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BREUSCH-PAGAN TEST: A COMPREHENSIVE EVALUATION OF ITS PERFORMANCE IN DETECTING HETEROSCEDASTICITY ACROSS LINEAR, EXPONENTIAL, QUADRATIC, AND SQUARE ROOT STRUCTURES USING MONTE CARLO SIMULATIONS

***[1]Akewugberu, H. O., [2]Umar, S. M., [1]Musa, U. M., [1]Ishaaq, O. O., [1]Auwalu Ibrahim, [1]Osi, A. A. and [1]Ganiyat, A. F.**

[1]Department of Statistics, Aliko Dangote University of Science and Technology, 3244, Wudil, Kano, Nigeria.
[2]Department of Mathematical Sciences, Bayero University, 3011, Kano.

*Corresponding authors' email: habeebullahakewugberu@gmail.com Phone: +2348161138369

ABSTRACT

This study provides a comprehensive evaluation of the Breusch-Pagan test's performance in detecting heteroscedasticity across various structures and levels, addressing a significant gap in existing literature. Through Monte Carlo simulations, we investigate the test's power, Type II errors ($\sigma = 0$), and Type I errors ($\sigma \neq 0$) in confirming homoscedasticity assumptions at different sample sizes (100, 250, and 500). Our objectives include assessing the test's ability to detect heteroscedasticity at various levels and structures, examining the impact of sample size on its performance, comparing its performance across different structures, and identifying its limitations and potential biases. Our findings reveal that the Breusch-Pagan test's performance varies across different heteroscedasticity structures and levels, with poor detection of low-level heteroscedasticity but improved performance at higher levels, particularly for exponential heteroscedasticity structures (EHS). While increased sample size enhances the test's performance, it remains inadequate for linear heteroscedasticity structures (LHS) and square root heteroscedasticity structures (SQRTHS). Based on our results, we recommend cautious use of the Breusch-Pagan test, especially when dealing with low-level heteroscedasticity or specific structures like LHS and SQRTHS. We suggest using the test with moderate to high sample sizes for improved performance, particularly for EHS and quadratic heteroscedasticity structures (QHS). For researchers with limited sample sizes or dealing with LHS and SQRTHS, alternative tests for heteroscedasticity may be considered. Finally, we emphasize the importance of assessing the underlying structure of heteroscedasticity in the dataset to choose the most suitable test and interpretation.

Keywords: Breusch-Pagan test, Heteroscedasticity, Monte Carlo simulations, Statistical inference, Econometrics, Regression analysis

INTRODUCTION

Every statistical procedure has some assumptions that must be at least approximately true before the procedure can thereafter produce reliable and accurate results (Ogunleye, Olaleye, and Solomon, 2014).

Researchers often adopt statistical procedure to their data without validating the assumptions of the procedure they chose to adopt. If one or more of the assumptions of a given statistical procedure are violated, it is likely to arrive at misleading results by that procedure (White, 1980).

In many real-life data and applications, variances of the errors vary from one observation to the other which is often regarded as heteroscedasticity. Since homoscedasticity is often unrealistic assumption, researchers have worked tirelessly to show the effect of heteroscedasticity on modelling and statistical inference. Even though the OLS estimates retain unbiasedness in the presence of heteroscedasticity, its estimates become inefficient. (Weerahandi, 1995).

Heteroscedasticity refers to the situation where the variance of the error term changes across different levels of the independent variable(s), leading to inefficient estimates and incorrect inference." (Muhammad et al., 2023)

Direct opposite in meaning to "homogeneity assumption" is "heterogeneity of error variances", which simply refers to a situation where the variances of the residuals are affected by at least one predictor variable leading to unequal magnitude in spread. Thus, heterogeneity problem may arise in most of the economic (econometric), experimental and agricultural modelling where specifically analysis of variance technique is applied. Hence, homogeneity of variance is a major assumption underlying the validity of many parametric tests. More importantly, it serves as the null hypothesis in substantive studies that focus on cross- or within-group dispersion. (Onifade et all, 2020).

The assumption of homoscedasticity, also known as constant variance or homogeneity of variances, is a fundamental requirement in regression analysis. It posits that the error term has a consistent variance across all levels of the predictor variables, meaning that the variation of each error around its zero mean is independent of the predictor values. In essence, this assumption ensures that the variance of each error term remains the same regardless of the size or value of the explanatory variables. If this assumption is violated, the error term is considered heteroscedastic, leading to potentially inaccurate estimates, biased standard errors, and incorrect conclusions in regression analysis. Heteroscedasticity can significantly affect the accuracy of predictions and policy analysis in linear regression models." (Sultana et al., 2024)

In line with existing literature, such as Wiedermann et al. (2017), who identified Breusch-Pagan, Bartlett's, Goldfeld-Quandt, White, and Koenker-Bassett tests as commonly used tests for heteroscedasticity in generalized linear models, another study carefully selects the Breusch-Pagan test due to its robustness and sensitivity in detecting heteroscedasticity, particularly in the presence of non-normal errors and outliers (Harvey, 1976; Breusch & Pagan, 1979). Recent studies (e.g., Zeileis, 2004; Hayes & Cai, 2007) also recommend the Breusch-Pagan test for its reliability and accuracy in detecting heteroscedasticity.

This study provides a comprehensive evaluation of the Breusch-Pagan test's performance in detecting heteroscedasticity across various structures and levels, filling a gap in existing literature, by investigating its Power, frequency of Type II errors (when $\sigma = 0$) and Type I errors (σ

≠ 0) in confirming homoscedasticity assumptions at different sample sizes (100, 250, and 500).

MATERIALS AND METHODS

Forms of heteroscedasticity

This study explores four distinct heteroscedastic structures derived from additive and multiplicative models. However, for the purpose of this research, we assume a specific heteroscedastic structure where the variance of the error term is directly proportional to the mean of the response variable. The two general forms of heteroscedasticity are:

1. Var(e,e) =$\sigma^2 e^{E(y_i)}$; Exponential form.
2. Var(e,e) = $\sigma^2 E(y_i)^g$; g > 0. Linear form.

However, from the two above, we considered four heteroscedastic structures as follows:

1. Exponential Form.
2. Linear Form.
3. Square-rooted Form, when g = 0.5.
4. Quadratic Form, when g = 2.

for confirming homoscedasticity assumption using Breusch-Pagan test when different heteroscedasticity levels are injected into the generalized linear models at 100, 250 and 500 sample sizes and standard deviation σ = 0, 0.1, 0.3, 0.5, 0.7 and 0.9 levels of heteroscedasticity.

Procedure for Monte Carlo Simulation Experiment:

Through Monte Carlo experiments, this research assesses the performance of test statistics in identifying heteroscedasticity under finite sample conditions. The simulation consisted of 1000 iterations, each with varying sample sizes of 100, 250 and 500 data points.

In each iteration, we generated two data sets, i.e Heteroscedastic data and Homoscedastic data. For the heteroscedastic data, We simulated x from a standard normal distribution x ~N(0,1) and y as a linear function of x with added noise, where the variance of the noise increased with x.

$$y = \beta_1 x + \varepsilon \quad (1)$$

However, for the homoscedastic data, We simulated x and y from same standard normal distribution, x,y ~ N(0,1), with no relationship between x and the variance of y.

The structure was then formulated thus:

Linear Form:

In this form, the variance of the dependent variable increases linearly as the independent variable increases.

$$\varepsilon \sim N(0, \sigma^2(x)) = N(0, \beta_1 x) \quad (2)$$

Equation (2) indicates that the error term ϵ follows a normal distribution with a mean of zero and a variance that changes linearly with the independent variable x, according to the parameter β_1.

Exponential Form:

In this form, the variance of the dependent variable increases exponentially as the independent variable increases. This means that small changes in the independent variable can lead to large changes in the variance of the dependent variable.

$$\varepsilon \sim N(0, \sigma^2(x)) = N(0, exp(\beta_1 x)) \quad (3)$$

Equation (3) indicates that the error term ϵ follows a normal distribution with a mean of zero and a variance that changes exponentially with the independent variable x, according to the parameter β_1.

Quadratic Form:

In this form, the variance of the dependent variable changes in a quadratic, or U-shaped, manner as the independent variable increases. This means that the variance may initially decrease, then increase, or vice versa, as the independent variable changes.

$$\varepsilon \sim N(0, \sigma^2(x)) = N(0, \beta_1 x^2) \quad (4)$$

Equation (4) indicates that the error term ϵ follows a normal distribution with a mean of zero and a variance that changes quadratically with the independent variable x, according to the parameter β_1.

Square root Form:

In this form, the variance of the dependent variable increases as the independent variable increases, but at a decreasing rate. This is because the square root function grows more slowly as the input value increases.

$$\varepsilon \sim N(0, \sigma^2(x)) = N(0, \beta_1 \sqrt{x}) \quad (5)$$

Equation (5) indicates that the error term ϵ follows a normal distribution with a mean of zero and a variance that changes with the square root of the independent variable x, according to the parameter β_1.

Which were then set to produce the three metrics used for the analysis i.e, Power, Type I error and Type II error.

Breusch-Pagan test

Breusch-Pagan (BP) test (1979). Consider the linear regression model:

$$y_i = \beta_0 + \beta_1 x_{1_i} + \cdots + \beta_k x_{k_i} + e_i \quad (6)$$

Assumed the error variance (var u_i) = σ_1^2 given as follows:

$$\sigma_1^2 = f(a_0 + a_1 w_{1_i} + \cdots + a_m w_{m_i} \quad (7)$$

that is, σ_1^2 is some function of the non-stochastic variables w's; some or all of the x's can serve as w's. Specifically, compute the following auxiliary regression:

$$\sigma_1^2 = a_0 + a_1 w_{1_i} + \cdots + a_m w_{m_i} \quad (8)$$

The null hypothesis of homoscedasticity is:

$H_0: a_1 = a_2 = \cdots = a_P = 0$

while the alternative hypothesis is given that at least one of the α's is not zero and that at least one of the w's affects the variance of the residuals, which will be different for respective i's.

RESULTS AND DISCUSSION

As stated earlier, Breusch-Pagan test for detecting heteroscedasticity is used in this study, with the use of three metrics, that is the number of time (frequency) the test commits type II error and type I error as the case may be and the power of the test such that the least frequency (type II error) when sigma = 0 shall be considered as the best among other structures and the highest frequency (type I error) when sigma ≠ 0 shall be considered as the best among others, likewise the structure with the highest power.

The null hypothesis is such that homoscedasticity assumption is upheld.

Table 1: Performance of Breusch-Pagan test when n=100 across different heteroscedasticity structures.

Performance of the test when error follows LHS at 5% level of significance

		Variations					
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=100	Power	0.055	0.051	0.053	0.047	0.041	0.044
	Type I Error	0.102	0.048	0.054	0.044	0.049	0.052
	Type II Error	0.945	0.949	0.947	0.953	0.959	0.956

Performance of the test when error follows EHS at 5% level of significance

		Variations					
		σ = 0	σ = 0.1	σ = 0.3	σ = 0.5	σ = 0.7	σ = 0.9
N=100	Power	0.427	0.409	0.420	0.470	0.467	0.441
	Type I Error	0.927	0.919	0.911	0.856	0.816	0.756
	Type II Error	0.573	0.591	0.580	0.530	0.533	0.559

Performance of the test when error follows QHS at 5% level of significance

		Variations					
		σ = 0	σ = 0.1	σ = 0.3	σ = 0.5	σ = 0.7	σ = 0.9
N=100	Power	0.119	0.129	0.129	0.148	0.146	0.137
	Type I Error	0.194	0.207	0.207	0.217	0.195	0.190
	Type II Error	0.881	0.871	0.871	0.852	0.854	0.863

Performance of the test when error follows SQRTHS at 5% level of significance

		Variations					
		σ = 0	σ = 0.1	σ = 0.3	σ = 0.5	σ = 0.7	σ = 0.9
N=100	Power	0.042	0.042	0.142	0.029	0.038	0.045
	Type I Error	0.048	0.048	0.204	0.044	0.049	0.052
	Type II Error	0.958	0.958	0.858	0.971	0.962	0.955

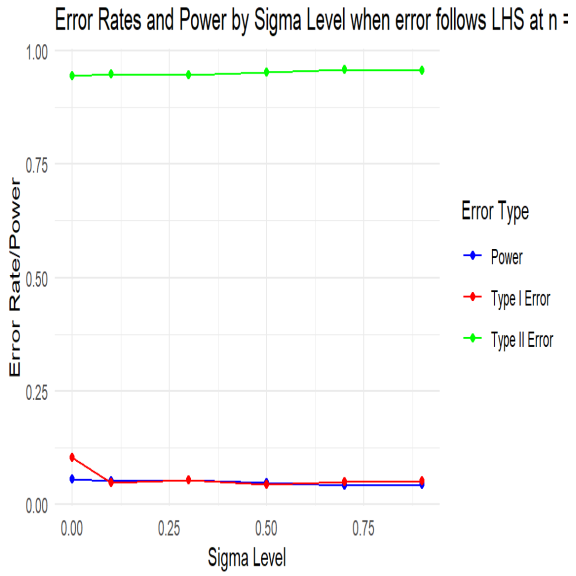

Figure 1: Performance of Breusch-Pagan test

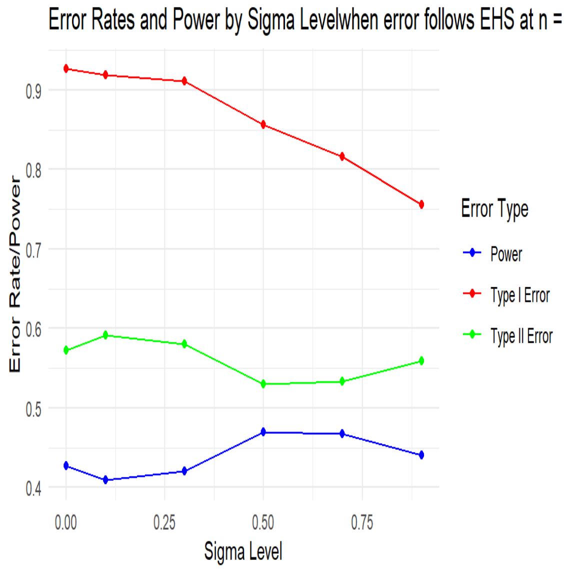

Figure 2: Performance of Breusch-Pagan test

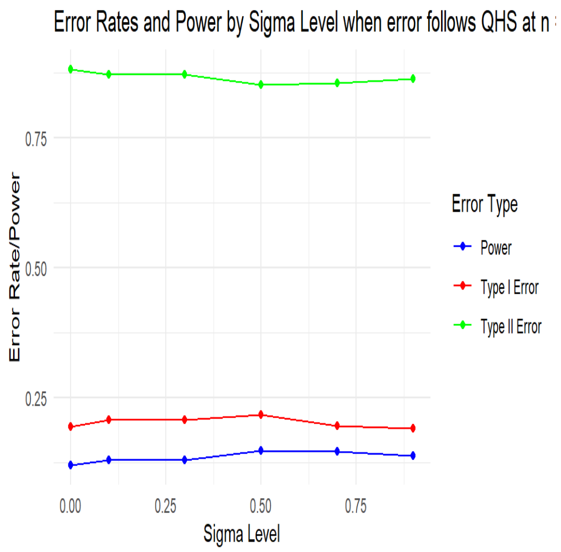

Figure 3: Performance of Breusch-Pagan test

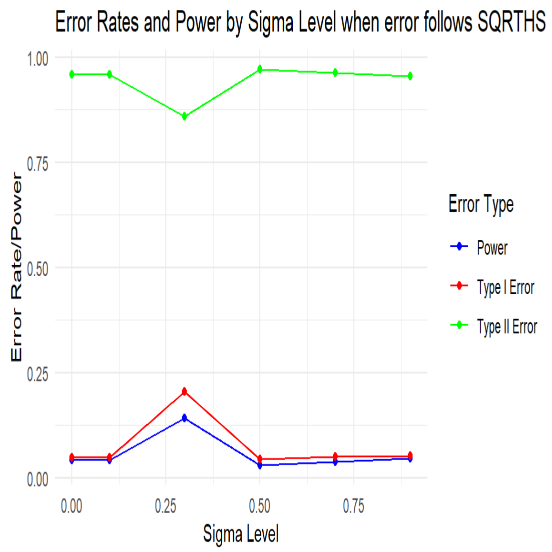

Figure 4: Performance of Breusch-Pagan test

The above Figures shows the performance of Breusch-Pagan test when n=100 across different heteroscedasticity structures.

Table 2: Performance of Breusch-Pagan test when n=250 across different heteroscedasticity structures.

Performance of the test when error follows LHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=250	Power	0.045	0.043	0.047	0.034	0.059	0.059
	Type I Error	0.067	0.051	0.055	0.041	0.057	0.057
	Type II Error	0.955	0.957	0.953	0.966	0.941	0.941
Performance of the test when error follows EHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=250	Power	0.751	0.773	0.776	0.802	0.791	0.776
	Type I Error	1.000	1.000	1.000	0.999	0.994	0.988
	Type II Error	0.227	0.227	0.224	0.198	0.209	0.224
Performance of the test when error follows QHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=250	Power	0.158	0.180	0.195	0.196	0.198	0.159
	Type I Error	0.221	0.225	0.246	0.225	0.243	0.228
	Type II Error	0.842	0.820	0.805	0.804	0.802	0.841
Performance of the test when error follows SQRTHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=250	Power	0.039	0.042	0.042	0.046	0.041	0.037
	Type I Error	0.118	0.092	0.092	0.047	0.051	0.041
	Type II Error	0.961	0.958	0.958	0.954	0.959	0.963

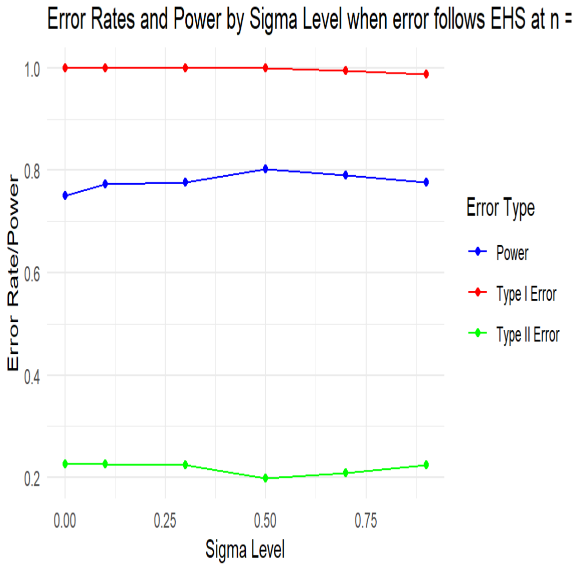

Figure 5: Performance of Breusch-Pagan test

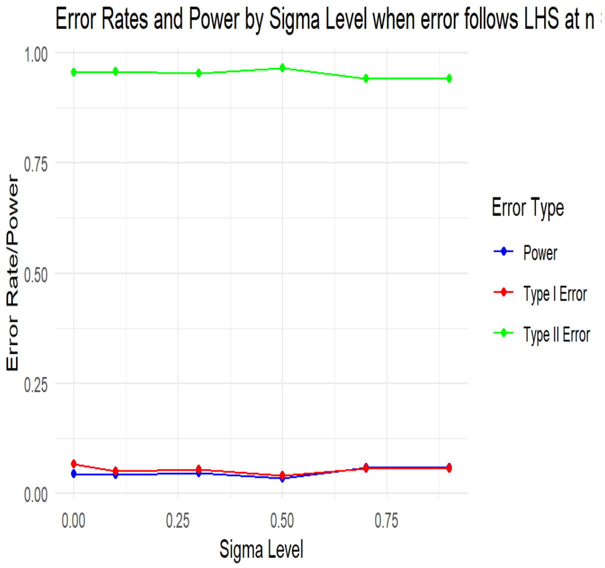

Figure 6: Performance of Breusch-Pagan test

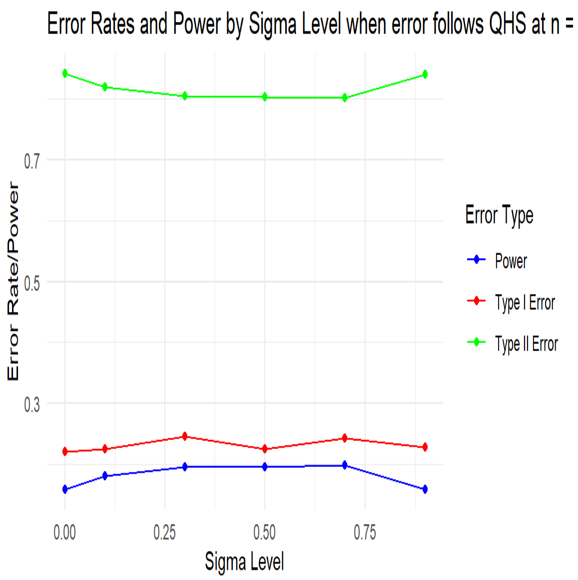

Figure 7: Performance of Breusch-Pagan test

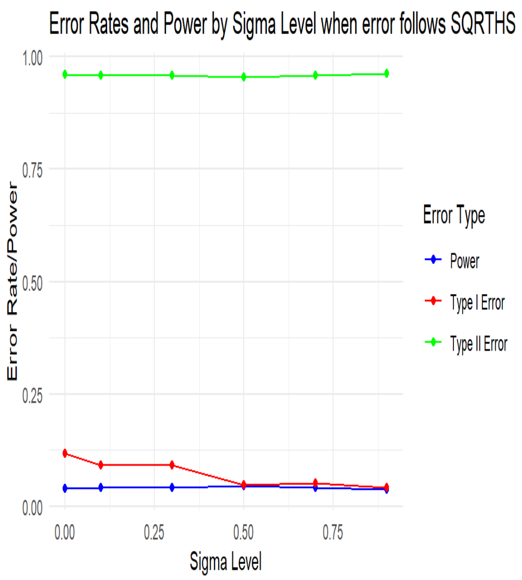

Figure 8: Performance of Breusch-Pagan test

Figures 5,6,7 and 8 above shows the performance of Breusch-Pagan test when n=250 across different heteroscedasticity structures.

Table 3: Performance of Breusch-Pagan test when n=500 across different heteroscedasticity structures.

Performance of the test when error follows LHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=500	Power	0.041	0.046	0.056	0.045	0.053	0.051
	Type I Error	0.062	0.062	0.056	0.046	0.056	0.046
	Type II Error	0.959	0.954	0.944	0.955	0.947	0.949
Performance of the test when error follows EHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=500	Power	0.943	0.948	0.952	0.950	0.961	0.951
	Type I Error	1.000	1.000	1.000	1.000	1.000	1.000
	Type II Error	0.057	0.052	0.048	0.050	0.039	0.049
Performance of the test when error follows QHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=500	Power	0.218	0.231	0.206	0.213	0.208	0.206
	Type I Error	0.261	0.262	0.298	0.276	0.283	0.264
	Type II Error	0.782	0.769	0.794	0.787	0.792	0.794
Performance of the test when error follows SQRTHS at 5% level of significance.							
				Variations			
		σ = 0	**σ = 0.1**	**σ = 0.3**	**σ = 0.5**	**σ = 0.7**	**σ = 0.9**
N=500	Power	0.052	0.054	0.056	0.050	0.039	0.061
	Type I Error	0.167	0.129	0.072	0.043	0.060	0.056
	Type II Error	0.948	0.946	0.944	0.950	0.961	0.939

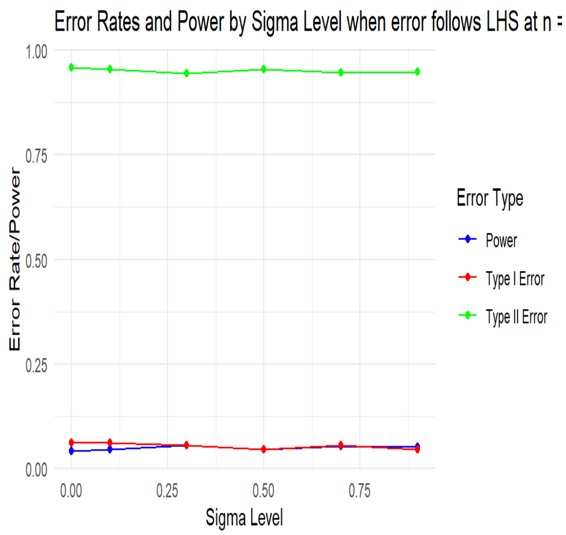

Figure 9: Performance of Breusch-Pagan test

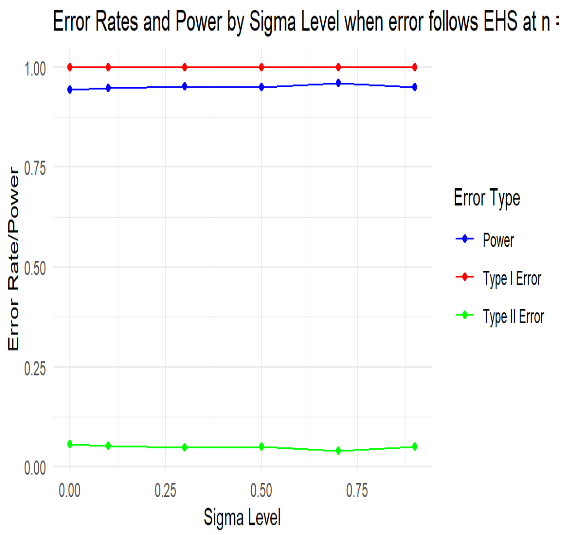

Figure 10: Performance of Breusch-Pagan test

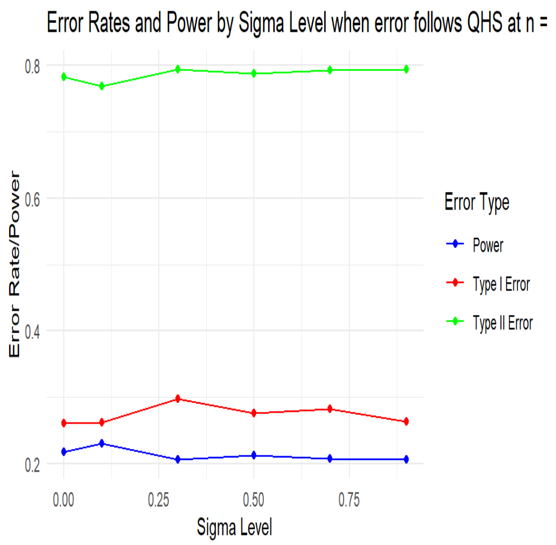

Figure 11: Performance of Breusch-Pagan test

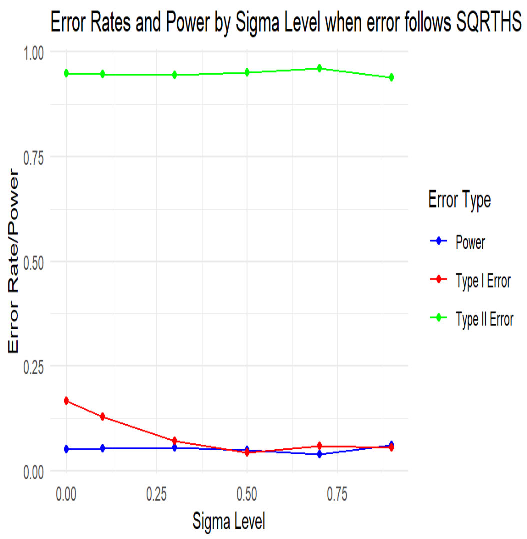

Figure 12: Performance of Breusch-Pagan test

Figure 9,10,11 and 12 above shows the performance of Breusch-Pagan test when n=500 across different heteroscedasticity structures.

Table 1 above shows the performance of Breusch-Pagan test across different heteroscedasticity structures and levels, it exhibits poor detection of non-existent heteroscedasticity (σ = 0) and low power ranging from 42.7% - 4.2% alongside high Type II error rates of about 95%. As heteroscedasticity increases (σ = 0.1, 0.3), the test's power improves but Type I error rates rise, indicating increased sensitivity but also more false positives. At higher heteroscedasticity levels (σ = 0.5, 0.7, 0.9), power increases while Type II error rates decrease, making the test more effective. However, performance differs across structures: LHS and SQRTHS show poor performance, QHS moderate performance, and EHS the best performance with high power and low Type II error rates at high heteroscedasticity levels, highlighting the test's variability in detecting heteroscedasticity across different scenarios.

Table 2 shows that the performance of Breusch-Pagan test varies across different heteroscedasticity structures and levels, with a sample size of n=250. At no heteroscedasticity (σ = 0), the test performs poorly, with low power of barely 4% - 7.5% and high Type II error rates 0f 95%. As heteroscedasticity increases from low to moderate levels (σ = 0.1, 0.3), there exist an improvement in the test's power, but Type I error rates also rise, indicating increased sensitivity in heteroscedasticity detection but also more false positives. At higher heteroscedasticity levels (σ = 0.5, 0.7, 0.9), the test becomes more effective, especially for EHS structures, showing high power 75% -80% and low Type II error rates of barely 20% - 25%. However, LHS and SQRTHS structures show poor performance, even at high heteroscedasticity levels.

Table 3 also shows that the performance of Breusch-Pagan test varies across different heteroscedasticity structures and levels, with a sample size of n=500. At no heteroscedasticity (σ = 0), the test performs poorly, with low power of 4% - 5% and high Type II error rates of about 95%. As heteroscedasticity increases from low to moderate levels (σ = 0.1, 0.3), power improves, but Type I error rates rise, indicating increased sensitivity in heteroscedasticity detection but also more false positives. At higher heteroscedasticity levels (σ = 0.5, 0.7, 0.9), power increases, and Type II error rates decrease, making the test more effective. The test performs best for EHS structures with optimum type I error rate detection, with high power 94% - 96% and low Type II error rates 3.9%, but poorly for LHS and SQRTHS structures, even at high heteroscedasticity levels, while QHS structures show moderate performance with increasing power.

CONCLUSION

Obviously, the Breusch-Pagan test performance varies across different heteroscedasticity structures and levels adopted for this research. While it shows poor performance in detecting low-level heteroscedasticity, it however, becomes more effective and shows improved performance at higher levels, especially for certain structures like EHS. As the sample size increases, the test exhibit improved performance, but it still performs poorly for LHS and SQRTHS structures.

RECOMMENDATIONS

i. Breusch-Pagan test should be used with caution, researchers should be aware of its limitations, especially when dealing with low-level heteroscedasticity or specific structures like LHS and SQRTHS.
ii. Researchers should use Breusch-pagan test with moderate to high sample sizes for improved performance, especially for EHS and QHS structures.
iii. Researchers with low sample sizes or LHS and SQRTHS might consider alternative test for heteroscedasticity for improved performance.
iv. Assess the underlying structure of heteroscedasticity in your dataset in other to choose the most suitable test and interpretation.

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