FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 8 No. 6, December, 2024, pp 186 - 195
DOI: https://doi.org/10.33003/fjs-2024-0806-2747

KUMARASWAMY TYPE II GENERALIZED TOPP-LEONE-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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ABSTRACT

In the field of reliability theory, practitioners have been working assiduously in recent years to propose new families of continuous probability distributions that extend the standard theoretical distribution that is currently in use. They have done this by hybridizing two or more probability models or by introducing one or more parameters to get more flexibility in fitting data from a variety of fields, including the environmental, economics, finance, and medical sciences. The T-X approach was used to establish the Kumaraswamy Type II Generalized Topp-Leone-G (K_wT_2GTL-G) family, which extends the Type II Generalized Topp-Leone-G family of distributions with extra shape parameters. A few statistical characteristics of the novel family were determined and examined. A sub-model emerged and MLE was used to estimate the model parameters. To demonstrate the value of the new family, two real-life data sets were used: a set that related to the relief times (in minutes) of patients taking an analgesic, and the other that related to the failure and service times for a windshield. The superior goodness-of-fits and empirical flexibility of the K_wT_2GTL-G distribution are demonstrated by comparisons with other distributions, including the Kumaraswamy Extension Exponential (K_wEEx), Kumaraswamy Exponential (KEx), Exponential Generalized Exponentiated Exponential (EGEEx), and Exponentiated Weibull-Exponential (EWEx) distributions.. In the second dataset, the K_wT_2GTLEx distribution achieved an AIC value of 38.0489, outperforming the EGEEx distribution which had an AIC value of 39.6708 next to it. These findings highlight the K_wT_2GTL-G family's potential to enhance lifetime data modeling, which would have a substantial impact on engineering, medicine, and other domains.

Keywords: Kumaraswamy, MLE, TIIGTL-G, K_wT_2GTL-G

INTRODUCTION

Many problems abound in various field of human endeavor cannot be perfectly and adequately handled by most commonly known conventional probability distributions available for modeling lifetime data sets such as Normal, Weibull, Pareto, Gomepertz, Rayleigh, Exponential, e.t.c, (Yahaya and Doguwa (2021)). According to Adekunle *et al* (2022), Numerous statistical distributions have been used widely to describe and forecast current occurrences in a variety of fields, including biology, engineering, economics, geography, and many more. However, the data in many of these areas typically exhibit complex behavior and a variety of forms that are linked to varying levels of skewness and kurtosis. There is a clear need for an extended version of these classical distributions to enhance their capability while also improving their goodness of fit. Various strategies for changing existing classical distribution to make them more flexible or inventing new statistical distributions for modeling data sets from various fields of study have flooded the statistical archive in recent decades. Majority of the strategies are focused toward constructing heavy-tailed distributions, monotonic and non- monotonic failure rates, a tractable cumulative distribution function (CDF) for easy simulation, and modeling data with varying degrees of skewness and kurtosis.

The Exponentiated Type II Generalized Topp-Leone-G Family by Kolawole *et al* (2023), the Beta-G by Eugene *et al*. (2002), the Weibull-X family of distributions of Alzaatreh *et al*. (2013), the Exponentiated Generalized class of Cordeiro *et al*. (2013), the Logistics-G introduced by Torabi and Montzeri (2014), the Gamma-X family of Alzaatreh *et al*. (2014), the Odd Generalized Exponential-G of Tahir *et al*. (2015), the Type I half- logistic family of Cordeiro *et al.* (2016), the Kumaraswamy-Weibull-Generated family of Hassan and Elgarhy (2016), the New Weibull-G family of Tahir *et al*. (2016), the Generalized Transmuted-G of Nofal *et al*. (2017), the New Generalized family of distributions of Ahmad *et al* (2018), the Topp-Leone Kumaraswamy- G family of distributions by Ibrahim *et al*. (2020), *Rayleigh*-Exponentiated Odd Generalized-*X* Family by Yahaya and Doguwa (2021), Type I Half Logistic Exponentiated-G family by Bello *et al.* (2021) are some well- known modified families of distributions in the literature proposed by different researchers to improve the standard theoretical distribution

The Generalized distribution can be used effectively in fitting lifetime datasets because it can accommodate monotonic and non-monotonic data characteristics.

Hence, this study proposed, Kumaraswamy Type II Generalized Topp-Leone-G family of distribution capable of modeling with monotonic and non- monotonic hazard functions that can provide better fits to medical and engineering data.

In this work, we define another new class of generalized family of distribution called Kumarasawamy Type II Generalized Topp-Leone-G family (K_wT_2GTL-G) from the Cumulative Distribution Function of Kumaraswamy-G (Kw-G)family defined by Cordeiro and de Castro (2011) as;

$$F_{kw}(x;\lambda,\theta,\xi) = 1-\left[1-G^{\lambda}(x;\xi)\right]^{\theta} \tag{1}$$

And pdf

$$f_{Kw}(x;\lambda,\theta,\xi) = \theta\lambda g(x;\xi)G^{\lambda-1}(x;\xi)\left[1-G^{\lambda}(x;\xi)\right]^{\theta-1} \tag{2}$$

Where $\theta > 0, and \lambda > 0$ are two shape parameters belonging to set of positive real numbers.

According to Alzaatreh *et al.* (2013), the cdf of the T-X family of distribution is given as

$$F(x) = \int_{\beta_1}^{N[G(x)]} r(t)dt = R\left[N[G(x)]\right] \tag{3}$$

Where $N[G(x)]$ must be found to satisfy the following conditions

i. $N[G(x)] \in [\beta_1, \beta_2]$

ii. $N[G(x)]$is differentiable and monotonically non-decreasing, and (4)

iii. $N[G(x)] \to \beta_1$ as $x \to -\infty$ and $N[G(x)] \to \beta_2$ as $x \to \infty$

Let $r(t)$ be the pdf of a random variable $T \in [\beta_1, \beta_2]$for $-\infty \leq \beta_1 < \beta_2 < \infty$and $N[G(x)]$ be a function of the cdf of a random variable X.

Then the pdf corresponding to equation (3) is given by;

$$f(x) = \left\{\frac{d}{dx} N[G(x)]\right\} r\{N[G(x)]\} \quad (5)$$

Let$X$be any arbitrary random variable with CDF:$G(x;\xi)$.Also, let $T \in (\beta_1, \beta_2)$ be a random variable with a PDF: $r(t)$. Furthermore, let our proposed link function be Type II Generalized Topp-Leone family of distribution, (Hassan *et al.*, 2019) and it is given as,

$$F_{T_2GTL-G}(t;\beta,\alpha,\xi) = 1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha} \quad (6)$$

And the Probability Density Function (pdf) is given as

$$f_{T_2GTL-G}(t;\beta,\alpha,\xi) = 2\alpha\beta h(t;\xi) H^{2\beta-1}(t;\xi)\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha-1} \quad (7)$$

where $\beta > 0, \alpha > 0; t > 0$and $F(t;\xi)$ and $f(t;\xi)$are the cdf and pdf of the baseline distribution with parameter vectorξ.

MATERIALS AND METHODS

Kumaraswamy Type II Generalized Topp-Leone-G Family of Distribution (K_wT_2GTL-G)

The CDF ofK_wT_2GTL-G family of distribution is given by;

$$F_{K_wT_2\text{GTL}-G}(t;\beta,\alpha,\theta,\lambda,\xi) = 1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta} \quad (8)$$

Proof

From equation (2), we can write;

$$F_{K_wT_2\text{GTL}-G}(t;\beta,\alpha,\theta,\xi) = \int_0^{F_{T_2GTL-G}(t;\beta,\alpha,\xi)} f_{K_w}(x;\lambda,\theta,\xi)dx \quad (9)$$

$$F_{K_wT_2\text{GTL}-G}(t;\beta,\alpha,\theta,\lambda,\xi) = \int_0^{1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}} \theta\lambda g(x;\xi) G^{\lambda-1}(x;\xi)\left[1 - G^{\lambda}(x;\xi)\right]^{\theta-1} dx$$

$Let\ y = 1 - G^{\lambda}(x;\xi), when\ x = 0, y = 1, when\ x = 1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}, y = 1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}$

$$\frac{dy}{dx} = -\lambda g(x;\xi) G^{\lambda-1}(x;\xi), dx = \frac{dy}{-\lambda g(x;\xi) G^{\lambda-1}(x;\xi)}$$

$$F_{K_wT_2GTL-G}(t;\beta,\alpha,\theta,\lambda,\xi) = \int_{1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}}^{1} \theta\lambda g(x;\xi) G^{\lambda-1}(x;\xi) y^{\theta-1} \frac{dy}{\lambda g(x;\xi) G^{\lambda-1}(x;\xi)}$$

$$F_{K_wT_2GTL-G}(t;\beta,\alpha,\theta,\lambda,\xi) = \int_{1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}}^{1} \theta y^{\theta-1} dy$$

$$F_{K_wT_2GTL-G}(t;\beta,\alpha,\theta,\lambda,\xi) = \theta\left[\frac{y^{\theta-1+1}}{\theta-1+1}\right]_{1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}}^{1}$$

$$F_{K_wT_2GTL-G}(t;\beta,\alpha,\theta,\lambda,\xi) = \left[y^{\theta}\right]_{1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}}^{1}$$

The cdf of the new family is as follows

$$F_{K_wT_2GTL-G}(t;\beta,\alpha,\theta,\lambda,\xi) = 1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}$$

Important Representation

The binomial expansion to expand cdf of K_wT_2GTL-G is given below as;

$$[1 - t]^{b} = \sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i} t^{i} \quad (10)$$

Using the series expansion in equation (10), then equation (8) becomes

$[F(t)]^{h}$

$$\left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]^{h} = \sum_{i=0}^{\infty}(-1)^{i}\binom{h}{i}\left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta i}$$

Consider;

$$\left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta i} = \sum_{j=0}^{\infty}(-1)^{j}\binom{\theta i}{j}\left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda j}$$

$$\left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda j} = \sum_{k=0}^{\infty}(-1)^{k}\binom{\lambda j}{k}\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha k}$$

$$\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha k} = \sum_{n=0}^{\infty}(-1)^{n}\binom{\alpha k}{n} H^{2\beta n}(t;\xi)$$

Therefore,

$[F(t)]^h = \sum_{i,j,k,n=0}^{\infty} (-1)^{i+j+k+n} \binom{h}{i}\binom{\theta i}{j}\binom{\lambda k}{k}\binom{\alpha k}{n} H^{2\beta n}(t;\xi)$

$[F(t)]^h = \sum_{i,j,k,n=0}^{\infty} \varphi_q H^{2\beta n}(t;\xi)$ (11)

Where;

$\varphi_q = (-1)^{i+j+k+n} \binom{h}{i}\binom{\theta i}{j}\binom{\lambda k}{k}\binom{\alpha k}{n}$

Also we have expansion for pdf as;

$f(t,\alpha,\beta,\theta,\lambda,\xi) = \sum_{k,l,m=0}^{\infty} \zeta_q h(t;\xi) H^{2\beta(m+1)-1}(t;\xi)$ (12)

Where;

$\zeta_q = 2\alpha\beta\theta\lambda(-1)^{k+l+m} \binom{\theta-1}{k}\binom{\lambda(k+1)-1}{l}\binom{\alpha(l+1)-1}{m}$

Statistical Properties of K_wT_2GTL-G Family of Distribution

In this section, some statistical properties of the K_wT_2GTL-G family of distributions were derived.

Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, r^{th}moment for the variable $T \sim K_wT_2GTL - G$, says $\mu^1{}_r$ is derived as follows;

$\mu_r^1 = E(t^r) = \int_{-\infty}^{\infty} t^r f(t)dt$ (13)

By using important representation in equation (12), we have

$= \int_0^1 t^r 2\alpha\beta\theta\lambda h(t;\xi) \sum_{k,l,m=0}^{\infty} (-1)^{k+l+m} \binom{\theta-1}{k}\binom{\lambda(k+1)-1}{l}\binom{\alpha(l+1)-1}{m} H^{2\beta(m+1)-1}(t;\xi)\, dt$ (14)

$= \int_0^1 t^r \sum_{k,l,m=0}^{\infty} \zeta_q h(t;\xi) H^{2\beta(m+1)-1}(t;\xi)\, dt$

Moment Generating Function (MGF)

The Moment Generating Function of $T \sim K_wT_2GTL - G$is given as:

$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(t)dt$ (15)

By using important representation in (12), we have

$= \int_0^{\infty} e^{tx} \sum_{k,l,m=0}^{\infty} \zeta_q h(t;\xi) H^{2\beta(m+1)-1}(t;\xi)\, dt$ (16)

$M_x(t) = \sum_{k,l,m=0}^{\infty} \zeta_q J_r$ (17)

Where;

$J_r = \int_0^1 e^{tx} \zeta_q h(t;\xi) H^{2\beta(m+1)-1}(t;\xi)dt$

Probability Weighted Moments (PWM)

The class of moment used to describe inverse form estimators for the parameters and quantiles of a distribution is known as Probability Weighted Moments (PWMs) and was proposed by Greenwood *et al*., (1979). The PWMs, represented by $\tau r, s$can be derived for a random variable T using the following relationship.

$\tau r, s = E[T^r F(t)^s] = \int_{-\infty}^{\infty} T^r f(t) \big(F(t)\big)^s dt$ (18)

The PMWs is derived by substituting equation (11) and (12) into equation (18) replacing h with s, we have

$\tau r, s = \int_0^1 T^r 2\alpha\beta\theta\lambda h(t;\xi) *$

$\sum_{i,j,k,n=0}^{s} \sum_{k,l.m=0}^{\infty} (-1)^{i+j+k+n} (-1)^{k+l+m} \binom{\theta-1}{k}\binom{\lambda(k+1)-1}{l}\binom{\alpha(l+1)-1}{m}\binom{h}{i}\binom{\theta i}{j}\binom{\lambda j}{k}\binom{\alpha k}{n} H^{2\beta(m+n+1)-1}(t;\xi)dt$ (19)

$\tau r, s = \int_0^1 T^r \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \varphi_q \zeta_q h(t;\xi) H^{2\beta(m+n+1)-1}(t;\xi)d\, t$ (20)

$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \varphi_q \zeta_q \int_0^1 T^r h(t;\xi) H^{2\beta(m+n+1)-1}(t;\xi)dt$ (21)

$\tau r, s = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \varphi_q \zeta_q \,.\, Tr, 2\beta(m+n+1)-1$ (22)

Where;

$\tau r, 2\beta(m+n+1)-1 = \int_0^1 T^r h(t;\xi) H^{2\beta(m+n+1)-1}(t;\xi)\, dt$

Quantile Function of K_wT_2GTL-G

The quantile function is an important tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For t, the quantile function is $F(t) = U$, where $U$is distributed as $U(0,1)$.The K_wT_2GTL-G family is easily simulated by inverting equation (8) which yields the Quantile function $Q(U)$ defined as

$$Q(u) = H^{-1}(t;\xi)\left[1 - \left[1 - \left[1 - [1-U]^{\frac{1}{\theta}}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}} \quad (23)$$

where $H(t;\xi)^{-1}$ is the quantile function of the baseline cdf$G(t;\xi)$. The first quartile, the median and the third quartile are obtained by seting$U = 0.25, U = 0.5, U = 0.75$, respectively in equation (23)

Where $t = Q^{-1}$ is the quantile function of the baseline distribution.

Survival Function

The Probability of an item not failing prior to some time is known as survival or reliability function and it is defined as;

$R(t;\beta,\alpha,\theta,\lambda,\xi) = P(T > t) = 1 - F(t;\xi)$

$$R(t;\beta,\alpha,\theta,\lambda,\xi) = P(T > t) = 1 - \left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right] \quad (24)$$

Hazard Function

The hazard function is the probability of an event of interest occurring within a relatively short time period and it is given as;

$T(t;\beta,\alpha,\theta,\lambda,\xi) = \frac{f(t;\alpha,\beta,\theta;\xi)}{R(t;\alpha,\beta,\theta;\xi)}$

$$T(t;\beta,\alpha,\theta,\lambda,\xi) = \frac{2\beta\alpha\lambda\theta h(t;\xi)H^{2\beta-1}(t,\xi)\left[1-H^{2\beta}(t;\xi)\right]^{\alpha-1}\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda-1}\left[1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1}}{1-\left[1-\left[1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]} \quad (25)$$

Distribution of Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $T_1, T_2 \ldots T_n$be independent and identically distributed $(i.i.d)$ random variables with their corresponding continuous distribution function $F(t)$. Let $T_{1:n} < T_{2:n} < \ldots < T_{n:n}$ the corresponding ordered random sample from a population of size n. Let$F_{r:n}(t)$ and $f_{r:n}(t), r = 1,2,3\ldots, n$denote the cdf and pdf of the r^{th}order statistics $T_{r:n}$respectively. David (1970) gave the probability density function of $T_{r:n}$ as;

$$f_{r:n}(t) = \frac{1}{B(r,n-r+1)} F^{r-1}(t)[1 - F(t)]^{n-r} f(t) \quad (26)$$

By substituting equation (8) and equation (9) into equation (26), we have;

$$f_{r:n}(t) = \frac{1}{B(r,n-r+1)}\left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]^{r-1}\left[1 - \left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]\right]^{n-r}$$

$$2\beta\alpha\lambda\theta h(t;\xi)H^{2\beta-1}(t,\xi)\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha-1}\left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda-1}\left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1} \quad (27)$$

The pdf of the maximum order statistics is obtained by setting $r = n$ in equation (27) as; $f_{n:n}(t;\beta,\alpha,\theta,\lambda,\xi) = 2n\beta\alpha\lambda\theta h(t;\xi)H^{2\beta-1}(t,\xi)\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha-1}\left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda-1}$

$$\left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1}\left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]^{r-1} \quad (28)$$

Also, the pdf of the minimum order statistics is obtained by setting $r = 1$in equation (27)

$$f_{1:n}(t;\beta,\alpha,\theta,\lambda,\xi) = 2n\beta\alpha\lambda\theta h(t;\xi)H^{2\beta-1}(t,\xi)\left[1 - H^{2\beta}(t;\xi)\right]^{\alpha-1}\left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda-1}$$

$$\left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1}\left[1 - \left[1 - \left[1 - \left[1 - \left[1 - H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{\theta}\right]\right]^{n-r} \quad 29)$$

Sub-Model

In this section, we provide sub-model of this family corresponding to the baseline Exponential (Ex) distribution to show the flexibility of the new family.

Kumaraswamy Type II Generalized Topp-Leone Exponential Distribution (K_wT_2GTLEx)

Let us consider the Exponential distribution which is the baseline distribution with parameter $\delta$with cumulative distribution and probability density functions given, respectively by;

$$H_{Ex}(t;\delta) = 1 - e^{-\delta t} \quad (30)$$

and

$$h_{Ex}(t;\delta) = \delta e^{-\delta t} \quad (31)$$

Where $t > 0, \delta > 0$

Then the cdf and pdf of the proposed K_wT_2GTLEx distribution with five parameters are obtained by inserting equation (30) into equation (8) and are respectively given as

$$F_{K_wT_2GTLEx}(t;\beta,\alpha,\theta,\lambda,\delta) = 1 - \left[1 - [1 - [1 - [1 - e^{-\delta t}]^{2\beta}]^{\alpha}]^{\lambda}\right]^{\theta} \quad (32)$$

And

$$f_{K_wT_2GTLEx}(t;\beta,\alpha,\theta,\lambda,\delta) = 2\beta\alpha\theta\lambda\delta e^{-\delta t}\left[1 - e^{-\delta t}\right]^{2\beta-1}\left[1 - \left[1 - e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}\left[1 - \left[1 - \left[1 - e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda-1}$$

$$\left[1 - \left[1 - \left[1 - \left[1 - e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1} \quad (33)$$

Where$t > 0, \alpha, \theta, \beta, \lambda > 0$ are shape paremeters $and. \delta > 0, is$ scale parameter

Furthermore, the following are the reliability function, hazard rate function and the quantile function respectively

$$R(t;\alpha,\beta,\theta,\lambda,\delta) = 1 - \left[1 - \left[1 - \left[1 - \left[1 - \left[1 - e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]\right]^{\theta} \quad (34)$$

$$r(t) = \frac{2\beta\alpha\theta\delta\lambda e^{-\delta t}\left[1-e^{-\delta t}\right]^{2\beta-1}\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda-1}\left[1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]^{\theta-1}}{1-\left[1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]^{\theta}} \quad (35)$$

$$t = Q(U) = \frac{1}{\delta}\left[-\log\left[1 - \left[1 - \left[1 - \left[1 - \left[1 - U\right]^{\frac{1}{\theta}}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}\right]\right] \quad (36)$$

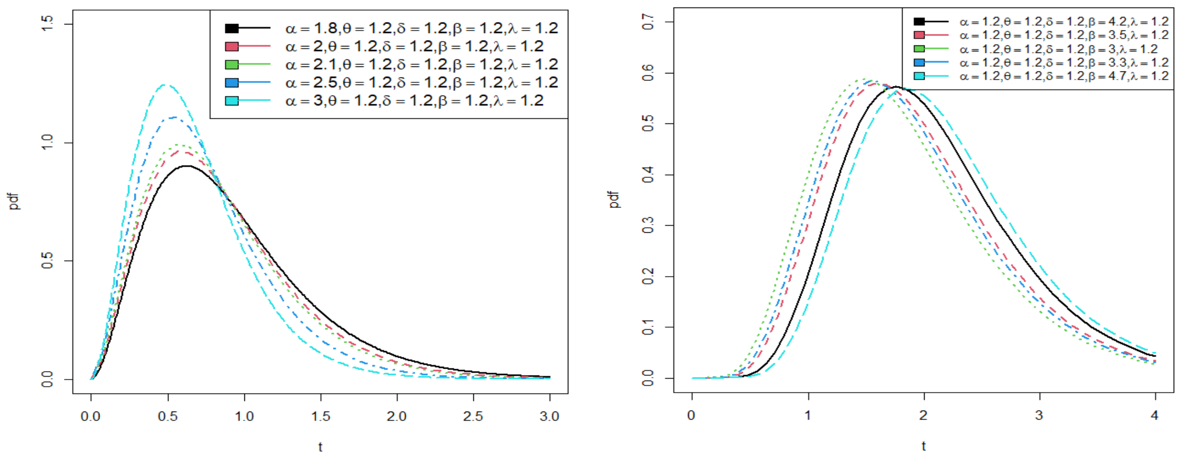

Figure 1: Plots of Pdf of the K_wT_2GTLEx distribution for different parameters values that shows the shape of the distribution at different parameter values which indicates that the K_wT_2GTLEx distribution can be used to model highly skewed data.

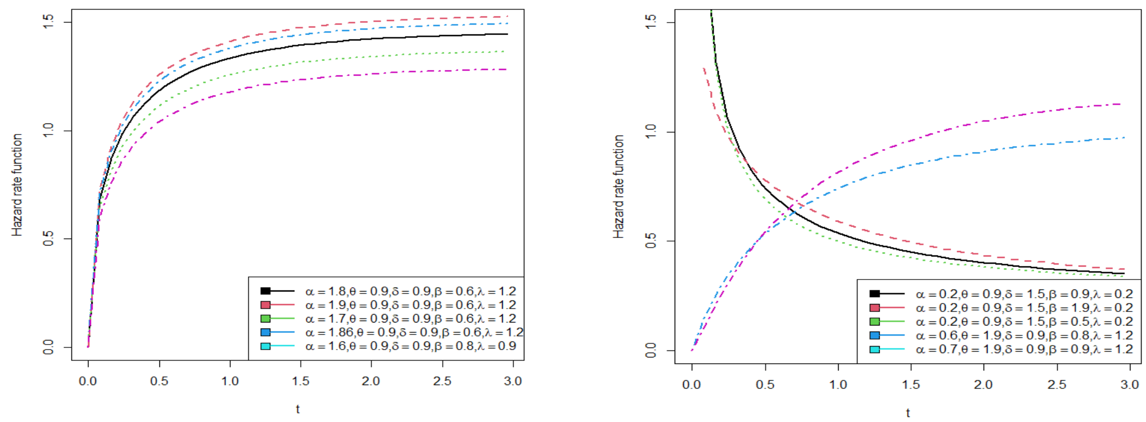

Figure 2: Plots of Hazard function of the K_wT_2GTLEx distribution for different parameters values which show different shapes of hazard function of the distribution. It can be deduced from the plot that the distribution has an increasing, and decreasing shapes which makes it a good distribution for modeling biomedical data and engineering data

Parameter Estimation

In this work, Maximum Likelihood Estimate is used to estimate the unknown parameter of K_wT_2GTL-G family for a complete data. Let $t_1, t_2 \ldots t_n$ be a random sample of size n from the K_wT_2GTL-G family. Then, the likelihood function based on observed sample for the vector Φ of parameter $(\alpha,\beta,\theta,\lambda,\delta)^T$ is given by;

$$LogL = n\log 2 + n\log\beta + n\log\alpha + n\log\theta + n\log\delta + n\log\lambda - \delta\sum_{i=1}^{\infty} t_i + (2\beta-1)\sum_{i=1}^{\infty}\log\left[1-e^{-\delta t}\right] + (\alpha-1)\sum_{i=1}^{\infty}\log\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right] + (\lambda-1)\sum_{i=1}^{\infty}\log\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right] + (\theta-1)\sum_{i=1}^{\infty}\log\left[1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right] \quad (37)$$

Differentiating the log-likelihood with respect to $\alpha,\beta,\theta,\delta,\lambda$ and equate the result to zero, we have

$$\frac{\partial(\log L)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{\infty}\log\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]-(\lambda-1)\sum_{i=1}^{\infty}\frac{\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\log\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]}{[1-[1-[1-e^{-\delta t}]^{2\beta}]^{\alpha}]}+$$
$$(\theta-1)\lambda\sum_{i=1}^{\infty}\frac{\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda-1}\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\log\left[1-[1-e^{-\delta t}]^{2\beta}\right]}{\left[1-\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]} \qquad (38)$$

$$\frac{\partial(\log L)}{\partial\beta}=\frac{n}{\beta}+2\sum_{i=1}^{\infty}\log\left[1-e^{-\delta t}\right]-$$
$$(\alpha-1)\sum_{i=1}^{\infty}\frac{\left[1-e^{-\delta t}\right]^{2\beta}\log\left[1-e^{-\delta t}\right]^{2}}{[1-[1-e^{-\delta t}]^{2\beta}]}+(\lambda-1)\alpha\sum_{i=1}^{\infty}\frac{\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}\left[1-e^{-\delta t}\right]^{2\beta}\log\left[1-e^{-\delta t}\right]^{2}}{[1-[1-[1-e^{-\delta t}]^{2\beta}]^{\alpha}]}+$$
$$(\theta-1)\alpha\lambda\sum_{i=0}^{\infty}\frac{\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda-1}\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha-1}[1-e^{-\delta t}]^{2\beta}\log[1-e^{-\delta t}]^{2}}{\left[1-\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]} \qquad (39)$$

$$\frac{\partial(\log L)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{\infty}\log\left[1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right] \qquad (40)$$

$$\frac{\partial(\log L)}{\partial\delta}=\frac{n}{\delta}-\sum_{i=1}^{\infty}t_i-(2\beta-1)\sum_{i=1}^{\infty}\left[\frac{e^{-\delta t}t_i}{[1-e^{-\delta t}]}\right]+(\alpha-1)\sum_{i=1}^{\infty}\left[\frac{2\beta\left[1-e^{-\delta t}\right]^{2\beta-1}t_ie^{-\delta t}}{[1-[1-e^{-\delta t}]^{2\beta}]}\right]-$$
$$(\lambda-1)\sum_{i=1}^{\infty}\left[\frac{\alpha\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}2\beta\left[1-e^{-\delta t}\right]^{2\beta-1}t_ie^{-\delta t}}{[1-[1-[1-e^{-\delta t}]^{2\beta}]^{\alpha}]}\right]+$$
$$2\beta\lambda\alpha(\theta-1)\sum_{i=1}^{\infty}\left[\frac{\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda-1}\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha-1}[1-e^{-\delta t}]^{2\beta-1}t_ie^{-\delta t}}{\left[1-\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]}\right] \qquad (41)$$

$$\frac{\partial(\log L)}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{\infty}\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}+(\theta-1)\sum_{i=1}^{\infty}\frac{\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda}\log\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]}{\left[1-\left[1-\left[1-[1-e^{-\delta t}]^{2\beta}\right]^{\alpha}\right]^{\lambda}\right]} \qquad (42)$$

Equating and solving these equations simultaneously yields the MLEs. Equation (38), (39), (40), (41) and equation (42) cannot be solved analytically, and analytical software is required to solve them numerically.

RESULTS AND DISCUSSION

Simulation Study

The quantile function of the KwT2GTLEx distribution was used to generate 1000 replicates with the sample size of n=20,50,100,250,500,1000 from the KwT2GTLEx distribution. From the 1000 replicates the estimates, bias and RMSE were computed, the result of which are presented in Table 1 showed the MLE parameter estimates, Bias and RMSE for the estimated parameters of KwT2GTLEx at the chosen values of $\delta=0.44$, $\alpha=1$, $\beta=0.52$, $\theta=0.8$, $\lambda=0.6$. The values of biases and RMSEs approach zero in the table, and the estimates tend to the true values as the sample size increases, indicating that the estimates are efficient and consistent;

Table 1: Monte Carlo simulation results for some values of parameters

N	Actual Parameter Value	Estimated Values	Bias	RMSE
20	$\delta=0.44$	0.5258	0.0858 0.016	0.2366 0.201
	$\alpha=1$	1.0162	2 0.0676 0.0	8 0.2018 0.2
	$\beta=0.52$	0.5876	266 0.0243	254 0.1315
	$\theta=0.8$	0.8266		
	$\lambda=0.6$	0.6243		
50	$\delta=0.44$	0.4715	0.0315 0.007	0.1391 0.151
	$\alpha=1$	1.0079	9 0.0218 0.0	3 0.1152 0.1
	$\beta=0.52$	0.5418	246 0.0095	933 0.0801
	$\theta=0.8$	0.8246		
	$\lambda=0.6$	0.6095		
100	$\delta=0.44$	0.4419	0.0019 0.012	0.0917 0.128
	$\alpha=1$	1.0125	5 0.0120 0.0	0 0.0777 0.1
	$\beta=0.52$	0.5320	313 0.0062	593 0.0603
	$\theta=0.8$	0.8313		
	$\lambda=0.6$	0.6062		
250	$\delta=0.44$	0.4355	-0.0045 0.0	0.0554 0.093
	$\alpha=1$	1.0217	217 0.0012	2 0.0438 0.1
	$\beta=0.52$	0.5212	0.0080 0.00	035 0.0407
	$\theta=0.8$	0.8080	54	

	$\lambda = 0.6$	0.6054		
500	$\delta = 0.44$	0.4307	-0.0093 0.0	0.0340 0.076
	$\alpha = 1$	1.0240	240 0.0007	7 0.0279 0.0
	$\beta = 0.52$	0.5207	0.0052 0.00	730 0.0322
	$\theta = 0.8$	0.8052	14	
	$\lambda = 0.6$	0.6014		
1000	$\delta = 0.44$	0.4311	-0.0089 0.0	0.0234 0.060
	$\alpha = 1$	1.0247	247 -0.0008	5 0.0193 0.0
	$\beta = 0.52$	0.5192	0.0013 0.00	503 0.0231
	$\theta = 0.8$	0.8013	11	
	$\lambda = 0.6$	0.6011		

Application to Real-Life Dataset
This section discusses real-life applications to data sets. For illustrative purposes, a comparison study with the fits of the Kumaraswamy Extension Exponential (K_wEEx) distribution by Elbatal, *et al.*, (2018).Kumaraswamy Exponential (K_wEx)distribution by Adepoju and Chukwu (2015), The Exponential Generalized Exponentiated Exponential (EGEEx) distribution by Bukoye and Oyeyemi, (2018), The Exponentiated Weibull-Exponential (EWEx) distribution by Elgarhy, *et al*. (2017). The versatility of the new distribution in empirically portraying real-life data was demonstrated in this application. All of the calculations are performed using the Adequacy Model package in R software.

The Comparators of K_wT_2GTLEx
The pdf of the comparators considered are:
i. The K_wEEx by Elbatal, *et al*., (2018) has probability density function given as:

$$f(x) = \delta\beta\alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}}\left[1-e^{1-(1+\lambda x)^{\alpha}}\right]^{\delta-1}\left[1-\left[1-e^{1-(1+\lambda x)^{\alpha}}\right]^{\delta}\right]^{\beta-1}$$

ii. The K_wEx by Adebayo and Chukwu (2015) has pdf defined as:

$$f(x) = \alpha\beta\lambda e^{-\beta x}\left[1-e^{-\beta x}\right]^{\alpha-1}\left[1-\left[1-e^{-\beta x}\right]^{\alpha}\right]^{\lambda-1}$$

iii. The EGEEx by Bukoye and Oyeyemi, (2018) has pdf defined as:

$$f(x) = \frac{\alpha\beta\lambda}{\theta}e^{-\frac{x}{\theta}}\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda-1}\left[1-\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda}\right]^{\alpha}\right]^{\beta-1}$$

And
iv. The EWEx by Elgarhy, *et al*. (2017). has pdf given as:

$$f(x) = \alpha\delta\beta\lambda\left[e^{\lambda x}-1\right]^{\beta-1}exp\left[-\left[\alpha\left(e^{\lambda x}-1\right)^{\beta}-\lambda x\right]\right]\left[1-exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{\alpha-1}$$

The two datasets that used as examples in the application demonstrate the new family of distributions' flexibility and 'best fit' compared to the above comparator distributions in modeling the data sets experimentally. The R programming language is used to carry out all of the computations.

Data Set 1
The first data set shown below represents the failure times of 84 Aircraft Windshield, previously used by Tahir *et al*., (2015): 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

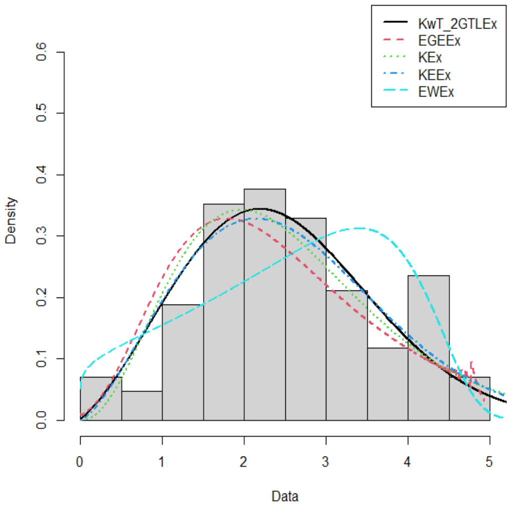

Figure 3: Fitted pdfs for the K_wT_2GTLEx, EGEEx, KEx, KEEx, and EWEx distributions to the data set 1

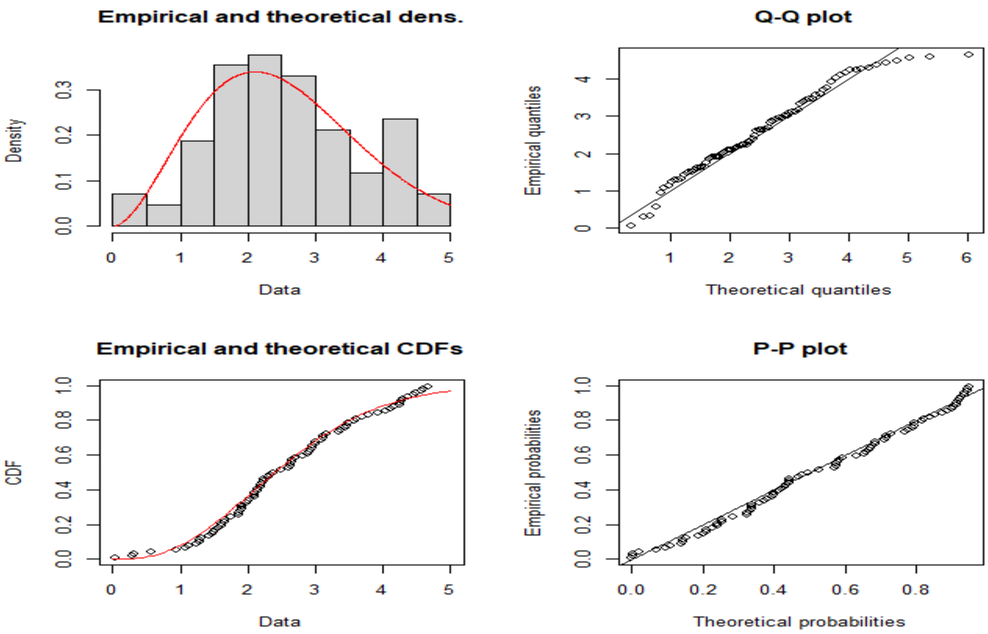

Figure 4: Estimated density plots for data set 1

Table 2: MLEs, Log-Likelihoods and Goodness of fits Statistics of the models based on the Data Sets 1

Distribution	β	α	θ	λ	δ	LL	AIC
K_wT_2GTLEx	0.0027	0.0617	4.3889	6.1127	6.4959	-129.4194	268.8387
EGEEx	5.2812	0.1066	0.1363	0.0728	-	-136.4105	280.8210
K_wEx	0.0216	2.4444	-	958.8999	-	-136.6001	279.2002
K_wEEx	6.4077	3.3169	-	0.0551	2.50991	-132.1957	272.3914
EWEx	0.8353	0.0216	-	1.3342	0.6922	-132.2505	272.5010

Table 2 presents the results of the MLE of the parameters of the proposed distribution and the four comparator distributions. The proposed distribution reported the lowest AIC value (268.8387) based on the goodness of fit measure; the visual inspection of the fit presented in figure 3 also confirms the superiority of the proposed distribution among its comparators; thus, the proposed distribution "best fit" the data sets among the range of distributions considered.

Data set 2

The second data set represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data set was given by Gross and Clark (1975). The data set consists of twenty (20) observations and it is as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

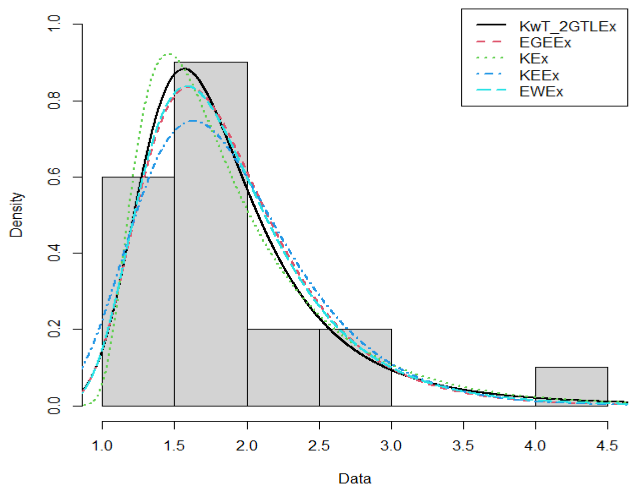

Figure 5: Fitted pdfs for the K_wT_2GTLEx, EGEEx, KEx, KEEx, and EWEx, distributions to the data set 2

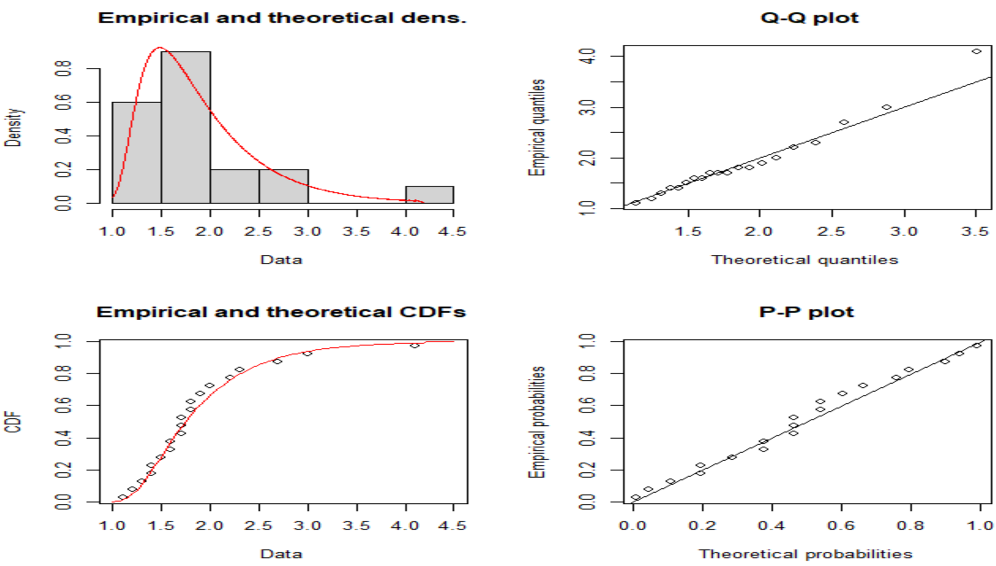

Figure 6: Estimated density plots for data set 2

Table 3: MLEs, Log-Likelihoods and Goodness of fits Statistics of the models based on the Data Sets 2

Distribution	β	α	θ	λ	δ	LL	AIC
K_wT_2GTLEx	38.4562	0.0565	6.0995	0.4160	1.9616	-15.0245	38.0489
EGEEx	13.4393	0.4599	0.2102	7.0755	-	-16.1561	40.3112
K_wEx	5.2384	918.4434	-	0.3021	-	-16.8354	39.6708
K_wEEx	1.0653	1.5201	-	0.7633	10.6729	-17.2540	42.5080
EWEx	0.7742	54.9415	-	0.0248	99.8325	-15.9689	39.9378

Table 3 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the four comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value (38.0489). The visual inspection of the fit presented in Figure 6, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution 'best fit' lifetime data relating to relief times (in minutes) of patients receiving an analgesic amongst the range of distributions considered.

CONCLUSION

In order to increase modeling flexibility for complex lifespan data sets, a novel family of continuous distributions known as the Kumaraswamy Type II Generalized Topp Leone-G (K_wT_2GTL-G) class has been introduced and examined in this paper. The K_wT_2GTL-G distribution integrates the cumulative distribution function of the Kumaraswamy-G family with a Type II Generalized Topp-Leone as a link function, resulting in a versatile model capable of representing both monotonic and non-monotonic hazard functions. The distribution of order statistics, quantile function, reliability function, hazard function, moments, moment generating function, probability weighted moments, and other important statistical features were all derived and thoroughly examined. A sub-model emerged known as K_wT_2GTLEx. Using the MLE method, the parameters of the K_wT_2GTLEx distribution were estimated using a package in *R* known as *AdequacyModel* and applied to real-life datasets; lifetime data relating to relief times (in minutes) of patients receiving an analgesic, and failure service times for a windshield and the results are presented in Table 1 and Table 2 respectively. Monte Carlo simulation was carried out to see the performance of MLEs of the K_wT_2GTLEx distribution and as expected, the MSEs of the estimated parameters decrease as the sample size *n* increases which proves the consistency of the estimators. The comparative assessments revealed that the K_wT_2GTL-G distribution outperformed existing distributions considered in this paper, beacuse it reported lower Akaike Information Criterion (AIC) values, indicating a better fit for the two datasets considered. The findings highlight the potential of the K_wT_2GTL-G family to enhance the modeling of lifetime data, contributing significantly to fields like medical and engineering. Future research should focus on exploring the application of the K_wT_2GTL-G family to a broader range of datasets and investigating the potential for further extensions or modifications to improve its flexibility and applicability in other domains.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous reviewers who helped to improve the work in its current state by offering enlightening critiques and recommendations.

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