FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645-2944
Vol. 8 No. 5, October 2024, pp 134 -142
DOI: https://doi.org/10.33003/fjs-2024-0805-2762

A MODIFIED DHILLON DISTRIBUTION: PROPERTIES AND APPLICATION

Iliyasu A. S., *Ishaq O.O, Abduhamid. A. O, Ibrahim A., Abubakar S.S, Musa U.M, Ahmed S, Usman A, and Abba B. M

Department of Statistics, Aliko Dangote University of Science and Technology Wudil, Kano state. Nigeria.
*Corresponding Author: ishaq.o.o@kustwudil.edu.ng

Abstract

There are still a lot of real-world issues where the observed facts cannot effectively fit into frequently used classical probability models. To solve this, it is imperative to provide probability models that accurately represent the behavior of certain real-world phenomena. having considered these problems, the study proposed a new lifetime distribution, the Modified Dhillon Distribution (MDD), developed using the Beta integrated model approach. The study examines the statistical properties of the new distribution such as the Quantile function, Moment, Moment generating function, Entropy, and reliability functions. Moreover, the maximum likelihood approach was used to estimate the distribution parameters. Using real data, the study demonstrates the applications of the MDD using two sets of real data sets, and it has the minimum value of AIC, BIC and CAIC. Therefore, based on the results the study concluded that the MDD offers the best fit out of all the competing distributions.

Keywords: Dhillon distribution, Distribution Properties, Maximum Likelihood Estimate, Real data Applications.

INTRODUCTION

Several researchers in the modern field of statistics have worked hard, and some are still working hard, to create distributions by extending the existence of classical distributions or combining some well-known classical distributions with others to produce a better fit than when they are used alone due to the added flexibility. The Weibull distribution (Weibull, 1951) is described as the most extensively used in many other disciplines outside reliability. It was named after the Swedish mathematician Waloddi Weibull, who provided a detailed description of it. On the other hand, the Weibull distribution's failure rate function can only be increasing, decreasing, or constant. Human mortality and the failure rate are examples of lifetime data that were not captured by it. Therefore, To satisfy the requirements, numerous Weibull distribution extensions, changes, and generalizations have been proposed (Xie et al., 2002). One of the extensions of the Weibull distribution was the modified Weibull (MW) distribution which is a notable distribution that has been used by hundreds of researchers and has been applied in different applications. The MW distribution was developed through the limiting distribution of the so-called Beta-integrated model (Lai et al., 1998).(Xie et al., 2002) Suggested models using bathtub-shaped failure rate functions for reliability analysis, which cannot be modeled by the standard Weibull distribution. (Lai et al., 2003) Proposed a novel lifetime distribution capable of simulating a hazard-rate function in the form of a bathtub, with parameters estimated using a Weibull probability paper plot.(Nassar and Eissa, 2003) Proposed the exponentiated Weibull distribution, which extends the Weibull family and includes various statistical measures. (Cordeiro et al., 2014) Introduced the exponential-Weibull distribution, a novel three-parameter model with various mathematical aspects obtained and evaluated for effectiveness through simulations.Xie and Lai (1996) Examined a basic model combining two Weibull survival functions to represent a bathtub-shaped failure rate. Lai et al. (2016) Introduced the integrated beta model for bathtub-shaped hazard rate data, specifically designed for modeling lifetime data with a finite range. Nadarajah and Haghighi (2011) Introduced the generalization of the exponential distribution where the generalization always has its mode at zero and allows for increasing, decreasing, and constant hazard rates. Abubakar et al. (2024) we introduce a new modified distribution called arcsine Rayliegh Pareto (ASRP) Distribution. Statistical properties, including survival function, hazard function, entropy, moment,

moment generating function, and order statistics were derived. Also, a maximum likelihood estimation was used to estimate the parameters of the distribution. Thach et al. (2020) Presented a nonlinear failure rate as a generalization of the linear failure rate model. This model was evaluated and applied to actual data sets, both censored and uncensored, demonstrating its flexibility and robustness.Almalki (2018) Hybridized the five-parameter new modified Weibull (NMW) distribution into a reduced variation, maintaining its desirable properties while using fewer parameters.

Moreover, flexible statistical distributions are always needed for practical purposes. Silva et al. (2010) extend the MW distribution to propose a five-parameter hybridized distribution known as the beta-modified Weibull distribution. Similarly in 2013, (Almalki and Yuan, 2013) introduced an additive distribution called the new modified Weibull NMW model by combining the MW and Weibull distribution. The NMW model was known for its great flexibility in modeling datasets characterized by increasing and bathtub failure rate shapes. (Zamani et al., 2021) Introduced a novel distribution tailored for specific data modeling needs. The new log-logistic distribution was designed to handle a wide range of hazard functions, including bathtub-shaped, unimodal, and monotone hazard functions. Al-Essa et al. (2023) The modified exponential-Weibull (MEW) distribution is a novel flexible four-parameter distribution that was proposed by the transfer function of the exponential and Weibull distribution using the odd function transformation. Anzagra et al. (2020) Developed a flexible generator of distributions within the odd Chen-G family, providing two distinct versions to address different modeling scenarios. Ghazal (2023) proposed a novel extension of the three-parameter modified Weibull distribution NMW3 called NMW3, In terms of reliability, it shows bathtub-shaped or increasing hazard rates, which can be advantageous. Its statistical properties were derived. Furthermore, the flatness of the NMW3 distribution's bathtub curve and the parameter sensitivity were examined. Pal et al. (2006) Introduced the family of distributions called exponentiated Weibull distribution EW. The distribution has three parameters: one scale and two shapes. The distributions' moments, survival function, and failure rate have all been determined through the application of specific special formulas. The failure rate's behavior has been examined and the distribution has been fitted to an actual data set with excellent results. Cordeiro et al. (2014) proposed a new three-parameter model called the exponential–Weibull EXW distribution, Some mathematical properties of the proposed distribution were investigated. The four explicit expressions for the generalized ordinary moments and a general formula for the incomplete moments based on infinite sums of Meijer's G functions were derived.

Several decades ago, (Dhillon, 1980) proposed the so-called hazard rate (HR) models in his effort to suggest models that are alternative to Weibull distribution and alike for modeling non-monotone failure rates. Among Dhillon's constructed HR models, one is identified as a suitable model for an inverted bathtub failure rate with its survival function given as

$$S(x) = e^{-\ln(\lambda t^{n}+1)}, \quad x > 0,\, \lambda,\, \eta > 0 \qquad (1)$$

Inspired by the great flexibility of MW distribution in various real data applications identified with non-monotone hazard rates, in this research, we propose to extend the HR model Dhillon (1980) by adopting the same approach as in the case of Weibull and MW distributions. Given that the author referred to the model as the HR model, in this study, we named it after the author for clarity. The author demonstrated the model's potential for describing various types of monotone HR (when $\eta \leq 1$) and non-monotone HR (when $\eta > 1$). The Dhillon model, despite its flexibility, lacks the recognition it merits. Therefore, in this work, we intend to propose yet another version of the Dhillon distribution using the methodology by Lai et al. (2003).

MATERIALS AND METHODS

Modified Dhillon Distribution

Lai et al. (2003) applied the idea of a Beta-integrated model by taking a suitable limit of Beta-integrated distributions developed by Lai et al. (1998), to extend the Weibull distribution Cumulative Density Function (CDF) and proposed the so-called modified Weibull (MW) distribution, with CDF

$$F(x) = 1 - e^{-ax^{b}e^{\lambda x}}, \quad x > 0 \qquad (2)$$

In essence, the term ax^b in the Weibull CDF $F(x) = 1 - e^{-ax^b}$ is replaced by the term $ax^b e^{\lambda x}$ to have the modified Weibull CDF (2). The resulting model in (2) has received wide attention in the literature due to its ability to overcome some of the Weibull's limitations, including the Weibull inability to model failure time found with the bathtub failure rate. The Dhillon model has its hazard rate and the reliability functions defined as

$$h(t) = \frac{\eta \lambda t^{\eta-1}}{\lambda t^{\eta}+1}, \qquad t, \lambda, \eta > 0. \qquad (3)$$

and

$$R(t) = e^{-ln(\lambda t^{\eta}+1)} \qquad \lambda, t > 0, \qquad (4)$$

where $\lambda > 0$ is the scale parameter and $\eta > 0$ represents the model's shape parameter (which controls the shape of the model).

In this study, we propose to extend the two-parameters classical distribution introduced by Dhillon (1980) by applying the same approach as in the case of Weibull and modified Weibull distributions.

Therefore, If X is an MDD random variable, it is said to have followed the cumulative distribution function (CDF) and The probability density functions (PDF) as;

$$F(x) = 1 - e^{-ln(\lambda x^{\alpha} e^{\beta x}+1)} \tag{5}$$

and

$$f(x) = \frac{(\alpha+\beta x)\lambda x^{\alpha-1}e^{\beta x}}{\lambda x^{\alpha}e^{\beta x}+1}e^{-ln(\lambda x^{\alpha} e^{\beta x}+1)} \tag{6}$$

Figure 1 below are CDF plots of some values of the parameters and it can be seen clearly that the CDF is increasing and converges to one. Figure 2 is the pdf plot Modified Dhillon distributions (MDD).

The hazard rate and survival function are expressed respectively as;

$$h(x) \ = \ \frac{(\alpha+\beta x)\lambda x^{\alpha-1}e^{\beta x}}{\lambda x^{\alpha}e^{\beta x}+1} \tag{7}$$

and

$$S(x) = e^{-ln(\lambda x^{\alpha} e^{\beta x}+1)} \tag{8}$$

Figure 3 It can be seen clearly that the survival function is monotonically decreasing. Figure 6 Shows that the hazard function is increasing, decreasing and bathtub as the α and β level changes.

Quantile Function

Suppose that X is a random variable, then the quantile function of X say $X_q(u)$ of the Modified Dhillon distribution is given as:

$$F(x) = 1 - e^{-log(\lambda x^{\alpha} e^{\beta x}+1)} \tag{9}$$

In this study, the quantile function close form can be obtained by substituting $k = x^{\alpha}e^{\beta x}$, it can be shown that $x_u = \sum_{i=1}^{\infty} a_i k^{\frac{i}{\alpha}}$. See Carrasco et al. (2008). Where $a_i = \dfrac{(-1)^{i+1}i^{i-2}}{i-1}(\dfrac{\alpha}{\beta})^{i-1}$ and the condition of convergence of this sum is $\dfrac{u-2}{\lambda} < a\left(\dfrac{\alpha}{\beta e}\right)^{\alpha}$. In that case, the quantile function $Q(u)$ is.

$$Q(u) = \sum_{i=1}^{\infty} a_i \left(\frac{u-2}{\lambda}\right)^{\frac{i}{\alpha}} \tag{10}$$

The skewness and the kurtosis are defined as

$$SK = \frac{Q\left(\frac{2}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{11}$$

and

$$KU = \frac{Q(\frac{7}{8}) - Q(\frac{3}{8}) - Q(\frac{3}{8}) + (\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})} \tag{12}$$

Moments

The r^{th} moment of the random variable X is defined as follows:

$$\mu_r' = \int_0^{\infty} \frac{x^r\lambda(\alpha+\beta x)x^{r+\lambda-1}e^{\beta x}}{(\lambda x^{\alpha}e^{\beta x}+1)^2}dx \tag{13}$$

It is obvious that equation 13 can not produce the closed-form solution, it can be simplified using a tailor series expansion as:

$$\mu_r' = \sum_{i=0}^{\infty}\int_0^{\infty} \frac{\lambda\beta^i i!(\alpha+\beta x)x^{r+\alpha+i-1}}{(\lambda\beta^i\sum_{i=0}^{\infty}x^{\alpha+i}+i!)^2}dx \tag{14}$$

where

$$I(x,r,\alpha,i,\lambda,\beta) = \int_0^{\infty} \frac{(\alpha+\beta x)x^{r+\alpha+i-1}}{(\lambda\beta^i\sum_{i=0}^{\infty}x^{\alpha+i}+i!)^2}dx \tag{15}$$

Therefore moment can be represented as:

$$\mu_r' = \sum_{i=0}^{\infty}\lambda\beta^i i! I(x,r,\alpha,i,\lambda,\beta) \tag{16}$$

Moment Generating Function (MGF)

The moment-generating function of the Modified Dhillon distribution is defined as

$$M_x(t) = \int_0^{\infty} \frac{e^{tx}\lambda(\alpha+\beta x)x^{\alpha-1}e^{\beta x}}{(\lambda x^{\alpha}e^{\beta x}+1)^2}dx \tag{17}$$

Using Tailor series expansion

$$e^{\beta x} = \sum_{j=0}^{\infty}\frac{\beta^j x^j}{j!} \tag{18}$$

apply the Tailor series expansion in 18 we have

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^{\infty} \frac{\lambda i!(\alpha+\beta x)t^i\beta^j x^{\alpha=i=j-1}}{(\lambda\beta^j\sum_{j=0}^{\infty}x^{\alpha+j}+j!)}dx \tag{19}$$

Then the MGF of MDD can be seen as:

$$M_x(t) = \sum_{i,j=0}^{\infty}\frac{\lambda i! t^i\beta^j}{j!}I(x,\alpha,\beta,t,m,i,j,\lambda) \tag{20}$$

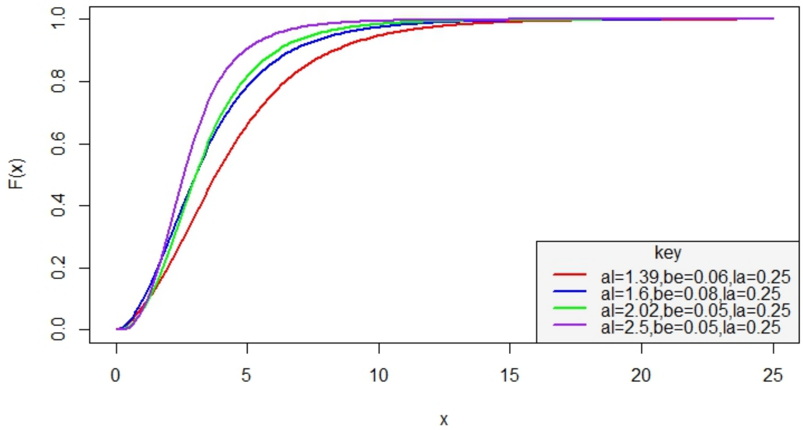

Figure 1: Plot of Modified Dhillon Cumulative density function (CDF)

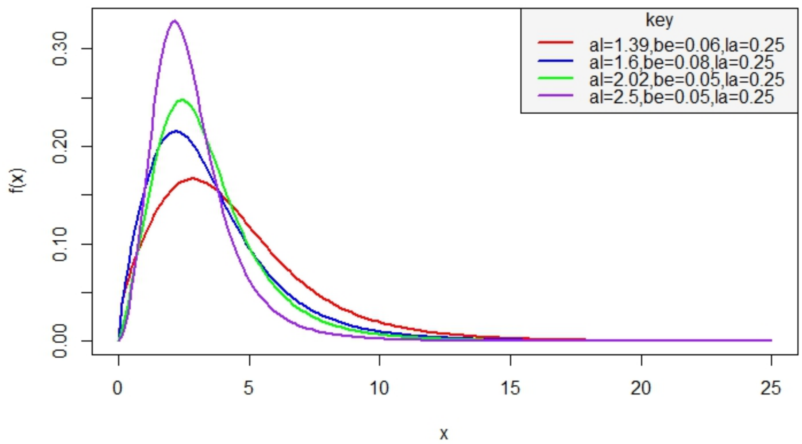

Figure 2: Plot of Modified Dhillon probability density function (pdf)

Where

$$I(x,\alpha,\beta,t,m,i,j,\lambda)=\int_0^{\infty}\frac{(\alpha+\beta x)x^{\alpha+i+j-1}}{(\lambda\beta^j\sum_{j=0}^{\infty}x^{\alpha+j}+j!)}dx \tag{21}$$

Order statistics

Suppose $x_1,...x_n$ is a random sample from the MDD and let $x_{i:n}<...<x_{n:n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-1)!}f(x)F(x)^{i-1}[1-F(x)]^{n-i} \tag{22}$$

Using binomial expansion

$$[i-F(x)]^{n-i}=\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}[F(x)]^j \tag{23}$$

Which is equivalent to

$$f_{i,n}(x)=\sum_{j=0}^{n-i}\frac{(-1)^j n!}{(i-1)(n-i-j)!j!}f(x)[F(x)]^{j+i-1} \tag{24}$$

substitute (5) and (6) into (24) gives:

$$f_{i,n}(x)=\sum_{j=0}^{n-i}(-1)^j n!(\alpha\lambda x^{\alpha-1}+\lambda\beta x^{\alpha})\times \frac{e^{\beta x}(1-(\lambda x^{\beta}x)^{-1})^{j+i-1}}{(i-1)(n-i-j)!j!(\lambda x^{\alpha}e^{\beta x}+1)^2} \tag{25}$$

The minimum order statistics are obtained as

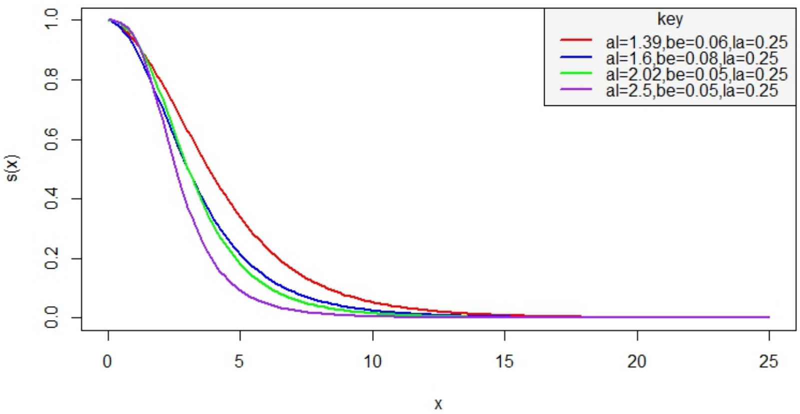

Figure 3: survival function plot of MDD

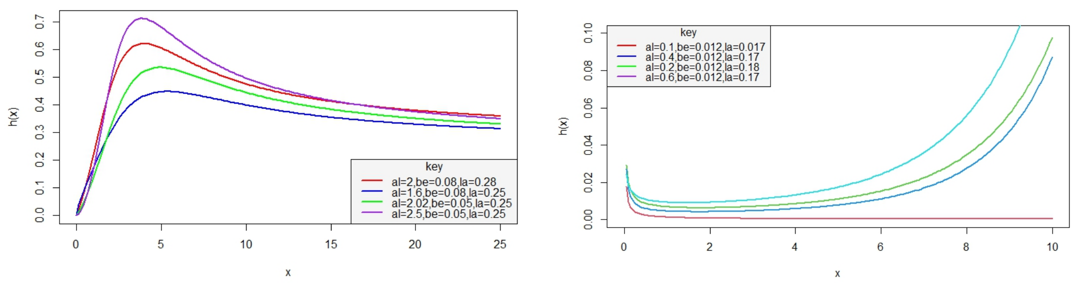

Figure 4: Hazard function plots of MDD

$$f_{i,n}(x) = \sum_{j=0}^{\infty} \binom{n-1}{j} (-1)^j (\lambda\alpha x^{\alpha-1} + \lambda\beta x^{\alpha}) e^{\beta x} (1 - (\lambda x^{\alpha} e^{\beta x} + 1)^{-1})^j \tag{26}$$

Also, the maximum order of statistics is obtained as

$$f_{n,n}(x) = n(\alpha\lambda x^{\alpha-1} + \lambda\beta x^{\alpha}) e^{\beta x} \times (1 - (\lambda x^{\alpha} e^{\beta x} + 1)^{-1})^{n-1} \tag{27}$$

Entropy

Entropy also known as Shanon entropy, is a measure of uncertainty or randomness of a probability distribution which is defined as.

$$I_y(x) = \frac{1}{1-\phi} log \int_{-\infty}^{\infty} f(y)^{\phi} dx \tag{28}$$

The Modified Dhillon distribution is defined as:

$$f(x)^{\phi} = \left(\frac{i!(\alpha\lambda x\alpha + i - 1 + \lambda\beta^{i+1} x^{\alpha+i})}{(\lambda x^{\alpha+i}\beta^{i} + 1)} \right)^{\phi} \tag{29}$$

The MDD Entropy is defined as:

$$I_y(x) = \frac{1}{1-\phi} log \int_{0}^{\infty} \frac{\left(i!(\alpha\lambda x\alpha + i - 1 + \lambda\beta^{i+1} x^{\alpha+i})\right)^{\phi}}{(\lambda x^{\alpha+i}\beta^{i} + 1)^{2\phi}} dx \tag{30}$$

Parameters estimation of the MDD

let $x_1, \therefore, x_n$ be a random sample of size n from the Modified Dhillon distribution (MDD) with parameter (α,β,λ). Then the log-likelihood function of MDD is expressed as

$$L(x,\theta) = \prod_{i=1}^{n} f(x,\theta) \tag{31}$$

Now, substitutes (6) into (31) gives;

$$L(x,\theta) = \prod_{i=1}^{n} \frac{\lambda(\alpha + \beta x) x^{\alpha-1} e^{\beta x}}{\lambda x^{\alpha} e^{\beta x} + 1} e^{-ln(\lambda x^{\alpha} e^{\beta x} + 1)} \tag{32}$$

Evaluate (32) it becomes;

$$Lf(x_i,\theta) = \lambda^n(\alpha+\beta x_i)^n x^{n(\alpha-1)} \times e^{\beta\sum_{i=0}^n x_i}(\lambda x_i^{\alpha} e^{\beta x_i}+1)^{-2n} \tag{33}$$

take the log of equation 33 we have;

$$logLf(x_i,\theta) = log\lambda^n + log(\alpha+\beta x_i)^n + log\left[\prod_{i=1}^n x_i^{n(\alpha-1)}\right] + loge^{e^{\beta\sum_{i=0}^n x_i}} + log(\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}+1)^{-2n} \tag{34}$$

evaluate equations (34) to gives;

$$nlog\lambda + nlog(\alpha+\beta x_i) + nlog\sum_{i=0}^n x_i - .. \; nlog\sum_{i=0}^n x_i - nlog\sum_{i=0}^n x_i + \beta\sum_{i=0}^n x_i - 2nlog(\lambda x_i^{\alpha} e^{\sum_{i=0}^n x_i}+1).. \tag{35}$$

differentiate equation (35) with respect to α we have

$$\frac{n}{\alpha+\beta x_i} + nlog\sum_{i=0}^n x_i - \frac{2n\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}}{(\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}+1)} \tag{36}$$

differentiate equation (35) with respect to β we have

$$\frac{dlogLf(x_i,\theta)}{d\beta} = \frac{nx_i}{\alpha+\beta x_i} + \sum_{i=0}^n x_i - \frac{2n\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i} * \sum_{i=0}^n x_i}{(\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}+1)} \tag{37}$$

differentiate equation (35) with respect to λ we have;

$$\frac{dlogLf(x_i,\theta)}{d\lambda} = \frac{n}{\lambda} - \frac{2nx_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}}{(\lambda x_i^{\alpha} e^{\beta\sum_{i=0}^n x_i}+1)} \tag{38}$$

RESULTS AND DISCUSSIONS

In this section, we used an actual data set that was fitted to the model along with some baseline generalizations to demonstrate the performance of the Modified Dhillon distribution.

The Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) are taken into consideration while evaluating the performance of our suggested distribution in comparison to alternative distributions. The values of log-likelihood functions (L), The Akaike Information Criterion (AIC), the Consistent Akaike Information Criterion (CAIC), and the Bayesian Information Criterion (BIC) are given in the Table for the models to verify which of the distributions best fit the dataset.

Dataset 1

The first data set utilized comes from (Dhillon, 1981) and has been extensively utilized in numerous literary works see (Rizvi et al., 2008; Dhillon, 2007; Sra and Dhillon, 2006), it represents the "Time to first failure (1000's hours) of 500 MW" generators. The data sets are given below;

$0.058, 0.070, 0.090, 0.105, 0.113, 0.121, 0.153, 0.159, 0.224, 0.421, 0.570, 0.596, 0.618, 0.834, 1.019, 1.104, 1.497, 2.027, 2.234, 2.372, 2.433, 2.505, 2.690, 2.877, 2.879, 3.166, 3.455, 3.551, 4.378, 4.872, 5.085, 5.272, 5.341, 8.952, 9.188, 11.399.$

Dataset 2

The second data is the data sets representing the failure and running times of thirty devices see (Meeker et al., 2022), the thirty sets of datasets that represent the failure and running times of thirty devices is given below;
$2, 10, 13, 23, 23, 28, 30, 65, 80, 88, 106, 143,$
$147, 173, 181, 212, 245, 247, 261, 266, 275,$
$293, 300, 300, 300, 300, 300, 300, 300, 300.$

Discussion

Table 2 Is the result obtained using time to first failure of 500 Mega Watts (MW) generator data sets, which indicates the corresponding values of L, AIC, CAIC, and BIC for each model. The table is clearly demonstrate that the modified Dhillon distribution (MDD) has the minimum value of L, AIC, CAIC, and BIC among others, this indicate MMD outperforms the others extension distributions in terms of performance. As a result, when it comes to fitting the same data set, it will considered as the best model among the four (4) distributions that were used to compare performance.
On the other hand table 5 Is the result obtained using the Meeker and Escobar data set, it is clearly shown that the proposed distribution: Modified Dhillon distributions (MDD) proved to be a better-fitted model with the least value of (L), (AIC), (AICC), and (BIC), Using the Meeker and Escobar data set. This makes it clearer that, out of the four models, MDD might be the best. Consequently, we conclude that the MDD offers the best fit among the studied distributions after taking into account

Table 1: Summary of time to failure MW Generator

Min	Q(1)	Median	Mean	Q(5)	Max
0.0580	0.3718	2.1305	2.5674	3.5674	11.3994

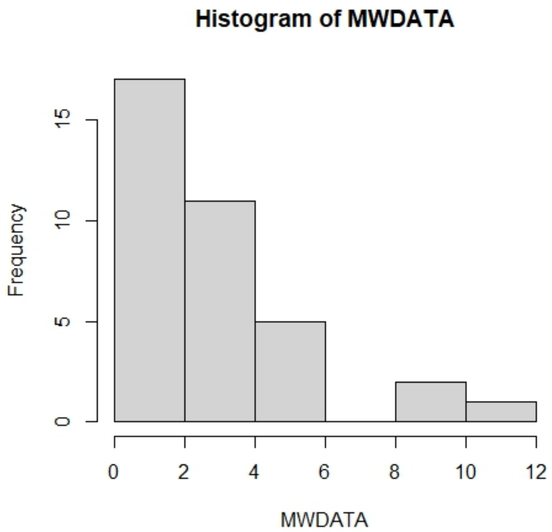

Figure 5: Histogram illustration of right-skewed Megawatts Generator data

Table 2: Models performance comparison using MW data sets

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{L}$	AIC	AICC	BIC
MDD	0.7041	0.2495	0.4930		-68.3577	142.7155	143.4655	147.4660
EXW	0.2245	1.3378	0.4754		-68.4044	142.8089	143.5589	147.5598
AW	0.0036	2.5145	0.4924	0.71985	-68.0764	144.1528	145.4431	150.4868
EW	0.7226	0.3532	0.1480		-68.6540	143.3080	144.0580	148.0586
DH	0.0021		0.0027		-72.0209	148.0417	148.4050	151.2087

Table 3: Normality test for the fitted models

Models	KS	A	W
MDD	0.10253	0.48643	0.070111
EXW	0.10245	0.46462	0.063375
AW	0.099371	0.4558	0.062381
EW	0.11789	0.55714	0.08505
DH	0.3499	1.1653	0.1932

Table 4: Summary of failure and running time data of thirty devices by Meeker and Escobar

Min	Q(1)	Median	Mean	Q(5)	Max
2.00	68.75	196.50	177.03	298.25	300.00

the outcomes of the two distinct data sets mentioned above.

CONCLUSION

From the foregone results, the Modified Dhillon Distribution (MDD) emerges as a superior model for time-to-failure data, showcasing its robustness and accuracy in comparison to other distributions. The MDD's ability to provide a better fit across different data sets underscores its practical utility in reliability analysis and decision-making processes.

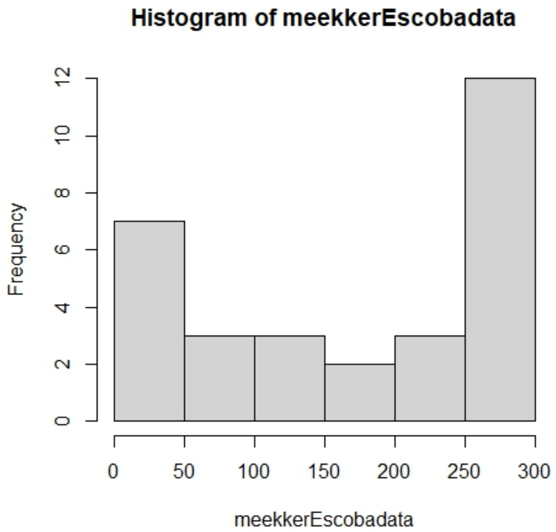

Figure 6: Failure and running time

Table 5: Model performance comparison using Meeker and Escobar data sets

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{L}$	AIC	AICC	BIC
MDD	0.67037	0.00442	0.014145		-142.2023	290.4046	291.3277	294.6082
EXW	0.00314	4.90531	0.186365		-142.5892	291.1783	292.1014	295.3819
AW	0.5432	1.3452	0.3421	0.7199	-162.3937	332.7874	334.3874	338.3922
EW	0.99396	0.11765	0.11950		-142.7917	290.5834	290.5065	294.7870
DH	0.0021		0.0027		-193.1844	390.3688	390.8132	393.1712

Table 6: Normality test for the fitted model's Meeker Escobar

Models	KS	A	W
MDD	0.29106	1.6521	0.27325
EXW	0.23415	1.3114	0.20497
AW	0.51366	0.0542	0.062381
EW	0.21875	0.1.8295	0.31136
DH	0.2069	2.1667	0.3797

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