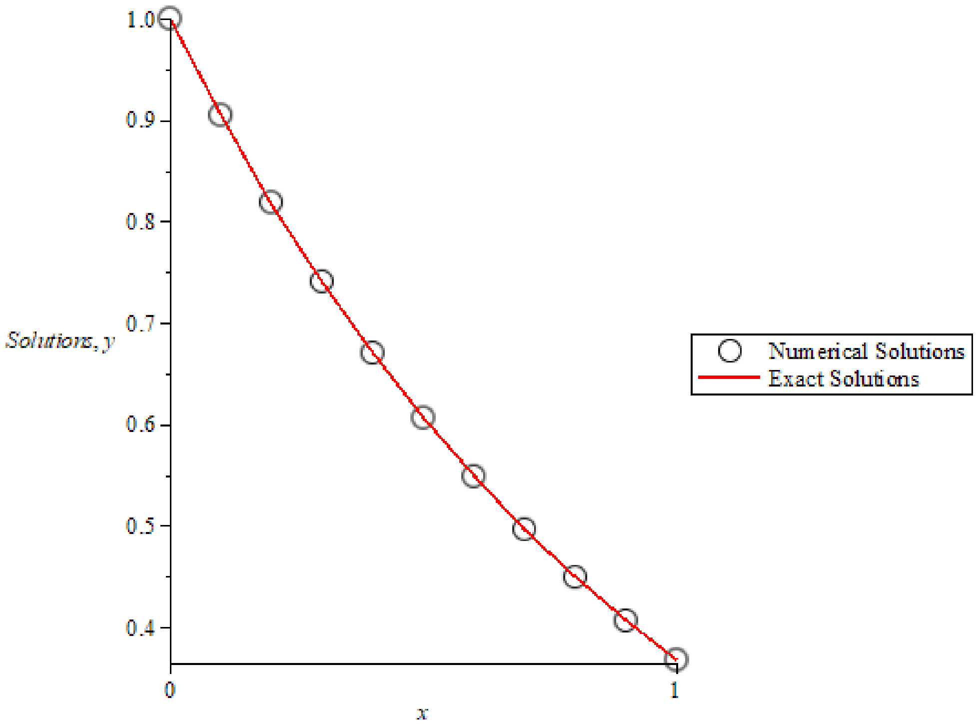

Figure 1: Showing the exact solutions and numerical solutions for example 1

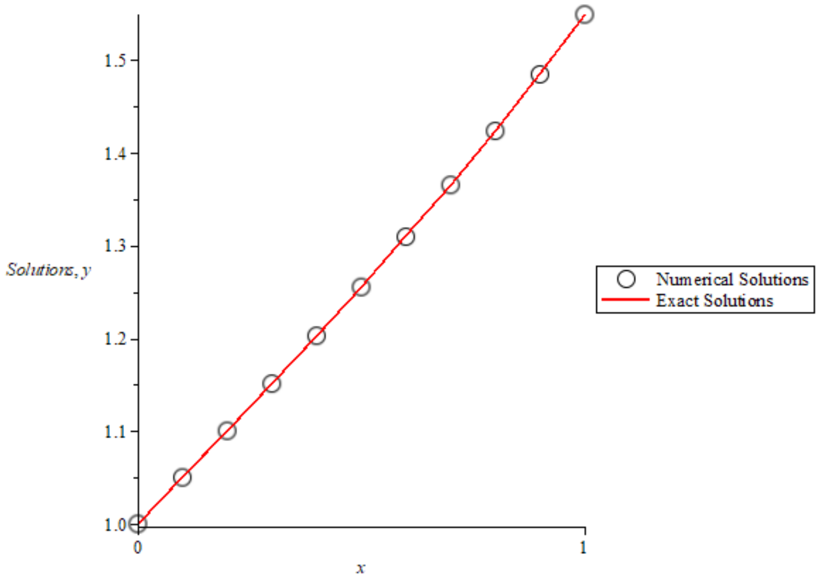

Figure 2: Showing the exact solutions and numerical solutions for example 2

Discussion

Table 1 shows the comparisons between the error in the proposed methods with the block extended trapezoidal rule of the second kind (Etr2) method in Atsi and Kumleng (2021). For the same particular step size, the new method has slightly smaller error compared to the three-step block methods in Atsi and Kumleng (2021).

In table 2, we can observe that the error in the new method compared to the error in Badmus and Yahaya (2019) at a step size, with total steps 5. For the comparison, we have used the similar step size and the result is comparable with lesser error in the new method. A close look at the tables presented in table 1 and 2, reveal that the newly proposed method performs better than those compared with.

The plots of step size vs. of numerical and exact solutions for example 1and are presented in figure 1 and 2. As it can be seen on the plots, the proposed method compete favorably with the exact. The x-axis is on a scale of 10 cm to 1 unit while the y-axis is on the scale of 2 cm to 0.1 unit.

Furthermore, convergence of the new method ensures a reduction in inherent error, in comparison with other existing methods.

CONCLUSION

Step number $k = 3$ has been presented in this research and following the technique adopted in Kumleng et. al. (2013). Eight discrete methods were constructed and implemented in solving two second order ordinary differential equations. The results obtained in examples 1and 2 perform better than the results obtained in the other literatures. Future work may address the construction block methods using higher step numbers.

ACKNOWLEDGEMENT

We are grateful to TETFUND IBR in funding and other support of this research work. Thanks be to the members of this research group for assisting in one way or the other, for the preparation of the work.

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