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# THE ODD LOMAX TOPP LEONE DISTRIBUTION: PROPERTIES AND APPLICATION

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**ABSTRACT**

Lifetime distributions are parametric models used in statistical analyses of time-to-event data. Several probability distributions have been very used in literature to model lifetime data sets which have been useful in the analysis of lifetime data, but in most cases, they are not flexible enough to analyze some complex lifetime data in practice. Due to the importance of these lifetime distributions in modeling real lifetime data, there has several modifications and generalization of lifetime distributions, particularly the Lomax distribution to develop more flexible distributions to address the drawback presented by some classical lifetime distributions including the Lomax distribution. Several attempts have been made by researchers to generalize classical lifetime distributions which offer more flexibility in modeling lifetime data. Our interest in this study is to introduce a new extension of Lomax distribution called the "The odd lomax Topp-Leone distribution" which is bounded on a unit interval data such that its flexibility can accommodate increasing, decreasing, right skewed, left skewed, symmetric and u-shaped data sets. An application to a real lifetime data set clearly shows that the proposed extension of the Lomax distribution is a better alternative to some existing distributions bounded on a unit interval.

**Keywords**: Odd lomax, Lifetime, Distribution, Topp-Leone, Unit interval, Newton-Raphson iterative scheme

## INTRODUCTION

Lifetime data analysis is a statistical method for analyzing time-to-event data. This could be the time of development of disease, time to response-to-treatment, time to death or the working time (reliability) of a device before the worn-out period etc. In recent years, the study of survival data is centered on predicting the probability of response, survival or mean lifetime and comparing the survival distributions of human patients. These predictions are attainable through the use of statistical models known as Lifetime Distributions. In light of this, many probability distributions have been proposed by researchers in the field of statistics to handle such situation. Some example of classical lifetime distributions includes the Exponential distribution, Gamma distribution, Weibull distribution, Gompert distribution, Lomax distribution, Lindley distribution, Log-normal distribution amongst others. These parametric models have been extensively studied and applied in literature and have been proven to be sufficient in modeling lifetime data. Marshall and Olkin (1997) noted that the Exponential distribution plays a central role in analyses of lifetime or survival data due to its convenient statistical theory, "lack of memory" as well as its constant failure rate property.

Lomax (1954) introduced the Lomax distribution which is a heavy-tailed distribution as a special case of a *Pareto distribution* of the 2nd kind, hence called the Pareto type II distribution. Bryson (1974) said that the Lomax (or Type II Pareto) distribution as an alternative to the exponential distribution in the analysis of data with heavy-tailed. The Lomax distribution model has can be applied to many fields of study which includes income and wealth analysis, economics, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling and many more. Harris (1968) used Lomax distribution for the analysis of income and wealth data, Atkinson and Harrison (1978) used the Lomax distribution for modelling business failure data, Holland et al. (2006) used the Lomax distribution for modelling the distribution of the sizes of computer files on servers, Corbelini et al. (2007) used the Lomax distribution to model the size of firm and its queuing problems, amongst others. See Arnold (1983) and Johnson et al. (1994) for more details about the Lomax distribution and Pareto class of distributions.

Ahsanulla (1991) studied the distributional properties of the Lomax distribution with regard to record values pointed out that: (1) the Lomax distribution reverse J-shape and can be used for modelling situation that shows an improved performance in the system as development continues over time, (2) the Lomax distribution has a linear residual life function instead of constant which makes it a better alternative to exponential distribution in reliability studies. Other form of statistical treatment of the Lomax distribution was presented which revealed that the Lomax distribution is a better for modelling waiting time data than the exponential distribution.

In the last few decades, following the impact of the Lomax distribution in modelling time to event data as establishes by Ahsanulla (1991), the interest of researchers in the field of statistics have been drawn to developing modified (extended) forms of the Lomax distribution with the aim of increasing its flexibility in modeling real life data. Some examples of the modified and extended version of the Lomax distribution include the extended Lomax distribution by Lemonte and Cordeiro (2011), exponentiated Lonax distribution by Salem (2014), weighted Lomax distribution by Kilany (2015), power Lomax distribution by Rady et al. (2016), five parameter Lomax distribution by Mead (2016), generalization of the Lomax distribution by Oguntunde et al. (2017), and so on. Several methods of generating new probability distributions and family of probability distributions have been established in literature. Some of the most popularly and currently used generators in the literature include the Marshall-Olkin family by Marshall and Olkin (1997), the Beta-G family by Eugene et al. (2002), the transmuted-G family by Shaw and Bulkley (2007), the transformed-transformer (*T-X*) family by Alzaatreh et al. (2013), etc. Using these ideas, we have the exponential Lomax distribution by El-Bassiouny et al. (2015), the gamma Lomax distribution by Cordeiro et al. (2015), transmuted Lomax distribution by Ashour and Eltehiwy (2013), Poisson Lomax distribution by Al-Zahrani and Sagor

(2014), Weibull Lomax distribution Tahir (2015), Modified Kies–Lomax Distribution by Alsubie (2021), Marshall-Olkin Extended Power Lomax Distribution by Gillariose and Tomy (2020), the Lomax-Exponential Distribution by Nasrin et al. (2016),Applications of Half Logistic Marshall-Olkin Lomax-X family of Distributions to Time Series, Acceptance Sampling and Stress-strength Parameter by Tomy and Jose (2022), A New Transmuted Generalized Lomax Distribution byWael et al. (2021) and Cordeiro et al. (2019) presented the odd Lomax-G family of distributions. Ogunde et al. (2023) developed the Kumaraswamy Generalized Inverse Lomax (KGIL) distribution. Other distributions such as Weibll Dal and Burrxii-Dal and Weibull together with their properties and applications have been studied extensively (Nawaz S. et al 2021a,Nawaz S. et al 2021b,and Nawaz S. et al 2021c).

In this study, we present a new probability distribution as an extension of Lomax distribution called "The Odd Lomax Topp Leone distribution" which is more flexible lifetime distribution than some existing lifetime distributions.

## MATERIALS AND METHODS

### The Formation of Odd Lomax Topp Leone (OLxTL) Distribution

A random variable is said to follow the Topp Leone (TL) distribution if the cumulative distribution function (CDF) and probability density function (pdf) are respectively given as

$$\boldsymbol{G(x) = x^{\theta}(2-x)^{\theta} = [1-(1-x)^2]^{\theta} \quad 0 < x < 1, \theta > 0} \tag{1}$$

And

$$\boldsymbol{g(x) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1} = 2\theta(1-x)[1-(1-x)^2]^{\theta-1}} \tag{2}$$

Based on the framework of the T-X family presented by Azaatreh et al. (2013), Cordeiro et al, (2019) presented the Odd Lomax-G (OLx-G) family of distributions. The cumulative distribution function (cdf) of the function OLx-G family is defined as

$$F(x;\alpha,\beta,\Phi) = \alpha\beta^{\alpha}\int_0^{\frac{G(x;\Phi)}{\bar{G}(x;\Phi)}}(\beta+t)^{-(\alpha+1)}dt = 1-\beta^{\alpha}\left\{\beta+\frac{G(x;\Phi)}{\bar{G}(x;\Phi)}\right\}^{-(\alpha+1)} \tag{3}$$

Where $G(x;\Phi)$ is the cdf of a baseline distribution and $\bar{G}(x;\Phi) = 1-G(x;\Phi)$ is the survival function with $\Phi$ as vector of unknown parameters, $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters respectively from the Lomax distribution. The corresponding pdf of the OLx-G family, which is obtained by differentiating (3) with respect to $x$ is defined as

$$f(x;\alpha,\beta,\Phi) == \frac{\alpha\beta^{\alpha}g(x;\Phi)}{[\bar{G}(x;\Phi)]^2}\left\{\beta+\frac{G(x;\Phi)}{\bar{G}(x;\Phi)}\right\}^{-(\alpha+1)} \tag{4}$$

The cdf of the proposed odd Lomax Topp Leone (OLxTL) distribution model is obtained by substituting (1) into (3) to get

$$F(x;\alpha,\beta,\theta) = 1-\beta^{\alpha}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-\alpha}, \quad \boldsymbol{\alpha,\beta,\theta > 00 < x < 1} \tag{5}$$

Also, the pdf of the OLxTL distribution is derived by substituting (1) and (2) into (4) to get

$$f(x;\alpha,\beta,\Phi) == \frac{2\theta\alpha\beta^{\alpha}(1-x)[1-(1-x)^2]^{\theta-1}}{\{1-[1-(1-x)^2]^{\theta}\}^2}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)} \tag{6}$$

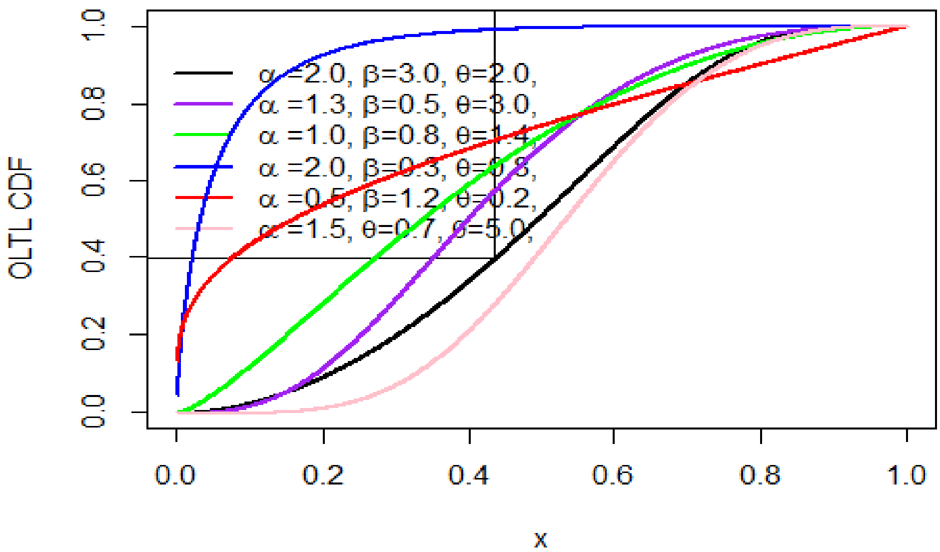


Figure 1: Cumulative Distribution Function of the OLxTL Distribution

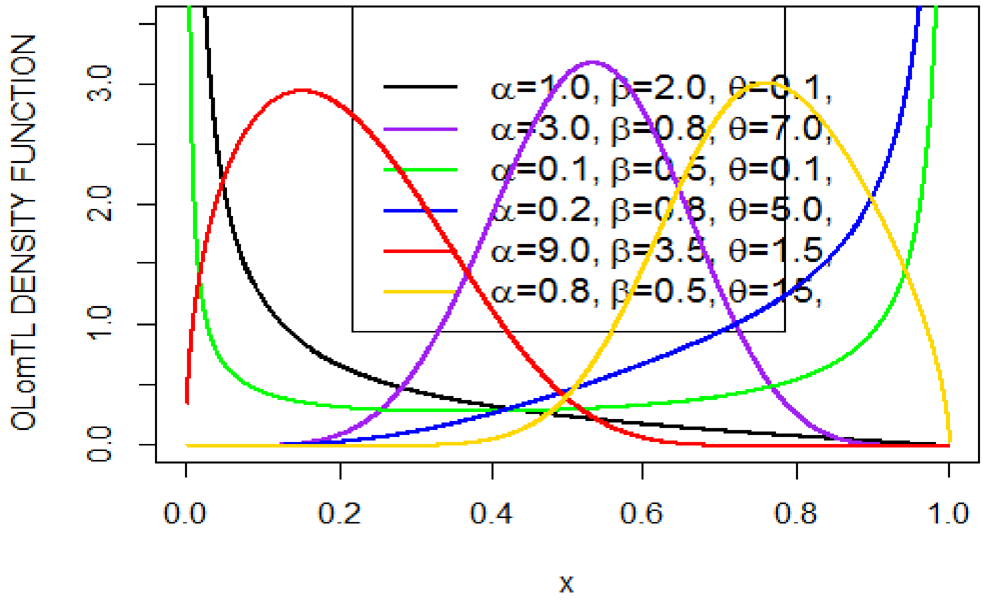


Figure 2: Probability Density Function of the OLxTL Distribution

**The Reliability Functions of** OLxTL **Distribution**

Some of the reliability characteristics that are of great significance in the study of lifetime models are derived for the OLxTL distribution in this section. These include the survival function$S(x;\alpha,\beta,\theta)$, the hazard rate function$h(x;\alpha,\beta,\theta)$, the inverse hazard rate function $\gamma(x;\alpha,\beta,\theta)$ and the cumulative hazard rate function$H(x;\alpha,\beta,\theta)$. These functions are respectively given as

$$S(x;\alpha,\beta,\Phi)=\beta^{\alpha}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)} \qquad (7)$$

$$h(x;\alpha,\beta,\Phi)=\frac{2\alpha\beta^{\alpha}\theta(1-x)[1-(1-x)^2]^{\theta-1}}{\left(1-[1-(1-x)^2]^{\theta}\right)^2}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-1} \qquad (8)$$

$$\gamma(x;\alpha,\beta,\Phi)=\frac{2\alpha\beta^{\alpha}\theta(1-x)[1-(1-x)^2]^{\theta-1}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)}}{\left(1-[1-(1-x)^2]^{\theta}\right)^2\left(1-\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-\alpha}\right)} \qquad (9)$$

and

$$H(x;\alpha,\beta,\Phi)=-Log\left(\beta^{\alpha}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)}\right) \qquad (10)$$

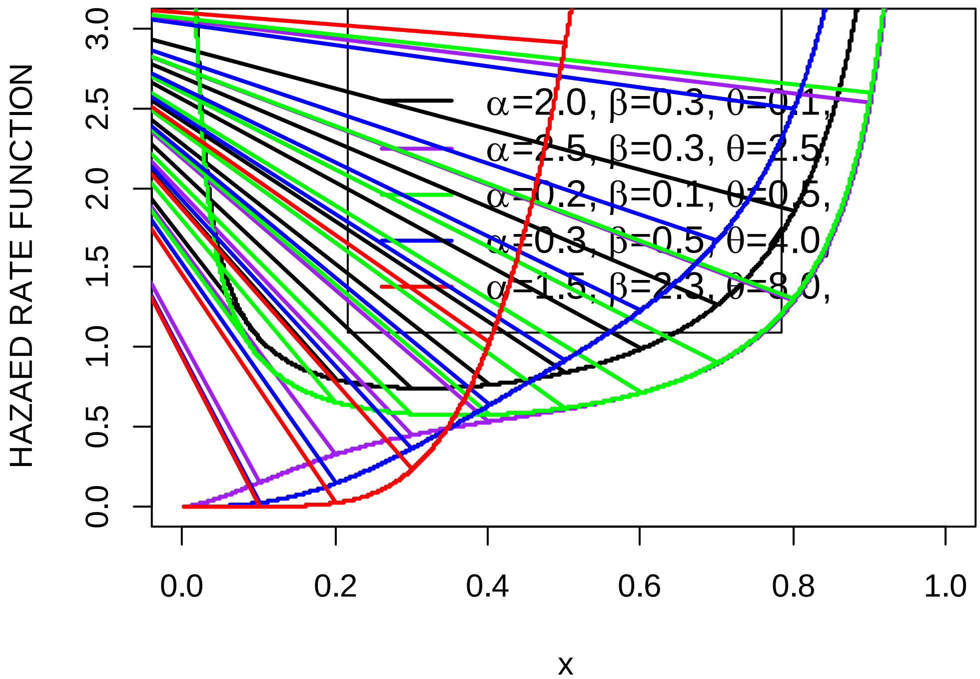


Figure 3: Hazard Rate Function of the OLxTL Distribution

**The Series Expansion of The CDF and pdf**

Using the following series expansion Logits given by Prudnikov et al. (1986) as$(a-x)^n=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}x^i$ and$(a-x)^{-n}=\sum_{j=0}^{n}\frac{\Gamma(n+j)a^{n-j}}{j!\Gamma(n)}x^i$, the cumulative distribution function of the OLxTL distribution in (3.5) can be expressed as

$$F(x;\alpha,\beta,\theta)=1-\beta^{\alpha}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-\alpha}=1-A(i,j,k,p)x^p \quad (11)$$

where$A(i,j,k,p)=\sum_{i-0}^{\infty}\sum_{j=0}\sum_{k=0}\sum_{p=0}\frac{\beta^{2\alpha-1}(-1)^{k+p}\Gamma(\alpha+i)\Gamma(i+j)}{i!j!\Gamma(\alpha)\Gamma(i)}\binom{\theta(i+j)}{k}\binom{2k}{p}$

similarly, the corresponding probability density function of the OLxTL distribution can be expressed as

$$f(x;\alpha,\beta,\Phi)=\frac{2\alpha\beta^{\alpha}\theta(1-x)[1-(1-x)^2]^{\theta-1}}{(1-[1-(1-x)^2]^{\theta})^2}\left\{\beta+\frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)}=B(i,j,k,p)x^p \quad (12)$$

where $B(i,j,k,p)=\sum_{i-0}^{\infty}\sum_{j=0}\sum_{k=0}\sum_{p=0}\frac{2\alpha\beta^{\alpha}\theta(-1)^{k+p}\Gamma(\alpha+i+1)\Gamma(2+i+j)}{i!j!\Gamma(\alpha+1)\Gamma(2+i)}\binom{\theta(i+j+1)-1}{k}\binom{2k+1}{p}$

**The Quantile Function of the OLxTL Distribution**

The quantile function of a probability distribution is simply the inverse expression of the cumulative distribution function (CDF). The mathematical expression of the quantile function is given as

$$q=F^{-1}(x)=F\big(Q(q)\big) \quad (13)$$

for all $Q(q)$ and $0<q<1$. Then the quantile function of the OLxTL distribution model can be derived as follows

$$q=1-\beta^{\alpha}\left\{\beta+\frac{\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}{1-\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}\right\}^{-\alpha}$$

$$\left(\frac{1-q}{\beta^{\alpha}}\right)=\left\{\beta+\frac{\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}{1-\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}\right\}^{-\alpha}$$

$$\beta(1-q)^{-\frac{1}{\alpha}}=\beta+\frac{\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}{1-\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}$$

$$\frac{\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}{1-\left[1-\left(1-Q(q)\right)^2\right]^{\theta}}=\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]$$

$$\left[1-\left(1-Q(q)\right)^2\right]^{\theta}=\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]-\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]\left[1-\left(1-Q(q)\right)^2\right]^{\theta}$$

$$\left[1-\left(1-Q(q)\right)^2\right]^{\theta}\left(1+\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]\right)=\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]$$

$$\left[1-\left(1-Q(q)\right)^2\right]^{\theta}=\frac{\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}{1+\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}$$

$$\left(1-Q(q)\right)^2=1-\left\{\frac{\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}{1+\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}\right\}^{\frac{1}{\theta}}$$

$$Q(q)=1-\left(1-\left\{\frac{\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}{1+\beta\left[(1-q)^{-\frac{1}{\alpha}}-1\right]}\right\}^{\frac{1}{\theta}}\right)^{\frac{1}{2}} \quad (14)$$

**The Moment and Related Measure**

The r[th] raw moment of a continuous random variable X, denoted by $\mu_r'$ is defined as

$\mu_r'=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx$, therefore for a continuous random variable X having the OLxTL distribution, the r[th] raw moment can be obtained as

$$\mu_r'=E(X^r)=B(i,j,k,p)\int_0^1 x^{r+p}\,dx \quad (15)$$

where $B(i,j,k,p)=\sum_{i-0}^{\infty}\sum_{j=0}\sum_{k=0}\sum_{p=0}\frac{2\alpha\beta^{\alpha}\theta(-1)^{k+p}\Gamma(\alpha+i+1)\Gamma(2+i+j)}{i!j!\Gamma(\alpha+1)\Gamma(2+i)}\binom{\theta(i+j+1)-1}{k}\binom{2k+1}{p}$

But

$$\int_0^1 x^{r+p}\,dx=\left.\frac{x^{r+p+1}}{r+p+1}\right|_0^1=\frac{1}{r+p+1}$$

Hence,

$$\mu_r'=E(X^r)=B(i,j,k,p)\frac{1}{r+p+1} \quad (16)$$

From (3.16), we obtain the first four raw moments of the Odd Lomax Topp Leone (OLxTL) distribution as

$\mu_1' = E(X^1) = \frac{B(i,j,k,p)}{p+2}$ $\mu_2' = E(X^2) = \frac{B(i,j,k,p)}{p+3}$
$\mu_3' = E(X^3) = \frac{B(i,j,k,p)}{p+4}$ $\mu_4' = E(X^4) = \frac{B(i,j,k,p)}{p+5}$
Similarly, the central moment of a random variable X is defined by
$\mu_r = E(X-\mu)^r = E\left\{\sum_{i=1}^{r}\binom{r}{i}X^{r-i}(-\mu)\right\} = \sum_{i=1}^{r}\binom{r}{i}(-1)^i E[X^{r-i}]E[\mu^i]$
$= \sum_{i=1}^{r}\binom{r}{i}(-1)^i \mu_{r-1}\mu^i$ (17)
The first four central moments can be obtained from (3.17) as follows
$\mu_2 = \mu_2'$; $\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$ and $\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu - 3\mu^4$
Then some statistical measures that are related to the moment of a probability distribution which are the mean ($\mu$), variance ($\sigma^2$), coefficient of variation ($CV$), skewness ($Sk$) and kurtosis ($Ks$) respectively be obtained using the expressions as follows:
$\mu = \mu_1' = E(X^1)$; $\sigma^2 = \mu_2 - \mu^2$; $CV = \frac{\sigma}{\mu}$; $Sk = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$ and $Ks = \frac{\mu_4}{(\mu_2)^2}$

**Remark:** It is important to note that the standard deviation ($\sigma$) of any random variable say X is the square root of variance i.e. $\sigma = \sqrt{\mu_2 - \mu^2}$
Due to the fact that the moment of some probability distributions does not exist or may be difficult to evaluate, Moor (1988) developed the method of obtaining the skewness and kurtosis of any distribution whose quantile function is well defined (3.18) and (3.19) respectively as

$$Sk = \frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{2}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \quad (18)$$

and

$$Ks = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \quad (19)$$

**The Moment Generating Function of OLxTL Distribution**
The moment generating function (mgf) of a random variable X with pdf $f(x)$ is defined as $M_X(T) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$. Then the moment generating function of the Odd Lomax Topp Leone (OLxTL) distribution can be derived as follows

$$M_X(T) = E(e^{tx}) = B(i,j,k,p)\int_0^1 e^{tx}x^p dx$$

But$e^Z = \sum_{i=0}^{\infty}\frac{z^i}{i!}$, then we have
$E(e^{tx}) = B(i,j,k,p,s)\int_0^1 x^{p+s}\,dx$ $= B(i,j,k,p,s)\frac{x^{p+s+1}}{p+s+1}\Big|\begin{matrix}1\\0\end{matrix}$
$= B(i,j,k,p,s)\left(\frac{1}{p+s+1}\right)$ (20)
Where $B(i,j,k,p,s) = \sum_{i-0}^{\infty}\sum_{j=0}\sum_{k=0}\sum_{p=0}\sum_{s=0}\frac{2\alpha\beta^\alpha\theta(-1)^{k+p}t^s\Gamma(\alpha+i+1)\Gamma(2+i+j)}{i!j!s!\Gamma(\alpha+1)\Gamma(2+i)}\binom{\theta(i+j+1)-1}{k}\binom{2k+1}{p}$

**Renyi Entropy**
Entropy is an important concept in statistics. It is used to measure the level of uncertainty with respect to a random variable X. Renyi (1961) defined the entropy of a random variable X with pdf $f(x)$ as
$J_R(s) = \frac{1}{1-s}Log\varphi(s)$, $s > 0 and s \neq 1$ (21)
where $\varphi(s) = \int_0^\infty f^s(x)\,dx$
Using the pdf in (6) and the concept of the series expansion in section (3), we derive the Renyi entropy of the OLxTL distribution as follows

$$\varphi(s) = (2\alpha\beta^\alpha\theta)^s\int_0^\infty \frac{(1-x)^s[1-(1-x)^2]^{s(\theta-1)}}{(1-[1-(1-x)^2]^\theta)^{2s}}\left\{\beta + \frac{[1-(1-x)^2]^\theta}{1-[1-(1-x)^2]^\theta}\right\}^{-s(\alpha+1)} dx$$

$$= (2\alpha\beta^\alpha\theta)^s\sum_{i=0}^{\infty}\frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)}{i!\,\Gamma(\alpha s+s)}\int_0^1 (1-x)^s[1-(1-x)^2]^{s(\theta-1)}\left(1-[1-(1-x)^2]^\theta\right)^{-(2s+i)} dx$$

$$= (2\alpha\beta^\alpha\theta)^s\sum_{i,j=0}^{\infty}\frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)\Gamma(2s+i+j)}{i!\,j!\,\Gamma(\alpha s+s)\Gamma(2s+i)}\int_0^1 (1-x)^s[1-(1-x)^2]^{\theta(s+i+j)-s} dx$$

$$= (2\alpha\beta^\alpha\theta)^s\sum_{i,j,k=0}^{\infty}\frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)\Gamma(2s+i+j)}{i!\,j!\,\Gamma(\alpha s+s)\Gamma(2s+i)}\binom{\theta[s+i+j]-s}{k}(-1)^k\int_0^1 (1-x)^{2k+s} dx$$

$$= (2\alpha\beta^\alpha\theta)^s\sum_{i,j,k,p=0}^{\infty}\frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)\Gamma(2s+i+j)}{i!\,j!\,\Gamma(\alpha s+s)\Gamma(2s+i)}\binom{\theta[s+i+j]-s}{k}\binom{2k+s}{p}(-1)^k\int_0^1 x^p\,dx$$

$$= (2\alpha\beta^\alpha\theta)^s\sum_{i,j,k,p=0}^{\infty}\frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)\Gamma(2s+i+j)}{i!j!\Gamma(\alpha s+s)\Gamma(2s+i)}\binom{\theta[s+i+j]-s}{k}\binom{2k+s}{p}(-1)^k\frac{1}{p+1} \quad (22)$$

Hence, the Renyi entropy of OLxTL distribution is given as

$$J_R(s) = \frac{1}{1-s} Log(2\alpha\beta^{\alpha}\theta)^s \left\{\sum_{i,j,k,p=0}^{\infty} \frac{\beta^{s(\alpha+1)-1}\Gamma(\alpha s+s+i)\Gamma(2s+i+j)}{i!j!\Gamma(\alpha s+s)\Gamma(2s+i)} \binom{\theta[s+i+j]-s}{k}\binom{2k+s}{p}(-1)^k \frac{1}{p+1}\right\} \tag{23}$$

**Maximum Likelihood Estimation (MLE)**

Let $x_1, x_2, x_3, \cdots, x_n$ be a random sample of size n taken from the OLxTL distribution with pdf in (3.6), then the likelihood function denoted by $L(x; \alpha, \beta, \theta)$ is given as

$$L(x;\alpha,\beta,\theta) = \prod_{i=1}^{n} f(x)$$

$$= (2\alpha\beta^{\alpha}\theta)^n \prod_{i=1}^{n} \frac{(1-x)[1-(1-x)^2]^{(\theta-1)}}{(1-[1-(1-x)^2]^{\theta})^2} \left\{\beta + \frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\}^{-(\alpha+1)} \tag{24}$$

Taking the log of the likelihood function, we have

$$ln\, L(x;\alpha,\beta,\theta) = n\, ln(2\alpha\beta^{\alpha}\theta) + \sum_{i=1}^{n} ln(1-x) + (\theta-1)\sum_{i=1}^{n} ln[1-(1-x)^2] - 2\sum_{i=1}^{n} ln\left(1-[1-(1-x)^2]^{\theta}\right) - (\alpha+1)\sum_{i=1}^{n} ln\left\{\beta + \frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\} \tag{25}$$

The partial derivatives of (25) with respect to$\alpha$, $\beta$ and $\theta$ are respectively given as

$$\frac{\partial\, ln\, L}{\partial\alpha} = \frac{n}{\alpha} + n\, ln\, \beta - \sum_{i=1}^{n} ln\left\{\beta + \frac{[1-(1-x)^2]^{\theta}}{1-[1-(1-x)^2]^{\theta}}\right\} \tag{26}$$

$$\frac{\partial\, ln\, L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left\{\frac{1}{\beta(1-[1-(1-x)^2]^{\theta}) + [1-(1-x)^2]^{\theta}}\right\} \tag{27}$$

$$\frac{\partial\, ln\, L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln[1-(1-x)^2] + 2\sum_{i=1}^{n}\left\{\frac{[1-(1-x)^2]^{\theta}\, ln[1-(1-x)^2]}{1-[1-(1-x)^2]^{\theta}}\right\} - (\alpha-1)\sum_{i=1}^{n}\left\{\frac{[1-(1-x)^2]^{\theta}\, ln[1-(1-x)^2]}{\beta(1-[1-(1-x)^2]^{\theta}) + [1-(1-x)^2]^{\theta}}\right\} \tag{28}$$

The maximum likelihood estimates (MLEs) $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ of $(\alpha, \beta, \theta)$ can be obtained by equating (26) – (28) to zero and solving the nonlinear system of equation simultaneously. Since the system of equation cannot be solved analytically, we achieve the solution by using an iterative method called the Newton-Raphson iterative scheme with the help of R-Software package.

**Newton Raphson Iterative Method**

The Newton-Raphson method is an iterative scheme that is used to determine the value $\hat{\varphi}$ of $\varphi$ that maximizes a function of $\varphi$.

Let $\varphi_k$ be the k$^{th}$ approximation of $\hat{\varphi}$, where $k = 0,1,2, ....$ According to Agresti (1990), this method requires an initial guess, $\varphi_0$, that will maximize the function of $\varphi$. At the k$^{th}$ step, in the iterative process is maximized by

$$\hat{\varphi}_{k+1} = \varphi_k - H^{-1}(\varphi_k)U(\varphi_k) \tag{29}$$

where$U(\varphi)$ is the first derivative of the log-likelihood function of the model defined as

$$U(\varphi) = \frac{\partial\, ln\, L}{\partial\varphi_i} = \begin{pmatrix} \frac{\partial\, ln\, L}{\partial\alpha} \\ \frac{\partial\, ln\, L}{\partial\beta} \\ \frac{\partial\, ln\, L}{\partial\theta} \end{pmatrix} \tag{30}$$

and $H(\varphi)$ is the Hessian matrix and the elements of the Hessian matrix can be derived from the second derivatives of the log-likelihood function given by

$$H(\varphi) = \frac{\partial^2\, ln\, L}{\partial\varphi_i\partial\varphi_j} = \frac{\partial U(\varphi)}{\partial\varphi} = \begin{pmatrix} \frac{\partial^2\, ln\, L}{\partial\alpha^2} & \frac{\partial^2\, ln\, L}{\partial\alpha\partial\beta} & \frac{\partial^2\, ln\, L}{\partial\alpha\partial\theta} \\ \frac{\partial^2\, ln\, L}{\partial\beta\partial\alpha} & \frac{\partial^2\, ln\, L}{\partial\beta^2} & \frac{\partial^2\, ln\, L}{\partial\alpha\partial\theta} \\ \frac{\partial^2\, ln\, L}{\partial\beta\partial\theta} & \frac{\partial^2\, ln\, L}{\partial\theta\partial\beta} & \frac{\partial^2\, ln\, L}{\partial\theta^2} \end{pmatrix} \tag{31}$$

**Application**

In this section, we present the application of the odd Lomax Topp Leone (OLxTL) distributionby considering a real data, as well as to compare the OLxTL distribution model with some competing distribution models namely the Marshall-Olkin Topp Leone (MOTL) distribution model and the beta distribution model. The probability density function (pdf) of MOTL and beta distributions are given as;

Marshall-Olkin Topp Leone (MOTL) distribution;

$$f(x) = \frac{2\alpha\theta(1-x)[1-(1-x)^2]^{\theta}}{\left[1-\bar{\alpha}\{1-[1-(1-x)^2]^{\theta}\}\right]^2},\ 0 < x < 1, \alpha, \theta > 0$$

Beta distribution;

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \qquad 0 < x < 1, \alpha, \theta > 0$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma\beta}{\Gamma(\alpha+\beta)}$

For the data set, the estimates of the parameters of OLxTL, MOTL and Beta distributions was computed by the maximum likelihood estimation method. The maximum likelihood estimates (MLEs) of the parameters by using the Newton-Raphson iterative scheme with the help of R-Software package.

For purpose of comparison, we use the information criterion statistics which includes negative log – likelihood value (-*l*), the Akaike Information Criterion (AIC) and the Bayesian information criterion (BIC), as well as the goodness-of-fit statistics which includes the Kolmogorov-Smirnov (K-S) statistic, the Cramer-Von Mises (CVM) statistic and the Anderson-Darling (AD) statistics. Generally, the smaller the value of the goodness of fit criteria/statistics from a model, the better the fit of the model to the data set.

Data analysis and results

**Real Data Set On Rock Samples from a Petroleum Reservoir**

We consider an uncensored data set observed from measurements on petroleum rock samples. The data contains 48 rock samples from a petroleum reservoir as reported in Cordeiro et al (2012). The observed data set is as presented in table 1 below

Table 1present the maximum likelihood estimates (MLEs) of the unknown parameters with the corresponding standard errors (S.Es) enclosed in parentheses for Dataset above. Table 3shows the summary statistics: *l, AIC, BIC, CVM, AD* and *KS* Values for all the models under consideration.

**Table 1: Rock Samples from a Petroleum Reservoir**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0903296 | 0.2036540 | 0.2043140 | 0.2808870 | 0.1976530 | 0.3286410 | 0.1486220 |
| 0.1623940 | 0.2627270 | 0.1794550 | 0.3266350 | 0.2300810 | 0.1833120 | 0.1509440 |
| 0.2000710 | 0.1918020 | 0.1541920 | 0.4641250 | 0.1170630 | 0.1481410 | 0.1448100 |
| 0.1330830 | 0.2760160 | 0.4204770 | 0.1224170 | 0.2285950 | 0.1138520 | 0.2252140 |
| 0.1769690 | 0.2007440 | 0.1670450 | 0.2316230 | 0.2910290 | 0.3412730 | 0.4387120 |
| 0.2626510 | 0.1896510 | 0.1725670 | 0.2400770 | 0.3116460 | 0.1635860 | 0.1824530 |
| 0.1641270 | 0.1534810 | 0.1618650 | 0.2760160 | 0.2538320 | 0.2004470 | |

**Table 2: The *MLEs* and *S.Es* (in parentheses) for Data Set 1**

| Distributions | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ |
|---|---|---|---|
| MOTL | 5.5829 (*0.6957*) | | 0.3542 (*0.1472*) |
| Beta | 5.9416 (*1.1813*) | 21.2078 (*4.3473*) | |
| OLxTL | 4.3503 (*4.5400*) | 0.2956 (*0.5118*) | 3.0199 (*0.6451*) |

**Table 3: Summary Statistics for the Data Set 1**

| Model | $\ell$ | *AIC* | *BIC* | *CVM* | *AD* | *KS* |
|---|---|---|---|---|---|---|
| MOTL | 58.14939 | -107.20043 | -96.12975 | 0.149734 | 1.059758 | 0.14265 |
| Beta | 90.60239 | -14.29879 | -10.55630 | 15.89042 | 48.71644 | 0.99644 |
| OLxTL | **53.87168** | **-115.74340** | **-103.45802** | **0.129803** | **0.776148** | **0.13583** |

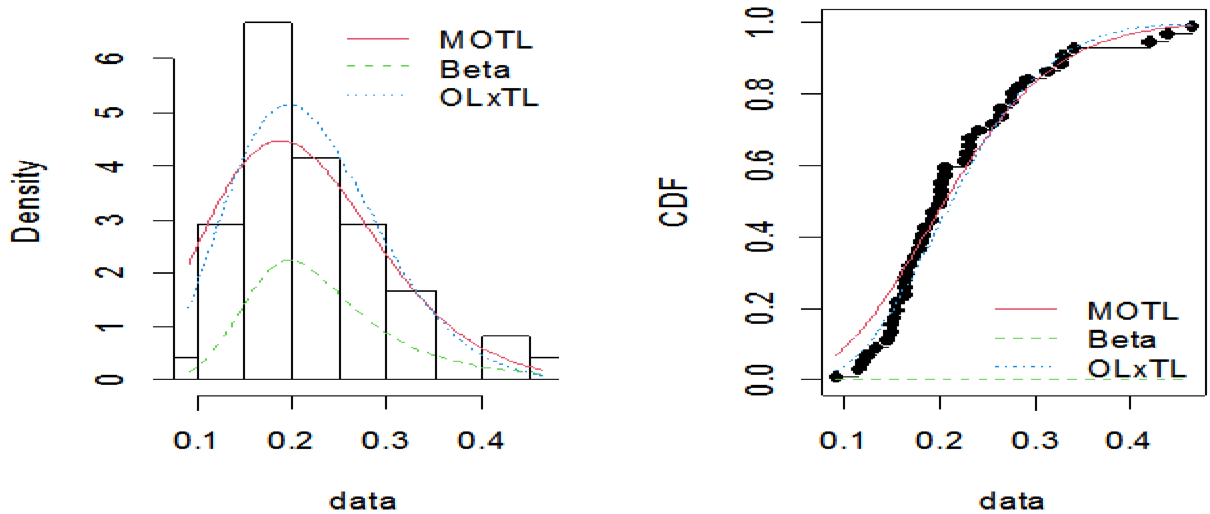


Figure 1: The probability density function (pdf) and the cumulative distribution function (CDF) of the fitted distributions for the data set

From the values of the summary statistics presented in Table 3, the OLxTL distribution has smaller values for all the statistics criteria. Hence, it shows that the OLxTL distribution performed better than the others. Also, the plot of the empirical density function and cumulative distribution function in Figure 4 indicated that the OLxTL distribution yields a better fit for the data set.

## CONCLUSION

In this study, we propose a new unit interval lifetime probability distribution, called the Odd Lomax Topp Leone (OLxTL) distribution. The flexibility of the proposed distribution in data analysis accommodates increasing, decreasing, right skewed, left skewed, symmetric and u-shaped in its pdf, as well as increasing and bathtub shapes in its hazard rate function. These attractive features of the OLxTL distribution makes it suitable for modelling several data on a unit interval in practical situations. Some of the Statistical properties of the OLxTL, such as the quantile function, moments with its related measures, moment generating function and Renyi entropy were derived. The maximum likelihood estimation (MLE) method was employed in estimating the parameters of OLxTL distribution. An application of the OLxTL distribution was illustrated with a real data set and the performance compared to that of existing distribution models and the result revealed that the proposed OLxTL distribution model performed better than the competing models.

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