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EFFECTS OF PUBLIC AWARENESS AND VACCINATION IN MODELING THE DYNAMICS OF MONKEY-POX TRANSMISSION DISEASE IN NIGERIA

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ABSTRACT

Monkey-pox disease is recognized as pathogens, disturbing animals and humans, it is among the family of orthopox virus and the disease causes lymph nodes to swell. In this paper, we developed a deterministic model system for Monkey pox infection by incorporating public awareness parameter and vaccination individual. The study verified the feasible region of the system equations and non-negativity of the solutions is achieved. The disease free and endemic equilibrium states have been obtained. The study computed and analyzed the reproduction number, R_e of the system equations. The study presented and analysed, the global stability of disease free equilibrium and endemic equilibrium state, it has been found that when there is no transmission between human and non-human (σ_{p_2}), then$\hat{G}(.X_1, X_2). = 0$ meaning it is globally asymptotically stable (GAS) at DFE and using nonlinear lyapunov function, the study shows that the endemic equilibrium state is GAS if $P < M$and unstable if $P > M$by comparison method of lyapunov functions. Numerical Simulations were done, it was found that the effective reproduction number decreases as vaccination of individual increases, varying the public awareness, the effective reproduction number reduces to zero and becomes stable as public awareness increases. It was discovered that effective reproduction number decreases as public awareness increases.

Keywords: Disease free equilibrium, Effective reproduction number, Endemic equilibrium, Mathematical Modeling, Monkey Pox Disease, Global Stability, Vaccination

INTRODUCTION

Pox disease is a family of orthopox viruses, which are monkey pox, small pox and cow pox diseases (Lasisi, Akinwande and Oguntolu, 2020; Bhunu and Mushayabase, 2011). Monkey pox cause infection in nonhumans and humans and causes lymph nodes to swell. The symptoms include fever, headache, muscle aches, and a feeling of discomfort. The infection spread to humans from an infected animal (rodents) through direct contact, animal bite and eating infected animal meats without proper done or cook. The disease also spread from infected person to person, less infectious than small pox virus, communicated through contact with body fluids of an infected individual, unclean and contaminated objects and sexual intercourse. The hazard factors for communication include sharing a room and bed, using the same tools as an infected person (Kantele, *et al.*, 2016).

Monkey-pox (MPX) epidemic was first discovered in 1958, which was later found evidence of monkeypox infection in a number of African rodents (Von Magnus, Andersen, Petersen, et al., 1959). In 1970, the virus was reported in humans (Alakunle, Moens, Nchinda *et al*., 2020; Jezek, Marennikova, Mutumbo, *et al.*, 1986). The incubation time is from 7–14days, the infection lasted for 2 to 4 weeks according to Centres for Disease Control (2003) and the fatality is 1% to 10% according to Rimoin, Kisalu, Kebela-Ilungam, *et al.* (2007). Monkeypox is endemic in Nigeria and in Congo Republic. The virus has been recounted in several countries in Africa, including Nigeria. At the start of the outbreak in 2017 to 2018, where 269 cases were suspected and 7 deaths were verified. In year 2018, there were 76 cases reported, 37 cases were confirmed and 2 deaths. In year 2022, Nigeria reported 558 suspected cases, where 231 were confirmed and No deaths were documented in 2022. Until year 2023, where the total confirmed cases is 988 from September 2017 to January 1, 2023 out of total suspected cases of 2635 (NCDC, 2023).

Deterministic models played a fundamental role in the disease transmission (Lasisi, *et al*., 2018; Lasisi, and Adeyemo, 2021; Lasisi and Suleiman, 2024). The objective of this paper is to the development and analyzed the stability of a modelling of transmission of monkey-pox virus in human and effects of public awareness and vaccination in Nigeria. Therefore, a mathematical modeling for monkey pox disease was developed by (Lasisi, *et al*., 2020; Bhunu and Mushayabase, 2011) with six (6) compartments. The study complements the work of the aforementioned author by having nine (9) compartments.

Model Formulation

The study formulates a model for the transmission of Monkey-pox infection in nonhuman and human population. The model is divided into epidemiological group see figure 1. The human populace is subdivided into five classes, namely; Susceptible class, S_h, Vaccinated class, V_h, Exposed asymptomatic class, E_h, Infected class, I_h, and Recovery class, R_h. The Nonhuman population model is subdivided into Susceptible class, S_p, Infected nonhuman class, I_p, Exposed asymptomatic class, E_p and Recovery class, R_p. As specified in the flowchart diagram in figure 1, individuals enter Susceptible class through birth and immigration ($\prod_h$), where a fraction of vaccinated human immigrants (f) enter class of vaccinated and proportion of unvaccinated immigrants (1-f) enter class of susceptible. The study does not consider the immigration of infection individual, because the study assumed that individuals coming from infection endemic zones have to be vaccinated. The susceptible persons vaccinated at the γ rate and loss the vaccination ω rate. Susceptible human, S_h are exposed to infection α_h rate and infected β_h rate, the natural death rate is μ_h and death due to the monkey pox disease is at rate of δ_h and recovery at the rate of ρ_h, $\emptyset$ is rate of public awareness. The Susceptible nonhuman class, S_p is produced from the daily recruitment of

persons through birth and immigration at the rate of Π_p, and natural death at μ_p rate. The nonhuman exposed to monkey-pox virus at the rate α_p, and move to infected class at the rate of β_p. Nonhuman infected death due to the infection at the rate of δ_p and recovery at the rate of ρ_p. Below is the flowchart representation of the model:

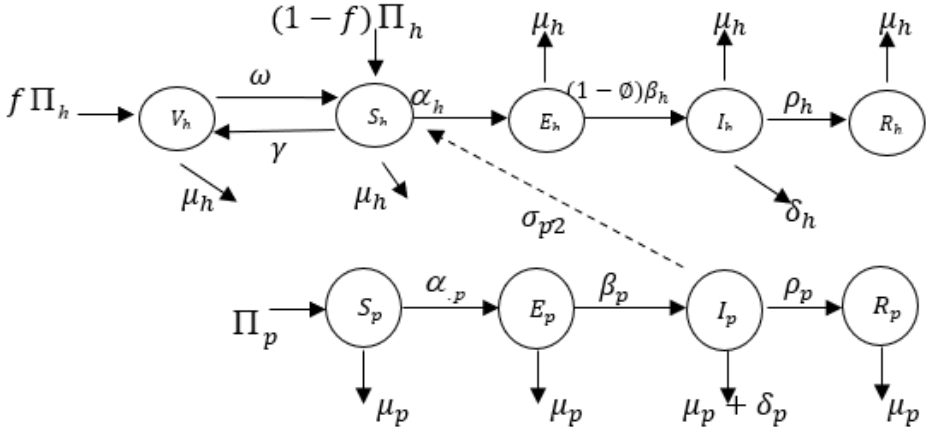


Figure 1: Flowchart representation of the Monkey-pox infection model

From the flow chart representation of the disease in figure 1 and assumptions, the dynamics of the monkey pox disease is described by ordinary differential equations below:

$$\frac{dS_h}{dt} = (1-f.)\Pi_h + \omega V_h - \gamma S_h - \alpha_h S_h - \mu_h S_h \quad (1)$$

$$\frac{dE_h}{dt} = \alpha_h S_h - (1-\varphi)\beta_h E_h - \mu_h E_h \quad (2)$$

$$\frac{dI_h}{dt} = (1-\varphi)\beta_h E_h - \rho_h I_h - \mu_h I_h - \delta_h I_h \quad (3)$$

$$\frac{dV_h}{dt} = f\Pi_h + \gamma S_h - \omega V_h - \mu_h V_h \quad (4)$$

$$\frac{dR_h}{dt} = \rho_h I_h - \mu_h R_h \quad (5)$$

$$\frac{dS_p}{dt} = .\Pi_p - \alpha_p . S_p - \mu_p . S_{p.} \quad (6)$$

$$\frac{dE_p}{dt} = \alpha_p S_p - \beta_p E_p - \mu_p E_p \quad (7)$$

$$\frac{dI_p}{dt} = \beta_p E_p - \rho_p I_p - \mu_p I_p - \delta_p I_p \quad (8)$$

$$\frac{dR_p}{dt} = \rho_p I_p - \mu_p R_p \quad (9)$$

Where,

$$N_h = S_h + E_h + I_h + V_h + R_h \quad (10)$$

$$N_p = S_p + E_p + I_p + R_p \quad (11)$$

$$\alpha_h = \frac{\sigma_{p_1}(\varepsilon_p E_p + I_p)}{N_p} + \frac{\sigma_h(\varepsilon_h E_h + I_k)}{N_h} \quad (12)$$

$$\alpha_P = \frac{\sigma_{p_2}(\varepsilon_p E_p + I_p)}{N_p} \quad (13)$$

S_h Becomes infected from both I_p and I_h, where σ_{p1} is effective contact product rate and probability of S_h becomes infected from I_p and σ_h is effective contact product rate and probability of S_h becomes infected from I_h. Correspondingly, the S_p becomes infected from infected nonhuman, where σ_{p2} *is* effective contact product rate and probability of nonhuman is becomes infected per contact with an infected I_p (Bhunu and Mushayabase, 2011). The adjustment parameter ε_h is the assumption that exposed human transmits at a rate lower than symptomatic humans. The adjustment parameter ε_p is for the assumption that exposed nonhuman transmits at a rate lower than symptomatic nonhuman and Monkey pox mortality is negligible due to human hunter

Analysis and Results of the Model

Theorem 1: The following biological feasible region of the equations (1.) - (9) $\Omega = \{S_h, E_{h.}, I_{h.}, V_{h.}, R_{h.}, S_{p.}, E_{p.}, I_{p.}, R_{p.}) \in \Re_+^9 : \{S_h + E_h + I_h + V_h + R_h \leq \frac{\Pi_h}{\mu_h}; S_p + E_p + I_p + R_p \leq \frac{\Pi_p}{\mu_p}\}$ is absolutely invariant and attracting.

Proof; The addition of all the model equations in (1) - (9) give

$$\frac{dN_h}{dt} = \Pi_h - \mu_h N_h - \delta_h I_h$$

And $$\frac{dN_p}{dt} = \Pi_p - \mu_p N_p - \delta_p I_p$$

So that $$\frac{dN_h}{dt} \leq \Pi_h - \mu_h N_h \text{ and } \frac{dN_p}{dt} \leq \Pi_p - \mu_p N_p \quad (14)$$

It follows from (Bauch and Earn, 2003), the gronwall inequality, that

$$N_h(t) \leq N_h(0)e^{-\mu_h(t)} + \frac{\Pi_h}{\mu_h}\{1 - e^{-\mu_h(t)}\}$$

And$$N_p(t) \leq N_p(0)e^{-\mu_p(t)} + \frac{\Pi_p}{\mu_p}\left\{1 - e^{-\mu_p(t)}\right\} \quad (15)$$

In specific, $N_h(t) \leq \frac{\Pi_h}{\mu_h}$ if $N_h(0) \leq \frac{\Pi_h}{\mu_h}$ if only $N_p(t) \leq \frac{\Pi_p}{\mu_p}$, also $N_p(0) \leq \frac{\Pi_p}{\mu_p}$. And Ω is positively invariant. Therefore, it is enough to consider the model equations dynamics (1) - (9) in Ω. In this region, the model system can be considered as been mathematically and epidemiologically well posed.

Theorem 2: (Non-negativity Solution of the Model system). Let $t_0 > 0$, the initial conditions satisfied $S_{.h}(0) > 0$, $E_{.h}(0) > 0$, $I_{.h}(0) > 0$, $V_h(0) > 0$, $R_{h.}(0) > 0$, $S_{p.}(0) > 0$, $E_{p.}(0) > 0$, $I_{p.}(0) > 0, R_{.p}(0) > 0$, then the solutions S_h, E_h, I_h, V_h, R_h, S_p, E_p, I_p, R_p of the model systems (1) - (9) are all nonnegative for $t. \geq 0$.

Proof:

To show that for all $t \in [0, t_0]$, S_h, E_h, I_h, V_h, R_h, S_p, E_p, I_p, R_p are nonnegative in $\Re_+^9$, note that the parameters used in the model are nonnegative. Thus, it is clear from model equation (1) that

$$\frac{d\,S_h}{dt} = (1-f)\Pi_h . + \omega V - \gamma S_h - \alpha_h S_h - \mu_h S_h \geq -(\gamma + \alpha_h + \mu_h)S_h$$

So that,

$S_h(t) \geq S_h(0)\mathrm{e}^{\{-\int(\gamma+\alpha_h+\mu_h)dt\}}$ (16)

Similarly, we can apply the approach to show that $E_h > 0$, $I_h > 0$, $V_h > 0$, $R_h > 0$, $S_p > 0$, $E_p > 0, I_p > 0, R_p > 0$. Hence, for all $t \in [0, t_0]$, S_h, E_h, I_h, V_h, R_h, S_p, E_p, I_p, R_p will be nonnegative and remain in $\Re_+^9$

The Equilibrium State

At equilibrium point, we setting the model equations to zero, we have

$\frac{dS_h}{dt.} = \frac{dE_h}{dt.} = \frac{dI_h}{dt.} = \frac{dV_h}{dt.} = \frac{dR_h}{dt} = \frac{dS_p}{dt} = \frac{dE_p}{dt} = \frac{dI_p}{dt} = \frac{dR_p}{dt} = 0.$

From (1)-(13), we have the following

$I_h = 0$ Or $\left(\frac{(\sigma_h\varepsilon_h(\rho_h+\mu_h+\delta_h)+\sigma_h\beta_h)S_h}{\beta_h N_h} - \frac{(\beta_h+\mu_h)(\rho_h+\mu_h+\delta_h)}{\beta_h}\right) = 0$ (17)

and

$I_p = 0$ or $\frac{(\sigma_{p2}\varepsilon_p(\rho_p+\mu_p+\delta_p)+\sigma_p\beta_p)S_p}{\beta_p N_p} - \frac{(\beta_p+\mu_p)(\rho_p+\mu_p+\delta_p)}{\beta_p} = 0$ (18)

Since $I_h = 0$, then we have $E_h = R_h = 0$ (19)

Finding V_h from the equations (4) and (1), we get

$V_h = \frac{f\Pi_h+\gamma S_h}{(\omega+\mu_h)} = \frac{\gamma S_h+\alpha_h S_h+\mu_h S_h-(1-f)\Pi_h}{\omega}$ (20)

Implies,

$S_h{}^0 = \frac{\Pi_h\omega+\Pi_h\mu_h-f\Pi_h\mu_h}{\gamma\mu_h+\mu_h\omega+\mu_h{}^2}$ (21)

As $\omega = \gamma = f = 0$ (no vaccination) then $S_h{}^0 = \frac{\Pi_h}{\mu_h}$

Making substitution of equation (25) into equation (4), gives

$V_h^0 = \frac{f\Pi_h\mu_h\omega+f\Pi_h\mu_h{}^2+\gamma\Pi_h\omega+\gamma\Pi_h\mu_h}{(\gamma\mu_h+\mu_h\omega+\mu_h{}^2)(\omega+\mu_h)}$ (22)

Since $I_p = 0$ then, we have $E_p = R_p = 0$

From (6), we have $S_p^0 = \frac{\Pi_p}{\mu_p}$

The DFE state is given by $E_0 = \{S_h{}^*, E_h{}^*, V_h{}^*, I_h{}^*, R_h{}^*, S_p{}^*, E_p{}^*, I_p{}^*, R_p{}^*\}$

$= \{\frac{\Pi_h\omega+\Pi_h\mu_h-f\Pi_h\mu_h}{\gamma\mu_h+\mu_h\omega+\mu_h{}^2}, 0, \frac{f\Pi_h\mu_h\omega+f\Pi_h\mu_h{}^2+\gamma\Pi_h\omega+\gamma\Pi_h\mu_h}{(\gamma\mu_h+\mu_h\omega+\mu_h{}^2)(\omega+\mu_h)}, 0, 0, \frac{\Pi_p}{\mu_p}, 0, 0, 0\}$ (23)

Effective Reproduction Number (R_e)

We compute the effective basic reproduction number according to (Van den Driessche and Watmough, 2002), using next generation matrix. Therefore, effective basic reproduction number is the spectral radius of FV^{-1}

$FV^{-1} = \left[\frac{\partial F_i(E^0)}{\partial x_i}\right]\left[\frac{\partial V_i(E^0)}{\partial x_i}\right]^{-1}$ (24)

Where, E^0 is the disease free equilibrium F_i is the new infection in compartment i and V_i is the movement of infection from one compartment i to another, so, $R_0 = \rho(FV^{-1})$ is spectral radius (ρ) of the next generation matrix FV^{-1} , the linearization of system (1)-(9) give F *and* V, obtained from the Jacobian matrix with the disease free equilibrium. Vector F *is the* inflow *and* V is the outflow from compartments E_h, E_p, I_h and I_P. We get

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \left(\frac{\sigma_{p1}(\varepsilon_p E_p+I_p)}{N_p} + \frac{\sigma_h(\varepsilon_h E_h+I_h)}{N_h}\right)S_h \\ \beta_h E_h \\ \frac{\sigma_{p2}(\varepsilon_p E_p+I_p)S_P}{N_p} \\ \beta_p E_p \end{bmatrix} \quad (25)$$

$$F = \begin{bmatrix} \frac{\sigma_h.\varepsilon_h S_h^0}{N_h^0} & \frac{\sigma_h S_h^0}{N_h^0} & \frac{\sigma_{p_1}\varepsilon_p S_h^0}{N_p^0} & \frac{\sigma_{p_1} S_h^0}{N_p^0} \\ \beta_h & 0 & 0 & 0 \\ 0 & 0 & \frac{.\sigma_{p_2}\varepsilon_p S_p^0.}{N_p^0} & \frac{.\sigma_{p_2} S_p^0.}{N_p^0} \\ 0 & 0 & .\beta_p & 0 \end{bmatrix} \quad (26)$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} Q_1 E_h \\ Q_2 I_h \\ Q_3 E_p \\ Q_4 I_p \end{bmatrix} \quad (27)$$

Where, $Q_1 = \beta_h + \mu_h$; $Q_2 = \rho_h + \mu_h + \delta_h$; $Q_3 = \beta_p + \mu_p$; $Q_4 = \rho_p + \mu_p + \delta_p$

$$V = \begin{pmatrix} Q_{1.} & 0 & 0 & 0 \\ 0 & Q_{2.} & 0 & 0 \\ 0 & 0 & Q_{3,} & 0 \\ 0 & 0 & 0 & Q_{4,} \end{pmatrix} \quad (28)$$

From (28), we have

$$V^{-1} = \begin{bmatrix} \frac{1}{Q_1} & 0 & 0 & 0 \\ 0 & \frac{1}{Q_2} & 0 & 0 \\ 0 & 0 & \frac{1}{Q_3} & 0 \\ 0 & 0 & 0 & \frac{1}{Q_4} \end{bmatrix} \quad (29)$$

At disease free equilibrium point, and since $N_h \leq \frac{\Pi h}{\mu_h.}$ and $N_p \leq \frac{\Pi p.}{\mu_p.}$ we get

$$F = \begin{bmatrix} \frac{.\sigma_h\varepsilon_h\mu_h S_h^0}{\Pi_h} & \frac{\sigma_h\mu_h S_h^0}{\Pi_h.} & \frac{\sigma_{p_1}\varepsilon_p\mu_p S_h^0}{\Pi_p} & \frac{\sigma_{p_1}\mu_p S_h^0}{\Pi_p.} \\ \beta_h. & 0 & 0 & 0 \\ 0 & 0 & \sigma_{p_2}\varepsilon_p. & \sigma_{p_2.} \\ 0 & 0 & \beta_p. & 0 \end{bmatrix} \quad (30)$$

$$FV^{-1} = \left[\frac{\partial F_i(E^0)}{\partial x_j}\right]\left[\frac{\partial V_i(E^0)}{\partial x_j}\right]^{-1} \quad (31)$$

Multiplying (29) and (30), we have

$$FV^{-1} = \begin{bmatrix} \frac{\sigma_h\varepsilon_h\mu_h S_h^0}{\Pi_h Q_1} & \frac{\sigma_h\mu_h S_h^0}{\Pi_h Q_2} & \frac{\sigma_{p_1}\varepsilon_p\mu_p S_h^0}{\Pi_p Q_3} & \frac{\sigma_{p_1}\mu_p S_h^0}{\Pi_p Q_4} \\ \frac{\beta_h}{Q_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_{p_2}\varepsilon_p.}{Q_3} & \frac{\sigma_{p_2.}}{Q_4} \\ 0 & 0 & \frac{\beta_p}{Q_3.} & 0 \end{bmatrix} \quad (32)$$

Characteristics Equation of (32), gives $/FV^{-1} - \lambda I/ = 0$

$$\begin{bmatrix} L_1 S_h^0 - \lambda & L_2 S_h^0 & L_3 S_h^0 & L_4 S_h^0 \\ L_5 & -\lambda & 0 & 0 \\ 0 & 0 & L_6 - \lambda & L_7 \\ 0 & 0 & L_8 & -\lambda \end{bmatrix} = 0 \quad (33)$$

Where, $L_1 = \frac{\sigma_h\varepsilon_h\mu_h S_h^0}{\Pi_h Q_1}, L_2 = \frac{\sigma_h\mu_h}{\Pi_h Q_2}, L_3 = \frac{\sigma_{p_1}\varepsilon_p\mu_p}{\Pi_p Q_3}, L_4 = \frac{\sigma_{p_1}\mu_p}{\Pi_p Q_4}$

$L_5 = \frac{\beta_h}{Q_1}, L_6 = \frac{\sigma_{p_2}\varepsilon_p}{Q_3}, L_7 = \frac{\sigma_{p_2}}{Q_4}, L_8 = \frac{\beta_p}{Q_3}$ (34)

Determinant of (33) gives

$(\lambda^2 - L_6\lambda - L_7 L_8) = 0 \text{or} (\lambda^2 - L_1 S_h^0\lambda - L_2 L_5 S_h^0) = 0$ (35)

To solve (35) with completing the square method, we have

$\lambda_1 = \frac{\frac{\sigma_{p_2}\varepsilon_p.}{(\beta_p.+\mu_p.)} \pm \sqrt{\frac{(\sigma_{p_2}\varepsilon_p.)^2}{(\beta_p.+\mu_p.)^2} + \frac{4\sigma_{p_2}\beta_p.}{(\beta_p.+\mu_p.)(\rho_p+\mu_p.+\delta_p)}}}{2}$ (36)

λ_1 is the Spectral Radius of $\rho(FV^{-1})$ and The reproduction number is given below

$R_p = \frac{\frac{\sigma_{p_2}\varepsilon_p}{(\beta_p+\mu_p)} + \sqrt{\frac{(\sigma_{p_2}\varepsilon_p)^2}{(\beta_p+\mu_p)^2} + \frac{4\sigma_{p_2}\beta_p}{(\beta_p+\mu_p)(\rho_p+\mu_p+\delta_p)}}}{2}$ (37)

And

$(\lambda^2 - L_1 S_h^0\lambda - L_2 L_5 S_h^0) = 0$

Implies, $\lambda_2 = \frac{L_1 S_h^0 \pm \sqrt{L_1^2 S_h^{0^2} + 4L_2 L_5 S_h^0}}{2}$ (38)

$\lambda_2 = \frac{\frac{\sigma_h\varepsilon_h\mu_h S_h^0}{\Pi_h(\beta_h.+\mu_h.)} \pm \sqrt{\frac{\sigma_h.^2\varepsilon_h^2\mu_h^2 S_h^{0^2}}{\Pi_h^2.(\beta_h+\mu_h.)^2} + \frac{4\sigma_h.\mu_h\beta_h S_h^0}{\Pi_h.(\beta_h+\mu_h)(\rho_h.+\mu_h+\delta_h)}}}{2}$ (39)

We have λ_2 as spectral radius (ρ) that is $R_e = \rho(FV^{-1})$.

$$R_h = \frac{\frac{\sigma_h.\varepsilon_h.\mu_h.S_h^0}{\Pi_h.(\beta_h.+\mu_h.)} + \sqrt{\frac{\sigma_h.^2\varepsilon_h.^2\mu_h.^2S_h^{0^2}}{\Pi_h^2.(\beta_h.+\mu_h.)^2} + \frac{4\sigma_h.\mu_h.\beta_h.S_h^0}{\Pi_h.(\beta_h.+\mu_h.)(\rho_h.+\mu_h.+\delta_h)}}}{2} \quad (40)$$

Hence, the effective reproduction number can be represented as,

$$R_e. = R_h. + R_p. \quad (41)$$

$$R_0 = \frac{\left(\frac{\sigma_h.\varepsilon_h.\mu_h.S_h^0}{\Pi_h.(\beta_h.+\mu_h.)} + \frac{\sigma_{p2.}\varepsilon_p.}{(\beta_p.+\mu_p.)} + \sqrt{\frac{\sigma_h.^2\varepsilon_h.^2\mu_h.^2S_h^{0^2}}{\Pi_h^2.(\beta_h.+\mu_h.)^2} + \frac{4\sigma_h.\mu_h.\beta_h.S_h^0}{\Pi_h.(\beta_h.+\mu_h.)(\rho_h.+\mu_h.+\delta_h.)}} + \sqrt{\frac{(\sigma_{p_2.}\varepsilon_p.)^2}{(\beta_p.+\mu_p)^2} + \frac{4\sigma_{p_2}\beta_p}{(\beta_p.+\mu_p.)(\rho_p.+\mu_p.+\delta_p.)}}\right)}{2} \quad (42)$$

Global Stability of DFE State

Theorem 3: The DFE of the model is globally asymptotically stable (GAS) if $R_e < 1$

Proof:

$$\frac{dX_1}{dt.} = F(.X_1, X_2.) \quad (43)$$

$$\frac{dX_2.}{dt.} = .G(X_1., X_2.); G(X_1., 0) = 0. \quad (44)$$

$X_1 = (S_h^0, V_h^0, R_h^0, S_p^0, R_p^0)$, AND

$$X_2 = (E_h^0, I_h^0, E_p^0, I_p^0) \quad (45)$$

The DFE is now represented as,

$$.E^0 = (X_{1.}^*, 0.) \text{where, } X_{1.}^* = (N^0, 0.) \quad (46)$$

Firstly, the condition that is GAS of X_1^*, gives

$$\frac{dX_1}{dt} = F(X_1, 0) = \begin{Bmatrix} (1-f)\Pi_h + \omega V_h^0 - \gamma S_h^0 - (0)S_h^0 - \mu_h S_h^0 \\ f\Pi_h + \gamma S_h^0 - \omega V_h^0 - \mu_h V_h^0 \\ -\mu_h R_h^0 \\ \Pi_p - (0)S_p^0 - \mu_p S_p^0 \\ \rho_p I_p^0 - \mu_p R_p^0 \end{Bmatrix} \quad (47)$$

A linear differential equation solving gives,

$$S_h^0(t) = \frac{(1-f)\Pi_h+\omega V_h^0}{(\gamma+\mu_h.)} - \frac{(1-f)\Pi_h+\omega V_h^0}{(\gamma+\mu_h)} * e^{-(\gamma+\mu_h)t.} + S_h^0(0) * e^{-(\gamma+\mu_h)t.} \quad (48)$$

$$V_h^0(t) = \frac{f\Pi_h+\gamma S_h^0}{(\omega+\mu_h.)} - \frac{f\Pi_h+\gamma S_h^0}{(\omega+\mu_h.)} * e^{-(\omega+\mu_h.)t} + V_h^0(0) * e^{-(\omega+\mu_h)t.} \quad (49)$$

$$R_h^0(t.) = R_h^0(0) * e^{-\mu_h t.} \quad (50)$$

$$S_p^0(t) = \frac{\Pi_P}{\mu_P} - \frac{\Pi_P}{\mu_P} e^{-\mu_P t} + s_P^0(0) e^{-\mu_P t} \quad (51)$$

$$R_p^0(t) = R_P^0(0) e^{-\mu_P t} \quad (52)$$

This showed that $S_h^0 + V_h^0 + R_h^0 + S_p^0 + R_p^0 -> N_0^{\square}$ as $t->\infty$ regardless of the value of $S_h^0, V_h^0, R_h^0, S_p^0 and R_p^0$. Therefore, $X_{1.}^* = (.N^0, 0.)$. is globally asymptotically stable.

To show if the second condition is true: $\hat{G}(.X_1, X_2) = AX_2 - G(.X_1, X_2)$

Where $X_2 = (E_h^0, I_h^0, E_p^0, I_p^0)$

$$A = \begin{bmatrix} (\frac{\sigma_h\varepsilon_h S_h^0}{N_h^0} - (.\beta_h + \mu_h.) & \frac{\sigma_h S_h^0}{N_h^0} & \frac{\sigma_p.\varepsilon_p.S_h^0}{.N_p^0} & \frac{\sigma_p.S_h^0}{.N_p^0} \\ .\beta_h & -(\rho_h. + \mu_h + \delta_h.) & 0 & 0 \\ 0. & 0 & (\frac{\sigma_p.\varepsilon_p.S_p^0}{.N_p^0}) - (\beta_p. + \mu_p.) & (\frac{\sigma_p.S_p^0}{.N_p^0} \\ 0 & 0 & \beta_p & -(.\rho_p + \mu_p. + \delta_p.) \end{bmatrix} \quad (53)$$

$$G(X_1, X_2) = \begin{bmatrix} (\frac{\sigma_p.(\varepsilon_p E_p^0 + I_p^0)}{N_p^0} + \frac{\sigma_h(\varepsilon_h E_h^0 + I_h^0)}{N_h^0}) S_h^0 - (\beta_h + \mu_h) E_h^0 \\ -(\beta_h + \rho_h + \mu_h + \delta_h) I_h^0 \\ (\frac{\sigma_p(\varepsilon_p E_p^0 + I_p^0)}{N_p^0}) S_p^0 - (\beta_p + \mu_p) E_p^0 \\ \beta_p E_p^0 - (\rho_p + \mu_p + \delta_p) I_p^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (54)$$

From, $.\hat{G}(.X_{1.}, X_{2.}) = A*X_{2.} - .G(X_{1.}, X_{2.})$

$$\begin{bmatrix} (\frac{\sigma_h\varepsilon_h S_h^0}{N_h^0} - (.\beta_h + \mu_h.) & \frac{\sigma_h^* S_h^0}{N_h^0} & \frac{\sigma_p\varepsilon_p S_h^0}{N_p^0} & \frac{\sigma_p.S_h^0}{N_p^0} \\ \beta_h & -(\rho_h. + \mu_h. + \delta_h.) & 0 & 0 \\ 0 & 0 & (\frac{\sigma_p^*\varepsilon_p^* S_p^0}{.N_p^0}) - (.\beta_p + \mu_p.) & (\frac{\sigma_p.S_p^0}{N_p^0} \\ 0 & 0 & \beta_p & -(.\rho_p + \mu_{p.} + \delta_{p.}) \end{bmatrix} \begin{bmatrix} E_h^0 \\ I_h^0 \\ E_p^0 \\ I_p^0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (55)$$

$$\hat{G}(.X_1, X_2.) = [.0 \quad 0 \quad 0 \quad 0.]^{T.}$$

It is clear that, $.\hat{G}(X_1, X_2). = 0$. Therefore, the proof is complete. It implies, disease free equilibrium of the model system is GAS if Re < 1.

Endemic Equilibrium (EE) State

At equilibrium state, we set (1) - (9) to zero, we have

$$\frac{dS_h}{dt.} = \frac{dE_h}{dt.} = \frac{dI_h}{dt.} = \frac{dV}{dt.} = \frac{dR_h}{dt.} = \frac{dS_p}{dt.} = \frac{dE_p}{dt.} = \frac{dI_p}{dt.} = \frac{dR_p}{dt.} = 0 \quad (56)$$

$$(1-f)\Pi_h + \omega V_h^* - \gamma S_h^* - (\frac{\sigma_{p_1.}(\varepsilon_p.{}^* E_{p..}^* + I_{p.}^*)}{N_p} + \frac{\sigma_{h..}(\varepsilon_{h..} E_{h..}^* + I_{h..}^*)}{N_h}) S_{h.}^* - \mu_h {}^* S_{h.}^*$$

$$= 0(\frac{\sigma_{p_1.}(\varepsilon_p. E_{p.}^* + I_{p.}^*)}{.N_p} + \frac{\sigma_{h.}(\varepsilon_{h.} E_{h.}^* + I_{h.}^*)}{.N_h}) S_{h.}^* - \beta_h E_{h.}^* - \mu_{h.} E_{h.}^* = 0.$$

$$\beta_h E_h^* - \rho_h I_h^* - \mu_h I_h^* - \delta_h I_h^* = 0$$

$$f\Pi_h + \gamma S_h^* - \omega V_h^* - \mu_h V_h^* = 0$$

$$\rho_h I_h^* - \mu_h R_h^* = 0 \quad (57)$$

$$\Pi_{p.} - (\frac{\sigma_{p_2.}(\varepsilon_p.E^*_{p.} + I^*_{p.})}{N_p})S^*_{p.} - \mu_p.S^*_{p.} = 0$$

$$(\frac{\sigma_{p_2.}(\varepsilon_p.E^*_{p.}+I^*_{p.})}{N_p})*S^*_{p.} - .\beta_{p.}*E^*_{p.} - \mu_{p.}*E^*_{p.} = 0$$

$$\beta_p*E^*_p - \rho_p*I^*_p - \mu_p*I^*_p - \delta_p*I^*_p = 0$$

$$\rho_p.I^*_{p.} - \mu_p.R^*_{p.} = 0.$$

Where,

$$A_1 = \beta_{h.} + \mu_{h.}, A_{2.} = \rho_{h.} + \mu_{h.} + \delta_{h.}, A_{3.} = \omega + \mu_{h.}, A_{4.} = \beta_{p.} + \mu_{p.}, A_{5.} = \rho_{p.} + \mu_{p.} + \delta_{p.} \quad (58)$$

From (57) and (58), we have the following:

$$E^*_{h.} = \frac{(\rho_{h.}+\mu_{h.}+\delta_{h.})*I^*_h}{\beta_h} = \frac{A_2*I^*_h}{\beta_{h.}} \quad (59)$$

$$V^*_h = \frac{f.\Pi_h+\gamma.S^*_h}{(\omega^*_{h.}+\mu_{h.})} = \frac{f.\Pi_{h.}+\gamma.S^*_{h.}}{A_{3.}} \quad (60)$$

$$.R^*_h = \frac{\rho_{h.}.I^*_{h.}}{\mu_{h.}} \quad (61)$$

$$E^*_{p.} = \frac{(\rho_{p.}+\mu_{p.}+\delta_{p.})*I^*_p}{\beta_{p.}} = \frac{A_5I^*_{p.}}{\beta_{p.}} \quad (62)$$

$$R^*_{p.} = \frac{\rho_{p.}.I^*_{p.}}{\mu_{p.}} \quad (63)$$

Substitute (62) into exposed class in (57), we get:

$$(\frac{\sigma_{p.}.(\frac{\varepsilon_p.A_5.I^*_p}{\beta_{p.}}+I^*_{p.})}{N_p}).S^*_{p.} = \frac{A_4.A_5.I^*_p}{\beta_{p.}} \quad (64)$$

From (64) we have:

$$.S^*_{p.} = \frac{\Pi_p.A_4.A_{5.}}{\mu_{p.}(\sigma_p.\varepsilon_p.A_{5.}+\sigma_p.\beta_{p.})} \quad (65)$$

From (57) we have the following:

$$I^*_{p.} = \frac{\{.\beta_p\Pi_p.(\sigma_p\varepsilon_p.A_5+\sigma_p\beta_{p.})-\beta_p.\Pi_p.A_4.A_5\}}{(\sigma_p.\varepsilon_pA_{5.}+\sigma_p\beta_p).A.A_5} \quad (66)$$

From(62) we get:

$$E^*_p = \frac{A_5.\{\beta_p.\Pi_p.(\sigma_p\varepsilon_pA_{5.}+\sigma_p\beta_{p.})-\beta_p.\Pi_p.A_4.A_5\}}{\beta_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p).A_4.A_5} \quad (67)$$

$$E^*_p = \frac{\{\Pi_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)-\Pi_p.A_4.A_5\}}{(\sigma_p.\varepsilon_p.B_5+\sigma_p.\beta_p)A_4} \quad (68)$$

Eq. (63) yield:

$$R^*_p = \frac{\rho_pI^*_p}{\mu_p} = \frac{\rho_p.\{\beta_p.\Pi_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)-\beta_p.\Pi_p.A_4.A_5\}}{\mu_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p).A_4.A_5} \quad (69)$$

From (69) we get:

$$R^*_p = \frac{\{\rho_p.\beta_p.\Pi_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_{p.})-\rho_p.\beta_p.\Pi_p.A_4.A_5\}}{\mu_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p).A_4.A_5} \quad (70)$$

From (57) we have:

$$(\frac{\sigma_p(\varepsilon_p.E^*_p+I^*_{p.})}{N_p} + \frac{\sigma_h.(\varepsilon_{h.}E^*_{h.}+I^*_{h.})}{N_h}).S^*_h = (.\beta_h + \mu_{h.}).E^*_h = A_{1.}*\frac{A_2.I^*_{h.}}{\beta_{h.}} \quad (71)$$

Eq. (71) yield:

$$I^*_h = \{\frac{\beta_h.\beta_h.\Pi_h.\mu_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_{p.}).I^*_p.S^*_h}{\beta_p.\Pi_p.[\beta_h.\Pi_h.A_1.A_2-\mu_h.\beta_h.(\sigma_h.\varepsilon_h.A_2+\sigma_h.\beta_h).S^*_h]}\} \quad (72)$$

From (57) we get:

$$(1-f).\Pi_h + \omega.(\frac{f.\Pi_{h.}+\gamma.S^*_h}{B_3}) =$$

$$[\gamma + \frac{\frac{\{\beta_p.\Pi_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)-\beta_p.\Pi_p.A_4.A_5\}}{(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p).A_4.A_5}(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)}{\beta_pN_p} + \frac{I^*_h.(\sigma_h.\varepsilon_h.A_2+\sigma_h.\beta_h)}{\beta_h.N_h} + \mu_h].S^*_h \quad (73) S^*_h = \frac{[(1-f).\Pi_h.A_3+\omega.f.\Pi_h]}{[\gamma.B_3+A_L.A_3+I^*_h.A_k.A_3+\mu_h.A_3-\omega.\gamma]} \quad (74)$$

Substitute (74) into (72) we have:

$$I^*_h = \{\frac{\beta_h.\beta_h.\Pi_h.\mu_p.(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p).I^*_p.S^*_h}{\beta_p.\Pi_p.[\beta_h.\Pi_h.A_1.A_2-\mu_h.\beta_h.(\sigma_h.\varepsilon_h.A_2+\sigma_h.\beta_h).S^*_h]}\} \quad (75)$$

Eq. (75) implies:

$$I^*_h = \frac{\beta_h.\Pi_h.\mu_p.\{(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)-A_4.A_5\}}{A_4.A_5.[\Pi_h.A_1.A_2-\mu_h.(\sigma_h.\varepsilon_h.A_2+\sigma_h.\beta_h)]} \quad (76)$$

From (59) we have:

$$E^*_h = \frac{A_2.I^*_h}{\beta_h} = \frac{A_2.\beta_h.\Pi_h.\mu_p\{(\sigma_p.\varepsilon_p.A_5+\sigma_p.\beta_p)-A_4.A_5\}}{\beta_h.A_4.A_5[\Pi_h.A_1.A_2-\mu_h(\sigma_h.\varepsilon_h.A_2+\sigma_h.\beta_h)]} \quad (77)$$

Implies:

$$E^*_h = \frac{A_2\Pi_h\mu_p\{(\sigma_p\varepsilon_pA_5+\sigma_p\beta_p)-A_4A_5\}}{A_4A_5[\Pi_hA_1A_2-\mu_h(\sigma_h\varepsilon_hA_2+\sigma_h\beta_h)]} \quad (78)$$

Substitute (76) into (60) we have:

$$S_h^* = \frac{[(1-f)\Pi_h A_3 + \omega f \Pi_h]}{[\gamma A_3 + A_L A_3 + I_h^* A_k A_3 + \mu_h A_3 - \omega\gamma]} \quad (79)$$

Implies:

$$S_h^* = \frac{[(1-f).\Pi_h.A_3 + \omega.f.\Pi_h]}{[\gamma A_3 + A_L A_3 + \frac{\beta_h.\Pi_h.\mu_p.A_k.A_3\{(\sigma_p.\varepsilon_p.A_5 + \sigma_p.\beta_p) - A_4.A_5\}}{A_4.A_5.[\Pi_h.A_1.A_2 - \mu_h.(\sigma_h.\varepsilon_h.A_2 + \sigma_h.\beta_h)]} + \mu_h.A_3 - \omega.\gamma]} \quad (80)$$

From (4) we have:

$$V_h^* = \frac{f.\Pi_h + \gamma.S_h^*}{(\omega_h^* + \mu_h)} = \frac{f.\Pi_h}{A_3} + \frac{\gamma.S_h^*}{A_3} \quad (81)$$

Implies:

$$V_h^* = \frac{f\Pi_h}{A_3} + \frac{\gamma[(1-f)\Pi_h A_3 + \omega f \Pi_h]}{A_3[\gamma A_3 + A_L A_3 + \frac{\beta_h \Pi_h \mu_p A_k A_3\{(\sigma_p \varepsilon_p A_5 + \sigma_p \beta_p) - A_4 A_5\}}{A_4 A_5[\Pi_h A_1 A_2 - \mu_h(\sigma_h \varepsilon_h A_2 + \sigma_h \beta_h)]} + \mu_h A_3 - \omega\gamma]} \quad (82)$$

Eq. (61) yield:

$$R_h^* = \frac{\rho_h.I_h^*}{\mu_h} = \frac{\rho_h.\beta_h.\Pi_h.\mu_p\{(\sigma_p.\varepsilon_p.A_5 + \sigma_p.\beta_p) - A_4.A_5\}}{\mu_h.A_4.A_5[\Pi_h.A_1.A_2 - \mu_h(\sigma_h.\varepsilon_h.A_2 + \sigma_h.\beta_h)]} \quad (83)$$

Implies:

$$R_h^* = \frac{\rho_h.\beta_h.\Pi_h.\mu_p\{(\sigma_p.\varepsilon_p.A_5 + \sigma_p.\beta_p) - A_4.A_5\}}{\mu_h.A_4.A_5[\Pi_h.A_1.A_2 - \mu_h.(\sigma_h.\varepsilon_h.A_2 + \sigma_h.\beta_h)]} \quad (84)$$

$$E_1 = \{S_h^* > 0,\ E_h^* > 0,\ V_h^* > 0,\ I_h^* > 0,\ R_h^* > 0, S_p^* > 0, E_p^* > 0, I_p^* > 0, R_p^* > 0\} \quad (85)$$

Global Stability of EE State

This study used Lyapunov function according to Lasisi, (2021); Lasisi, (2020); Lasisi *et. al.*,(2018) to examine the stability of the EE state, Lyapunov function $G(E_1)$ is defined by:

$$G = (S_h - S_h^*.\ln S_h) + (E_h - E_h^*.\ln E_h) + (I_h - I_h^*.\ln I_h) + (V_h - V_h^*.\ln V_h) + (R_h - R_h^*\ln R_h)$$
$$(S_p - S_p^*.\ln S_p) + (E_p - E_p^*.\ln E_p) + (I_p - I_p^*.\ln I_p) + (R_p - R_p^*.\ln R_p) \quad (86)$$

The derivation of (86) is setting as:

$$\frac{dG}{dt} = (\frac{S_h - S_h^*}{S_h})\frac{dS_h}{dt} + (\frac{E_h - E_h^*}{E_h})\frac{dE_h}{dt} + (\frac{I_h - I_h^*}{I_h})\frac{dI_h}{dt} + (\frac{V_h - V_h^*}{V_h})\frac{dV_h}{dt} + (\frac{R_h - R_h^*}{R_h})\frac{dR_h}{dt}$$
$$+(\frac{S_p - S_p^*}{S_p})\frac{dS_p}{dt} + (\frac{E_p - E_p^*}{E_p})\frac{dE_p}{dt} + (\frac{I_p - I_p^*}{I_p})\frac{dI_p}{dt} + (\frac{R_p - R_p^*}{R_p})\frac{dR_p}{dt} \quad (87)$$

Eq. (87) yield:

$$\frac{dG}{dt} = (\frac{S_h - S_h^*}{S_h})\{(1-f)\Pi_h + \omega V_h - (\frac{\sigma_p(\varepsilon_p E_p + I_p)}{N_p} + \frac{\sigma_h(\varepsilon_h E_h + I_k)}{N_h})S_h - A_h S_h\} + (\frac{E_h - E_h^*}{E_h})\{(\frac{\sigma_p(\varepsilon_p E_p + I_p)}{N_p} + \frac{\sigma_h(\varepsilon_h E_h + I_k)}{N_h})S_h - B_1 E_h\}$$
$$+(\frac{I_h - I_h^*}{I_h})\{\beta_h E_h - B_2 I_h\} + (\frac{V_h - V_h^*}{V_h})\{f\Pi_h + \gamma S_h - B_3 V_h\}$$
$$+(\frac{R_h - R_h^*}{R_h})\{\rho_h I_h - \mu_h R_h\} + (\frac{S_p - S_p^*}{S_p}).\{\Pi_p - (\frac{\sigma_p.(\varepsilon_p.E_p + I_p)}{N_p}).S_p - \mu_p.S_p\} + (\frac{E_p - E_p^*}{E_p}).\{(\frac{\sigma_p.(\varepsilon_p.E_p + I_p.)}{N_p}).S_p - B_4.E_p\} +$$
$$(\frac{I_p - I_p^*}{I_p})\{\beta_p.E_p - B_5.I_p\}$$
$$+(\frac{R_p - R_p^*}{R_p})\{\rho_p.I_p - \mu_p.R_p\} \quad (88)$$

At the endemic equilibrium points, we have:

$$\frac{dG}{dt} = (\frac{S_h - S_h^*}{S_h}).\{[\frac{\omega.V_h}{(1-f)} + (\alpha_p + \alpha_h).\frac{S_h^*}{(1-f)} + \frac{A_h.S_h^*}{(1-f)}] + \omega.V_h - (\alpha_p + \alpha_h).S_h - A_h.S_h\} + (\frac{E_h - E_h^*}{E_h}).\{(\alpha_p + \alpha_h).S_h -$$
$$\frac{(\alpha_p + \alpha_h).S_h^*}{E_h^*}.E_h\}$$
$$+(\frac{I_h - I_h^*}{I_h}).\{\beta_h.E_h - B_2.I_h\} + (\frac{V_h - V_h^*}{V_h}).\{f.\Pi_h + \gamma.S_h - B_3.V_h\}$$
$$+(\frac{R_h - R_h^*}{R_h}).\{\rho_h.I_h - \mu_h.R_h\} + (\frac{S_p - S_p^*}{S_p}).\{\Pi_p - (\alpha_p).S_p - \mu_p.S_p\} + (\frac{E_p - E_p^*}{E_p}).\{(\alpha_p).S_p - B_4.E_p\} + (\frac{I_p - I_p^*}{I_p}).\{\beta_p.E_p - B_5.I_p\}$$

$$+(\frac{R_p - R_p^*}{R_p}).\{\rho_p.I_p - \mu_p.R_p\} \quad (89)$$

Where, $\alpha_h = \frac{\sigma_p.(\varepsilon_p.E_p + I_p.)}{N_p} + \frac{\sigma_h.(\varepsilon_h.E_h + I_k.)}{N_h}$

$$\alpha_p = \frac{\sigma_p.(\varepsilon_p.E_p + I_p.)}{N_p} \quad (90)$$

From (89) we have:

$$\frac{dG}{dt} = (S_h - S_h^*).\{\frac{\omega.V_h}{(1-f).S_h} + (\alpha_p + \alpha_h).\frac{S_h^*}{(1-f).S_h} + \frac{A_h.S_h^*}{(1-f).S_h} + \frac{\omega.V_h}{S_h}\} - (S_h - S_h^*).\{(\alpha_p + \alpha_h) + A_h\} + (E_h - E_h^*).\{(\alpha_p +$$
$$\alpha_h).\frac{S_h}{E_h} - (E_h - E_h^*).\frac{(\alpha_p + \alpha_h).S_h^*}{E_h^*}\} + (\frac{I_h - I_h^*}{I_h})\beta_h E_h - (I_h - I_h^*)B_2 + (\frac{V_h - V_h^*}{V_h})(f\Pi_h + \gamma S_h) - (V_h - V_h^*)B_3 + (\frac{R_h - R_h^*}{R_h})\rho_h I_h - (R_h -$$
$$R_h^*)\mu_h + (\frac{S_p - S_p^*}{S_p}).\Pi_p - (S_p - S_p^*).[(\alpha_p) + \mu_p] + (\frac{E_p - E_p^*}{E_p})(\alpha_p).S_p - (E_p - E_p^*).B_4 + (\frac{I_p - I_p^*}{I_p}).\beta_p.E_p - (I_p - I_p^*).B_5 +$$
$$(\frac{R_p - R_p^*}{R_p})\rho_p I_p - (R_p - R_p^*)\mu_p \quad (91)$$

Eq. (91) yield:

$$\frac{dG}{dt} = (S_h - S_h^*)\{\frac{\omega V_h}{(1-f).S_h} + (\alpha_p + \alpha_h)\frac{S_h^*}{(1-f).S_h} + \frac{A_h S_h^*}{(1-f).S_h} + \frac{\omega V_h}{S_h}\} + (E_h - E_h^*).(\alpha_p + \alpha_h)\frac{S_h}{E_h} + (\frac{I_h - I_h^*}{I_h})\beta_h.E_h + (\frac{V_h - V_h^*}{V_h})(f.\Pi_h + \gamma S_h) + (\frac{R_h - R_h^*}{R_h})\rho_h.I_h + (\frac{S_p - S_p^*}{S_p})\Pi_p + (\frac{E_p - E_p^*}{E_p})(\alpha_p)S_p + (\frac{I_p - I_p^*}{I_p})\beta_p.E_p + (\frac{R_p - R_p^*}{R_p})\rho_p.I_p - (S_h - S_h^*)\{(\alpha_p + \alpha_h) + A_h\} - (E_h - E_h^*)\frac{(\alpha_p + \alpha_h).S_h^*}{E_h^*}\} - (I_h - I_h^*)B_2 - (V_h - V_h^*)B_3 - (R_h - R_h^*)\mu_h - (S_p - S_p^*).[(\alpha_p) + \mu_p] - (E_p - E_p^*)..B_4 - (I_p - I_p^*)B_5 - (R_p - R_p^*).\mu_p \quad (92)$$

Eq. (92) yields:

$$\frac{dG}{dt} = P - M \quad (93)$$

Where:

$$P = (S_h - S_h^*)\{\frac{\omega V_h}{(1-f).S_h} + (\alpha_p + \alpha_h)\frac{S_h^*}{(1-f).S_h} + \frac{A_h S_h^*}{(1-f).S_h} + \frac{\omega.V_h}{S_h}\} + (E_h - E_h^*).(\alpha_p + \alpha_h)\frac{S_h}{E_h} + (\frac{I_h - I_h^*}{I_h})\beta_h.E_h + (\frac{V_h - V_h^*}{V_h})(f.\Pi_h + \gamma.S_h) + (\frac{R_h - R_h^*}{R_h})\rho_h.I_h + (\frac{S_p - S_p^*}{S_p}).\Pi_p + (\frac{E_p - E_p^*}{E_p})(\alpha_p)S_p + (\frac{I_p - I_p^*}{I_p})\beta_p.E_p + (\frac{R_p - R_p^*}{R_p})\rho_p.I_p - (S_h - S_h^*)\{(\alpha_p + \alpha_h) + A_h\} - (E_h - E_h^*)\frac{(\alpha_p + \alpha_h)S_h^*}{E_h^*}\} \quad (94)$$

$$M = (I_h - I_h^*)B_2 + (V_h - V_h^*)B_3 + (R_h - R_h^*)\mu_h + (S_p - S_p^*)[(\alpha_p) + \mu_p] + (E_p - E_p^*)B_4 + (I_p - I_p^*)B_5 + (R_p - R_p^*)\mu_p \quad (95)$$

From (93) if P < M then $\frac{dG}{dt}$ will be negative definite, meaning that $\frac{dG}{dt} < 0$. Also it follows that $\frac{dG}{dt} = 0$ if and only if $S_h(t) = S_h^*, E_h(t) = E_h^*, I_h(t) = I_h^*, V_h(t) = V_h^*, R_h(t) = R_h^*, S_p(t) = S_p^*, E_p(t) = E_p^*, I_p(t) = I_p^*, R_p(t) = R_p^*$ The largest compact invariant set in $\{S_h^*, E_h^*, I_h^*, V_h^*, R_h^*, S_p^*, E_p^*, I_p^*, R_p^*) \in \Re^9 : \frac{dG}{dt} = 0\}$ is the singleton $\{E_1\}$ where E_1 is the EE of the model equations (1)-(13). From the principle of lasalle's invariant, it implies that E_1 is globally stable in $\Re^9$ if $P < M$. and unstable if $P > M$. This completed the proof.

Numerical Simulations

The calculation and estimation of the parameter values was done based on the availability of information from the Nigeria Centre for Disease Control (NCDC), Situation Report on Monkey-pox disease from 2017 to 2023 in Table 1.

Table 1: Update on Monkey-pox Disease in Nigeria from 2017 to 2023

Cases of Monkey pox	Number per year
Confirmed cases from Dec. 2017 to January 2023	988
Suspected cases from 2017 to 2023	2635
Deaths 2017–2023	15
Confirmed cases in 2017	88
Confirmed cases in 2018	49
Confirmed cases in 2019	47
Confirmed cases in 2020	8
Confirmed cases in 2021	34
Confirmed cases in February 2022	7
Confirmed cases in February 2023	762

(NCDC, 2023)

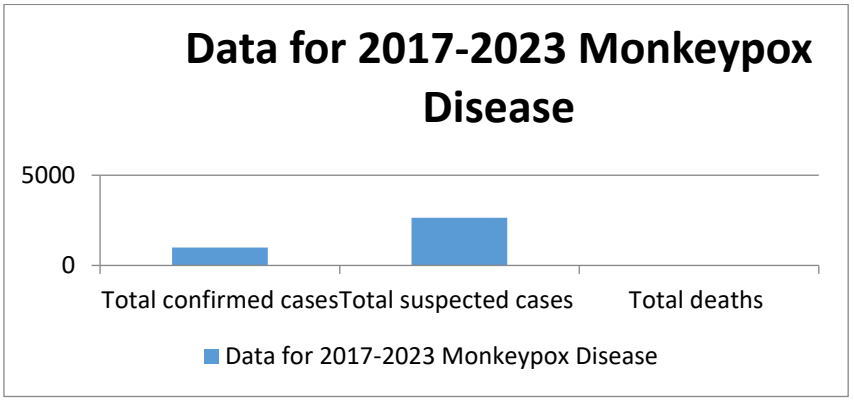


Figure 2: Graphical Representation of confirmed cases, suspected cases and death

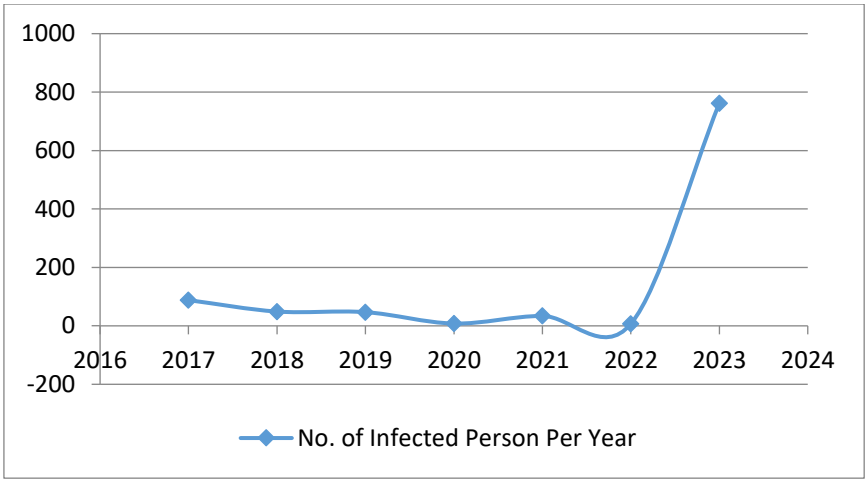


Figure 3: Graphical Representation of infected monkey pox individual per year

According to United Nation 2023 to 2024 report, the life expectancy for Nigerian at birth is 56.05 years. This gives the Natural Death rate as inverse of the life expectancy which is $\mu_h = \frac{1}{56.05} = 0.01784$ per year. The birth rate is 38.03 births per year per 1000 people; this gives the birth rate as $\frac{38.03}{1000} = 0.03803$/year. However, the recruitment rate due to birth in Nigeria is $\prod_h = N_h * \mu_h = 3{,}568{,}000$. According to NCDC (2023), there were 2635 suspected cases, where total confirmed cases were at 988, resulting in 15 deaths. This implies, Recovery rate is $\gamma_h = \frac{998-15}{998} = 0.985$. Also, Death rate due to the disease, it is cleared that 15 people out of 988 died of the infection of monkeypox in Nigeria between 2017 to 2023, which implies, $\delta_h = \frac{15}{998} = 0.0152$. We have infection rate = (confirmed cases / Total Population)*100 = 0.000494. The natural death rate of monkey, according to Primate Info Net (PIN), the life span of monkeys in the forest is 15-30 years, meanwhile, $\mu_m = \frac{1}{30}\ or\ \frac{1}{15} = (0.033\ or\ 0.067)$ is the Natural Death rate of Monkey. According to Pandrillus foundation (2008), it is about 8,000 drill monkey found in cross river state of Nigeria. However, 50,000 monkeys are estimated for Nigeria, hence, the recruitment rate of monkeys is given by $\prod_m = N_m * \mu_m = 1{,}665$. The vaccination rate of monkey pox infection is 10.1%, so fraction of vaccination against monkey pox (f) is $f = \frac{10.1}{100} = 0.0101$. Vaccination last upto 3-5 years and can also protect around 85% from monkey pox. So, $\varphi_1 = \frac{1}{5} = 0.2\ or\ \frac{1}{10} = 0.1\ and\ \varphi_2 = \frac{1}{85} = 0.012$. Other unavailable data have been assumed in the simulations.

Table 2: Symbols and Parameters Values of the Model

Parameter	Definition	Value	Source
β_h	Exposed Rate of Human	0.005	Assumed
α_h	Infection Rate of Human	0.000494	Table 1
Λ_h	Recruitment Rate of Human	3,568,000	Table 1
γ_h	Recovery Rate of Human	0.985	Table 1
μ_h	Natural Death Rate of Human	0.01784	Table 1
δ_h	Death Rate Due to Disease	0.0152	Table 1
φ_1	Loss of Vaccination Rate	(0.1 - 0.2)/year	Table 1
φ_2	Vaccination Rate	0.012/year	Table 1
f	Proportion of vaccinated h immigrants	0.0101	Table 1
∅	Effectiveness of Vaccination Drug	[0 - 1]	Assumed
α_m	Infection Rate of Monkey	0.004	Table 1
γ_m	Recovery Rate of Monkey	0.50	Assumed
Λ_m	Recruitment Rate of Monkey	1,665	Table 1
θ_{m2}	Exposed Rate of Monkey	0.003	Assumed
μ_m	Natural Death of Monkey	(0.033 – 0.067)	Table 1
δ_m	Death Due to infection of monkey	0.020	Assumed
τ_h	Exposed rate for human transmits lower than symptomatic humans	0.010	Assumed
τ_m	Exposed rate for non-human transmits lower than symptomatic non-humans	0.010	Assumed
N_h	Nigeria Population	200,000,000	Table 1
N_m	Population of Monkey in Nigeria	50,000	Table 1

The study used table 2 to simulate our model system with equation (45) and (48) by using Maple 17 Software for the graphic representation of the reproduction numbers,

Graphical Representation of the Model Equations for Monkey pox Disease

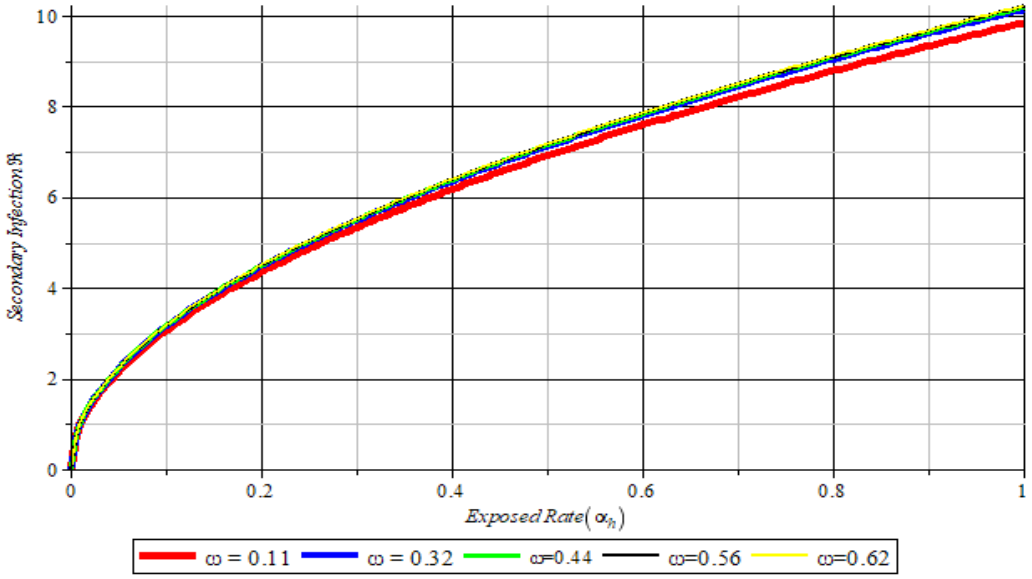
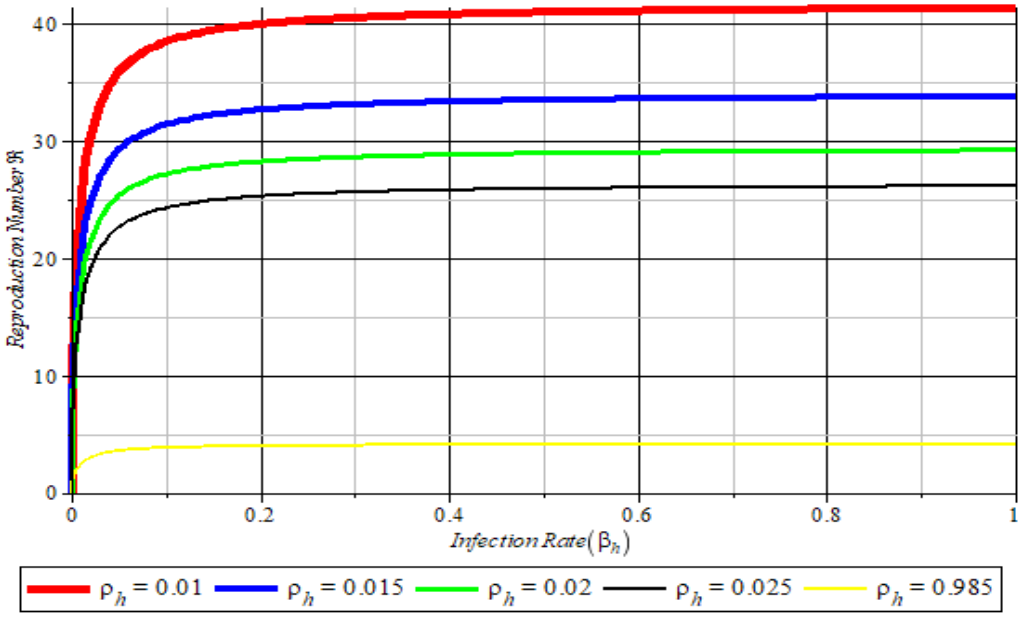


Figure 4: Effect of exposed rate on the effective reproduction number.

It was observed in figure 4 that rate of exposed increases with losing out of vaccination which leads to increases in effective reproduction number (secondary infection cases). This implies that, the two parameters increase the infection of monkey-pox disease in Nigeria.



e

Figure 5: Infection rate effect on effective reproduction number (R_e)

Figure 5 shows that as the infection rate increases, then effective reproduction number also increases, varying the rate of recovery of human, it is observed that as recovery individual increases, the infected individual decreases from the monkey-pox in Nigeria because the effective reproduction number decreases. This means that as more people are recovery from the monkey-pox disease, the secondary infection cases decreases.

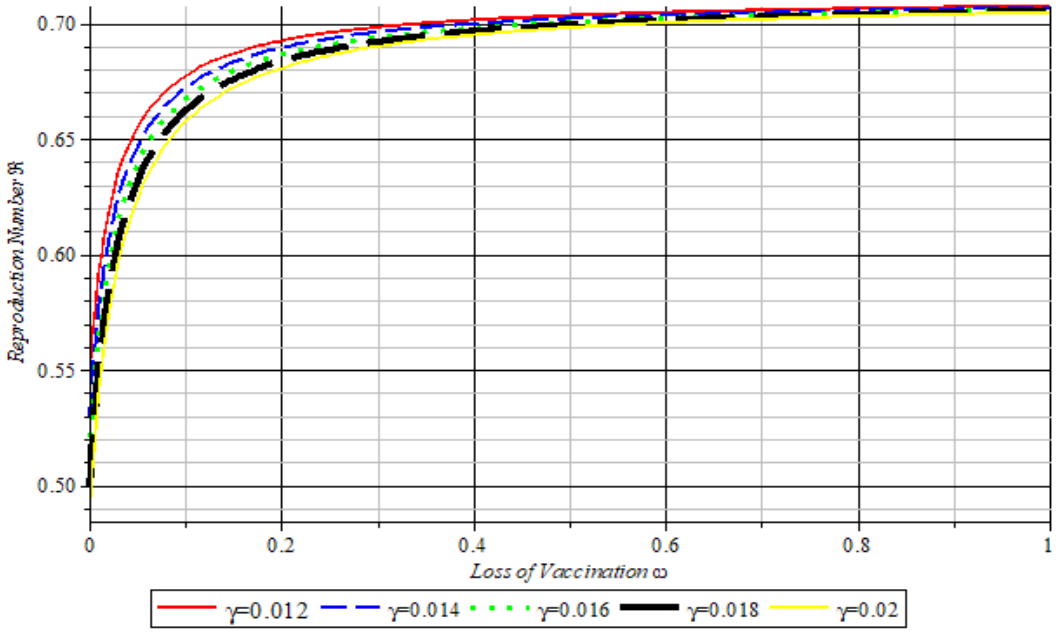


Figure 6: Effect of Loss of vaccination on the R_e

It is observed from figure 6 that as loss of vaccination increases, it increases R_e number. Also, Low recovery rate increases theR_e number of the infection.

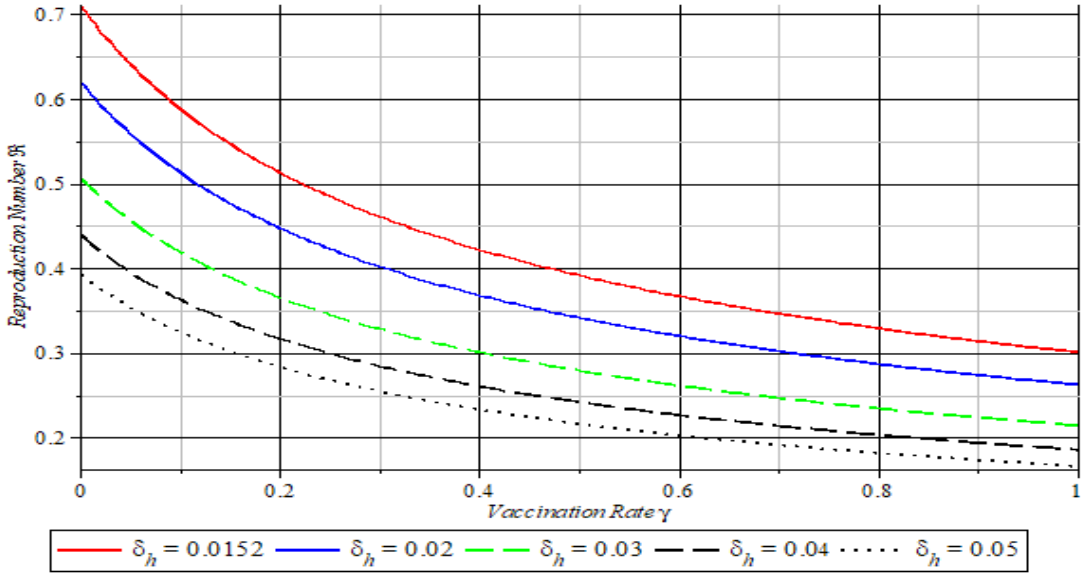


Figure 7: Effect of Vaccination on the R_e number of human

Figure 7 shows that as vaccination rate decreases, it also decreases R_e, by varying the rate of death due to monkey-pox, it is observed that effective reproduction number decreases as rate of death due to infection increases. This means that the higher the death due to the monkey-pox infection the lower the R_e. This is implies as more individuals die due to the monkey-pox, there will be less to be infected by monkey-pox.

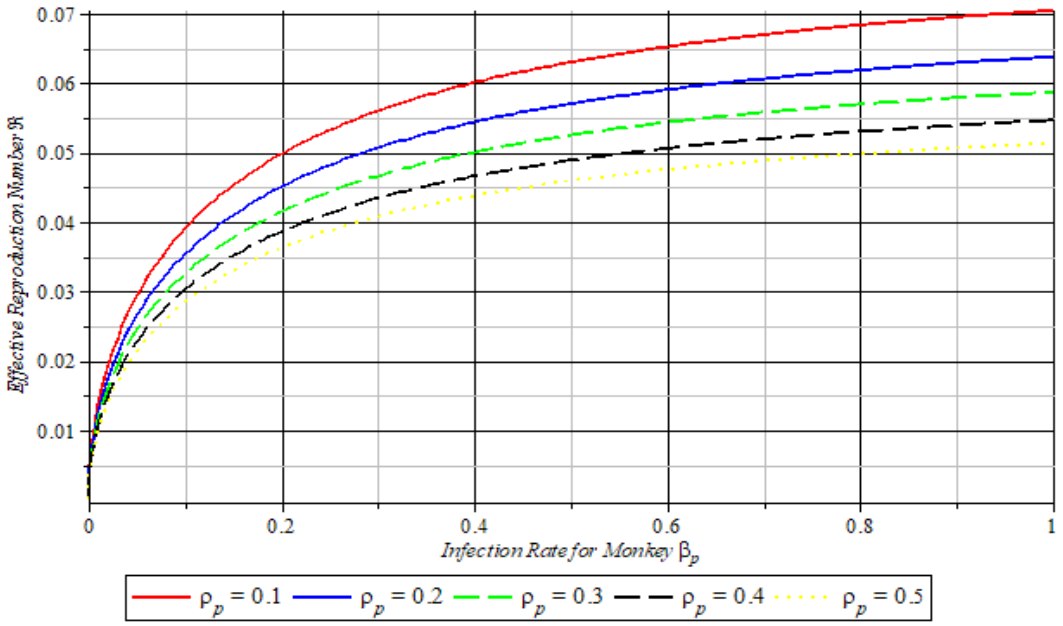


Figure 8: Infection rate effect of monkey pox on R_p of monkey (non-human)

It is shown in figure 8 that as rate of infection for non-human increases, it also increases R_e. Varying the rate of recovery, we found that the effective reproduction number decreases as recovery rate increases

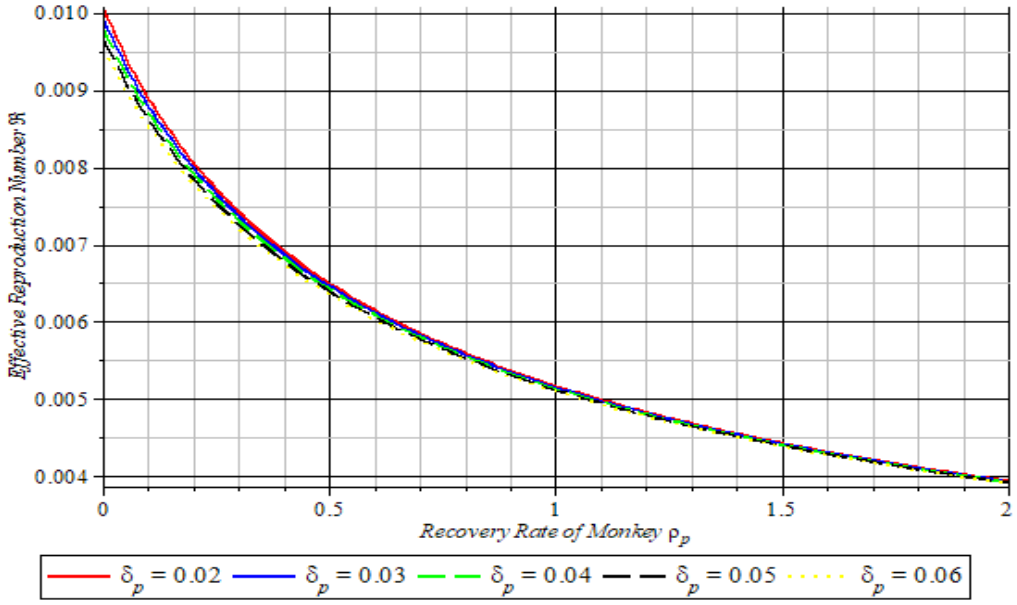


Figure 9: Recovery rate effect of monkey pox on the R_p of Monkey

Figure 9 shows the simulation of recovery rate of non-human with effective reproduction number of non-human, it is observed that effective reproduction number of non-human decreases as recovery rate increases.

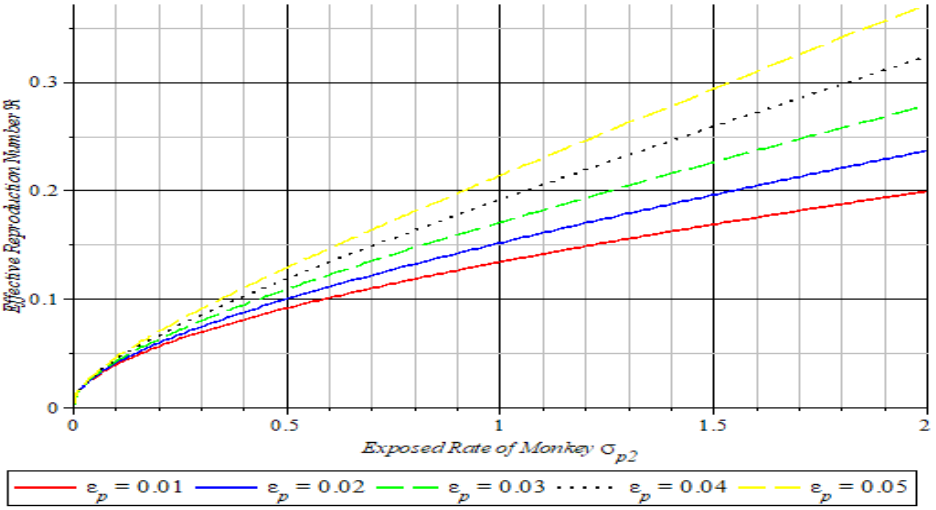


Figure 10: Effect of exposed rate of Monkey on the R_p of Monkey

Figure 10 shows the simulation of exposed rate of non-human with effective reproduction number of non-human, it is observed that R_p of non-human increases as exposed individual of non-human increases. It is also found that the R_p decreases due to the decreases in exposed of non-human transmit at a rate lower than symptomatic non-humans.

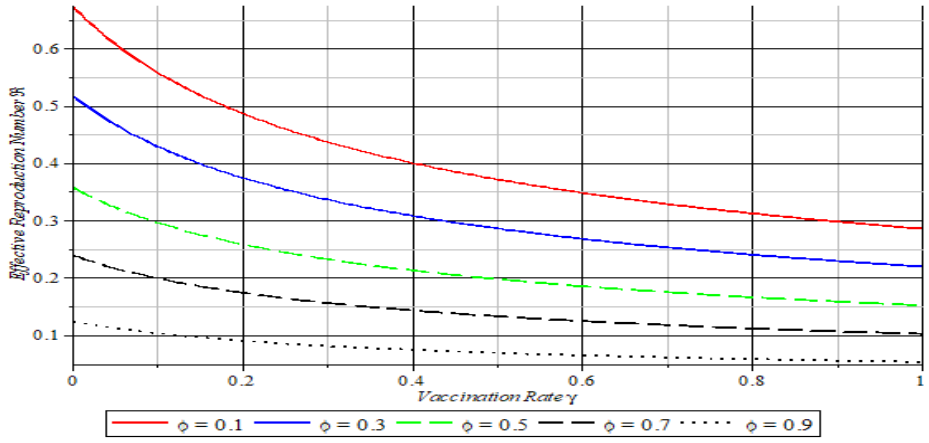


Figure 11: Effect of Vaccination on the R_e

We found in figure 11, that R_e decreases as vaccination of individual increases, varying the public awareness, we observed that effective reproduction number reduces to zero and becomes stable as public awareness increases.

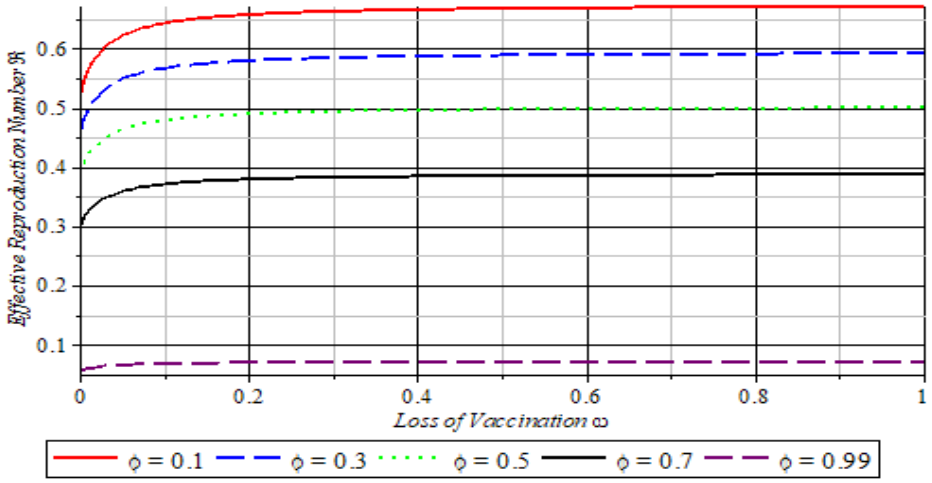


Figure 12: Effect of Loss of vaccination on the effective reproduction number (R_e), varying public

Awareness parameter

It is observed from figure 12 that, low loss of vaccination rate increases the R_e, it is also shown that, the R_e approaches zero and becomes stable as public awareness increases.

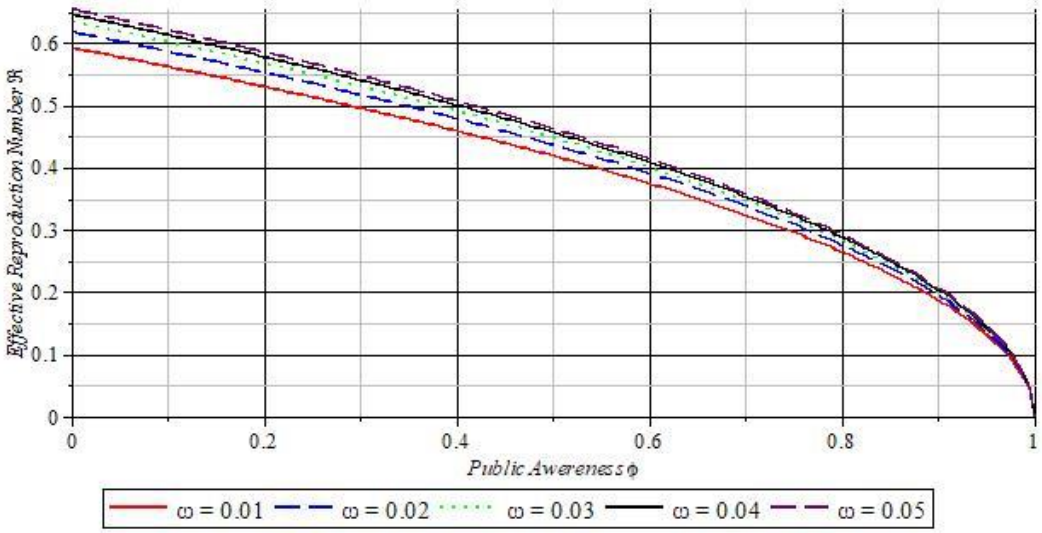


Figure 13: Effect of Public Awareness on the effective reproduction number

In figure 13, we observed that effective reproduction number decreases as public awareness increases, varying loss of vaccination rate, it is observed that as loss of vaccination rate decreases, it decreases the secondary cases of the infection.

CONCLUSION

This research work have simulated and analyzed a mathematical model of monkey-pox transmission disease, the study incorporated Vaccination and public awareness in the model. The analyses of the model showed invariant region and dynamics of model equations is in the region Ω, the model was considered been mathematically and epidemiologically well posed. The non-negativity of the solutions for the model implies that the solutions were non-negative and remains in $\mathbb{R}^9$. The global stability of the DFE was done and it was found that when there is no transmission between human and non-human (α_h), then $\hat{G}(X_1, X_2) = 0$ implies, that the DFE would be asymptotically globally stable. The endemic equilibrium state was obtained and we found that the endemic equilibrium state was globally asymptotically stable if$P < M$and unstable if $P > M$ by comparison method of lyapunov functions. Simulations of the model equations on effective reproduction number were done and the study found that the exposed rate increases with the loss of vaccination which leads to increases in secondary infection cases of the monkey-pox. It is found that as recovery individual increases, the infected individual decreases from the monkey-pox in Nigeria, it implies, that as more people recovered from the monkey-pox disease, the secondary infection cases decreases. It was also found that, low recovery rate increases the R_e of the infection. The study shown that as vaccination rate decreases, it decreases the R_e. It was observed that R_e decreases as death rate due to infection increases, this implies that the higher the death due to the monkey-pox infection the lower R_e would be. This is because as more people die due to the monkey-pox, there will be less to be infected of monkey-pox virus. The study noted that R_e decreases as vaccination of individual increases, varying the public awareness it was found that effective reproduction number (R_e) reduces to zero and becomes stable as public awareness increases. the study discovered that effective reproduction number decreases as public awareness increases.

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