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AN EXPLORATION OF ANTIMULTIGROUP EXTENSIONS

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ABSTRACT

This research paper pioneers an innovative extensions of antimultigroup theory by seemingly integrating the concept of cuts and comultiset, thereby revolutionizing the field. Notably, we demonstrate that the root sets of antimultigroup sums and differences are subgroups, uncovering a profound connection. Furthermore, we establish that if H is a complete sub-antimultigroup of G such that all the counts in H are factors of their corresponding counts in G. Then $|H|/|G|$. Finally, we prove that the cuts of antimultigroup unions and intersections also form subgroups, further enriching our understanding of these complex structures.

Keywords: Multisets, Multigroup, Group, Antimultigroup

INTRODUCTION

George Cantor developed the foundational ideas of set theory, which became the basis for group theory (Kleiner, 1986). Cantor assumed in his set theory that it is not permissible for objects in a set to be repeated; nevertheless, this does not align with ideas in practical situations. Cantorian set theory served as the foundation for group theory's development, and several findings and deductions pertaining to group theory based on classical set theory have been made. With the advancement of study and mathematical development, some algebraists realized that Cantor's premise needed to be addressed. To address the limitation in Cantor's assumption, the idea of multiset was developed. DeBruijn originally proposed the word "multiset" to Knuth in a private correspondence as a generalization of Cantor's crisp set theory (Knuth, 1981). In contrast to Cantorian set theory, a multiset is an unorganized arrangement of objects where repetition of elements is permitted. Due to its practicality and wide range of applications in biological systems, database systems, web information retrieval, membrane computing, etc., multiset has become extremely significant. Multisets' origins and evolution are covered in (Blizard, 1991; Singh 1994; Singh et al., 2007 & 2008). Given that a multiset generalizes a set, it follows that the concept of group must also be generalized to include multigroups. Dresher and Ore (1938) defined multigroups as algebraic systems that satisfy all of the group theory axioms, with the exception that multiplication is multivalued. This definition is incompatible with the idea of a multiset and does not align with other non-classical groups, such as fuzzy groups, soft groups, intuitionistic fuzzy groups, fuzzy soft groups, etc. (Rosenfeld 1971; Aktas & Cagman, 2007; Biswas, 1989; Nazmul & Samanta, 2011 & 2015; Shinoj et al., 2015; Shinoj & Sunil, 2015).

The notion of a multigroup was presented by Nazmul *et al.* (2013) using multisets, and their structures were explained. This definition is more appealing since it incorporates the notion of multiplicities as count functions, is based on the multiset concept, and is consistent with the methodology used by other non-classical groups. Several group theory structures, such as submultigroup, normal submultigroup, comultisets, factor multigroup, and commutative multigroup, have been extended to multigroups using multisets in light of this notion. As an extension of the idea of a multigroup in reverse order, Ejegwa (2020) established the concept of antimultigroup and clarified some of its features.

In this paper, we explain certain results and offer an extension on the topic of antimultigroup. The remainder of the paper is arranged as follows: materials and methods of approach are reported in section 2. In section 3, results and discussion are presented in relation to antimultigroup, and certain conclusions are drawn. Section 4 presents a conclusion and a few recommendations.

Preliminaries And Basic Definitions

This section provides an overview of relevant existing research, synthesizing key definitions and findings to establish a foundation for our work. Additionally, we augment this foundation with novel definitions and results that will be pivotal to our subsequent contributions, thereby extending the existing knowledgebase and paving way for our innovative approaches.

Definition 1: (Singh *et al.* 2007): Suppose $X = \{x_1, x_2, \dots, x_j, \dots\}$ is a set. A multiset A over X is a function that maps each element of X to a non-negative integer i.e., $A : X \longrightarrow N = \{0,1,2,\dots\} \ni$ for $x \in Dom(A)$, $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ is the frequency of x in the multiset A. The collection X is referred to as the root set from which all possible multisets are derived and it is represented as $MS(X)$.

Definition 2: (Syropoulos, 2001): Let A and B be two multisubsets, A is called an multisubset or a submultiset of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$ $\ni$ the root set of B is D. Also, if $A \subseteq B$ and $A \neq B$ then A is a proper submultiset of B.

Definition 3: (Nazmul *et al.* 2013): Consider $A \in MS(X)$. Accordingly, A_* and A^* are given as follows: $A_* = \{ x \in X \mid m_A(x) > 0 \}$ and $A^* = \{ x \in X \mid m_A(x) = m_A(e)\}$ where X has e as its identity.

Definition 4: (Nazmul *et al.* 2013): Consider a group X. A multiset A over X is said to be a multigroup of X if its multiplicity function m_A fulfill the conditions below:

i. $m_A(xy) \geq m_A(x) \wedge m_A(y)\ \forall x, y \in X$
ii. $m_A(x^{-1}) \geq m_A(x)$

It follows immediately that,
$m_A(x^{-1}) = m_A(x)$
since from (ii),
$m_A(x) = m_A((x^{-1})^{-1}) \geq m_A(x^{-1})$
The collection of all multisets over X that forms a multigroup is represented as $MG(X)$.

Proposition 5 : (Nazmul *et al*. 2013): Consider $A \in AMG(X)$. Thus X contains A_* and A^* as its subgroups.

Definition 6: (Nazmul *et al*., 2013): Consider a group X. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in X$ given as $m_{yA}(x) = m_A(y^{-1}x)\ \forall\, x \in A_*$ is referred as left comultiset of A. Also, the submultiset Ay of G for $y \in X$ given as $m_{Ay}(x) = m_A(xy^{-1})\ \forall\, x \in A_*$ is referred as right comultiset of A.

Definition 7: (Nazmul *et al*., 2013): Consider a group X such that G is multigroup of X. The cardinality of G is equal to the sum of the multiplicities of each distinct element in G, denoted as $|G| = \sum_{i=1}^{n} m_G(x_i)\ \forall x_i \in X$.

Definition 8: (Ejegwa, 2020): A multiset A of X is said to be an antimultigroup if it fulfills the conditions below:

i. $m_A(xy) \le m_A(x) \vee m_A(y)\ \forall x, y \in X$
ii. $m_A(x^{-1}) \le m_A(x) \forall x \in X$

The collection containing all antimultigroup of X is represented as $AMG(X)$.

Definition 9: (Ejegwa, 2020): Let $A \in AMG(X)$. The set $A_{[n]}$, defined as $A_{[n]} = \{x \in X |\ m_A(x) \le n, n \in \mathbb{N}\}$ is called the cut of A.

Theorem 10: (Ejegwa, 2020): Let $A \in AMG(X)$. Then X contains $A_{[n]}, n \in \mathbb{N}$ as a subgroup such that $n \ge m_A(e)$, where X has e as its identity.

Proposition 11: (Ejegwa, 2020): If $A \in AMG(X)$ then $\forall x \in X, n \in \mathbb{N}$, the assertions below are valid:

i. $m_A(e) \le m_A(x)$ where X has e as its identity.
ii. $m_A(x^n) \le m_A(x)$
iii. $m_A(x^{-1}) = m_A(x)$.

Definition 12: Consider $A \in AMG(X)$. A submultiset B of A is said to be a sub-antimultigroup of A represented as $B \le A$ if B forms an antimultigroup. A sub-antimultigroup B of A is a proper sub-antimultigroup represented as $B < A$, if $B \le A$ and $A \ne B$.

Example 13: Let $X = \{e, a, b, c, d\}$ be a Klein 4-group and $A = \{e^6, a^8, b^7, c^8\}$ be an antimultigroup generated from X. Then $A = \{e^6, a^8, b^7, c^8\}$, $B = \{e^5, a^7, b^6, c^7\}$, $C = \{e^4, a^6, b^5, c^6\}$, $D = \{e^3, a^5, b^4, c^5\}$ are sub-antimultigroups of A. But, $B = \{e^5, a^7, b^6, c^7\}$, $C = \{e^4, a^6, b^5, c^6\}$, $D = \{e^3, a^5, b^4, c^5\}$ are proper sub-antimultigroups of A.

Remark 14: If $A \in AMG(X) \ni B \le A$, thus $B \in AMG(X)$.

Proposition 15: Let $A \in MS(X)$.Then the following hold

i. $A_* \cap B_* = (A \cap B)_*$
ii. $A_* \cup B_* = (A \cup B)_*$

Proof:

i. Suppose $x \in A_* \cap B_* \Longrightarrow x \in A_*$ and $x \in B_*$. Then by definition of A_*, we have
$A_* = \{\, x \in X \mid m_A(x) > 0 \,\}$ and also $B_* = \{\, x \in X \mid m_B(x) > 0 \,\}$. Thus
$A_* \cap B_* = \{\, x \in X \mid m_A(x) > 0 \,\} \cap \{\, x \in X \mid m_B(x) > 0 \,\}$
$\le \{\, x \in X \mid m_A(x) > 0 \wedge m_B(x) > 0\}$
$= \{\, x \in X \mid m_A(x) \wedge m_B(x) > 0\}$
$= \{\, x \in X \mid m_{A \cap B}(x) > 0\}$
$= (A \cap B)_*$
On the other hand, suppose $x \in (A \cap B)_*$. Then we have
$(A \cap B)_* = \{\, x \in X \mid m_{A \cap B}(x) > 0\}$
$= \{\, x \in X \mid m_A(x) \wedge m_B(x) > 0\}$
$\le \{\, x \in X \mid m_A(x) > 0 \wedge m_B(x) > 0\}$
$= \{\, x \in X \mid m_A(x) > 0 \,\} \cap \{\, x \in X \mid m_B(x) > 0 \,\}$
$= A_* \cap B_*$
ii. Follows immediately from i.

RESULTS AND DISCUSSIONS

This section introduces novel insights into the realm of antimultigroups, presenting new findings that expand our understanding of this mathematical concept. We contribute meaningfully to the existing body of knowledge, fostering a deeper comprehension of antimultigroup.

Proposition 16: If $A \in AMG(X) \Leftrightarrow A^{-1} \in AMG(X)$.
Proof: Suppose $A \in AMG(X)$. Then by definition of A, we have $m_A(xy^{-1}) \le m_A(x) \vee m_A(y)$.
$\Rightarrow m_{(A^{-1})^{-1}}(xy^{-1}) \le m_{(A^{-1})^{-1}}(x) \vee m_{(A^{-1})^{-1}}(y)$
Since $m_A(x) = m_A(x^{-1}) = m_{A^{-1}}(x) \Rightarrow m_{(A^{-1})^{-1}}(x) = m_{A^{-1}}(x)$
$\Rightarrow m_{A^{-1}}(xy^{-1}) \le m_{A^{-1}}(x) \vee m_{A^{-1}}(y)$
Hence, $A^{-1} \in AMG(X)$.
On the other hand, suppose $A^{-1} \in AMG(X)$ and $\forall x, y \in X$, it follows that
$m_{A^{-1}}(xy^{-1}) \le m_{A^{-1}}(x) \vee m_{A^{-1}}(y)$
$\Rightarrow m_A([xy^{-1}]^{-1}) \le m_A(x^{-1}) \vee m_A(y^{-1})$
$\Rightarrow m_A(xy^{-1}) \le m_A(x) \vee m_A(y)$
Hence, $A \in AMG(X)$.

Proposition 17: Consider $A, B \in AMG(X)$ such that $A_* = B_*$, the assertions below holds

i. X contains $(A + B)^*$ as its subgroup.
ii. X contains $(A + B)_*$ as its subgroup.

Proof.
(i) Suppose $x, y \in (A + B)^*$. We have $m_{A+B}(x) = m_{A+B}(y) = m_{A+B}(e)$. Since $A, B \in AMG(X)$, then
$m_{A+B}(xy^{-1}) = m_A(xy^{-1}) + m_B(xy^{-1})$
$\le \big[\big(m_A(x) \vee m_A(y)\big) + \big(m_B(x) \vee m_B(y)\big)\big]$
$\le \big[\big(m_A(x) + m_B(x)\big) \vee \big(m_A(y) + m_B(y)\big)\big]$
$\le \big[\big(m_{A+B}(x)\big) \vee \big(m_{A+B}(y)\big)\big]$
$= \big[\big(m_{A+B}(e)\big) \vee \big(m_{A+B}(e)\big)\big]$
$= m_{A+B}(e)$
Thus $m_{A+B}(xy^{-1}) \le m_{A+B}(e)$ and also $m_{A+B}(e) \le m_{A+B}(xy^{-1})$ from proposition 2.11. So $m_{A+B}(xy^{-1}) = m_{A+B}(e)$. Since $x, y \in (A + B)^* \Rightarrow xy^{-1} \in (A + B)^*$. Hence X contains $(A + B)^*$ as its subgroup.
(ii) Suppose $x, y \in (A + B)_*$. We have $m_{A+B}(x) > 0$ and $m_{A+B}(y) > 0$. Since $A, B \in AMG(X)$, then
$m_{A+B}(xy^{-1}) = [m_A(xy^{-1}) + m_B(xy^{-1})]$
$\le \big[\big(m_A(x) \vee m_A(y)\big) + \big(m_B(x) \vee m_B(y)\big)\big]$
$\le \big[\big(m_A(x) + m_B(x)\big) \vee \big(m_A(y) + m_B(y)\big)\big]$
$\le \big[\big(m_{A+B}(x)\big) \vee \big(m_{A+B}(y)\big)\big] > 0$
$= m_{A+B}(x) > 0 \ \vee\ m_{A+B}(y) > 0$
Therefore, $x, y \in (A + B)^* \Rightarrow xy^{-1} \in (A + B)^*$. Hence X contains $(A + B)^*$ as its subgroup.

Proposition 18: Let $A, B \in AMG(X)$ such that $B_* \subseteq A_*$, the assertions below holds.

i. X contains $(A - B)^*$ as its subgroup.
ii. X contains $(A - B)_*$ as its subgroup.

Proof: Similar to theorem 3.2

Proposition 19: Let $A \in AMG(X)$ and B be a multiset of A. The assertions below are all equal such that $\forall x, y \in X$,

i. B is a sub-antimultigroup of A.
ii. $m_B(xy) \leq m_B(x) \vee m_B(y)$ and $m_B(x^{-1}) = m_B(x)$.
iii. $m_B(xy^{-1}) \leq m_B(x) \vee m_B(y)$.

Proof.
(i) ⇒ (ii) If B is a sub-antimultigroup of A. From remark 2.14, it follows that $B \in AMG(X)$. Thus, $m_B(xy) \leq m_B(x) \vee m_B(y)$ and $m_B(x^{-1}) = m_B(x)$.
(ii) ⇒ (iii) Since $B \in AMG(X)$, then it follows that $m_B(xy^{-1}) \leq m_B(x) \vee m_B(y)$.
(iii) ⇒ (i) Since $B \subseteq A$ and $B \in AMG(X)$. Then B is a sub-antimultigroup of A.

Comultisets of Antimultigroup

This section builds on the established concept of comultiset and also pioneer an innovative extension to antimultigroup thereby providing results that significantly augment the existing mathematical knowledge in this domain.

Definition 20: Suppose X is a group. Then any sub-antimultigroup A of an antimultigroup G of X , the submultiset yA of G for every $y \in X$ given as $m_{yA}(x) = m_A(y^{-1}x)\ \forall x \in A_*$ is referred to as left comultiset of A. Also, the submultiset Ay of G for every $y \in X$ given as $m_{Ay}(x) = m_A(xy^{-1})\ \forall x \in A_*$ is referred as right comultiset of A.

Example 21: Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a group of permutation on $S = \{1,2,3\} \ni \rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$.
Then $G = \{{\rho_0}^3, {\rho_1}^5, {\rho_2}^3, {\rho_3}^5, {\rho_4}^3, {\rho_5}^5\}$ is an antimultigroup of X and $H = \{{\rho_0}^2, {\rho_1}^4, {\rho_2}^2, {\rho_3}^4, {\rho_4}^2, {\rho_5}^4\}$ is a complete sub-antimultigroup of G.
The left comultisets of H are given by multiplying from the left every distinct member of X with H, that is
$\rho_0 H = [{\rho_0}^2, {\rho_1}^4, {\rho_2}^2, {\rho_3}^4, {\rho_4}^2, {\rho_5}^4]$
$\rho_1 H = [{\rho_2}^2, {\rho_0}^2, {\rho_1}^4, {\rho_5}^4, {\rho_3}^4, {\rho_4}^2]$
$\rho_2 H = [{\rho_1}^4, {\rho_2}^2, {\rho_0}^2, {\rho_4}^2, {\rho_5}^4, {\rho_3}^4]$
$\rho_3 H = [{\rho_3}^4, {\rho_5}^4, {\rho_4}^2, {\rho_0}^2, {\rho_2}^2, {\rho_1}^4]$
$\rho_4 H = [{\rho_4}^2, {\rho_3}^4, {\rho_5}^4, {\rho_1}^4, {\rho_0}^2, {\rho_2}^2]$
$\rho_5 H = [{\rho_5}^4, {\rho_4}^2, {\rho_3}^4, {\rho_2}^2, {\rho_1}^4, {\rho_0}^2]$
Similarly, the right comultisets of H are given below
$H\rho_0 = [{\rho_0}^2, {\rho_1}^4, {\rho_2}^2, {\rho_3}^4, {\rho_4}^2, {\rho_5}^4]$
$H\rho_1 = [{\rho_2}^2, {\rho_0}^2, {\rho_1}^4, {\rho_4}^2, {\rho_5}^4, {\rho_3}^4]$
$H\rho_2 = [{\rho_1}^4, {\rho_2}^2, {\rho_0}^2, {\rho_5}^4, {\rho_3}^4, {\rho_4}^2]$
$H\rho_3 = [{\rho_3}^4, {\rho_4}^2, {\rho_5}^4, {\rho_0}^2, {\rho_1}^4, {\rho_2}^2]$
$H\rho_4 = [{\rho_4}^2, {\rho_5}^4, {\rho_3}^4, {\rho_2}^2, {\rho_0}^2, {\rho_1}^4]$
$H\rho_5 = [{\rho_5}^4, {\rho_3}^4, {\rho_4}^2, {\rho_1}^4, {\rho_2}^2, {\rho_0}^2]$

Remark 22: By example 3.6, the sub-antimultigroup of an antimultigroup and its comultisets are the same since the multisets contain the same elements regardless of their order or arrangement. In essense, if A is a sub-antimultigroup of $B \in AMG(X)$, then $A = yA\ \forall\ y \in X$. Similarly, $A = Ay\ \forall\ y \in X$.

Proposition 23: If A is a sub-antimultigroup of $G \in AMG(X)$, then , $yA = Ay\ \forall\ y \in X$.
Proof: If A is a sub-antimultigroup of G. Thus $\forall\ x \in A_*$ it follows that
$m_{yA}(x) = m_A(y^{-1}x) \leq m_A(y) \vee m_A(x)$
$= m_A(x) \vee m_A(y)$
$= m_A(x) \vee m_A(y^{-1})$
Suppose by hypothesis, $m_A(x) \vee m_A(y) = m_A(xy)$. Then, $m_{yA}(x) \leq m_{Ay}(x)$
Similarly,
$m_{Ay}(x) = m_A(xy^{-1}) \leq m_A(x) \vee m_A(y)$
$= m_A(y) \vee m_A(x)$
$= m_A(y^{-1}) \vee m_A(x)$
Subsequently we have,
$m_{Ay}(x) \leq m_{yA}(x)$
Hence, $m_{yA}(x) = m_{Ay}(x)$ which implies that, $yA = Ay$.

Theorem 24: Let $G \in AMG(X)$. Any sub-antimultigroup A of G for every $z \in X$, the submultiset $zAz^{-1} \ni m_{zAz^{-1}}(x) = m_A(z^{-1}xz)$ for all $x \in X$ is a sub-antimultigroup of G.
Proof: Suppose $x, y \in X$ and $A \leq G$. We show that $zAz^{-1} \leq G\ \forall\ z \in X$. Thus,
$m_{zAz^{-1}}(xy^{-1}) = m_A(z^{-1}xy^{-1}z)$
$= m_A(z^{-1}xzz^{-1}y^{-1}z)$
$\leq m_A(z^{-1}xz) \vee m_A(z^{-1}y^{-1}z)$
$= m_{zAz^{-1}}(x) \vee m_{zAz^{-1}}(y^{-1})$
$= m_{zAz^{-1}}(x) \vee m_{zAz^{-1}}(y)\ \forall z \in X$
This implies that $m_{zAz^{-1}}(xy^{-1}) \leq m_{zAz^{-1}}(x) \vee m_{zAz^{-1}}(y)$. Hence, zAz^{-1} is a sub-antimultigroup of G.

Lemma 25: Consider a group X. If B is a sub-antimultigroup of a finite antimultigroup $A \in AMG(X)$, we have that $|B| = |xB|\ \forall x \in X$.
Proof: Let $A \in AMG(X)$. Given that A is finite and $B \leq A$, then $|A| = p$ and $|B| = q \ni q \leq p$. It follows that $|B|$ and $|xB|$ must be equal to q by remark 3.7. Thus, $|B| = |xB|\ \forall x \in X$.

Theorem 26: Suppose G is a finite antimultigroup of a group X and let H be a complete sub-antimultigroup of G such that all the counts in H are factors of their corresponding counts in G. Then $|H|\ /\ |G|$.
Proof: Suppose $|G| = p$ and $|H| = q$, then $q \leq p$ from lemma 4.5. We know that G is finite and H is a sub-antimultigroup of G, then H is also finite and so $G_* = H_*$. Next, we show that q is a factor of p. Since $H \leq G$, it follows that all the count in H are factors of their corresponding counts in G. Thus $q|p$ and hence the proof.

Extension on Cuts of Antimultigroup

This section aims to build on the existing concept of cuts in antimultigroup setting and also to pioneer novel insights and results, thereby propelling the field forward with innovative perspectives and meaningful contributions.

Definition 27: Let $A \in AMG(X)$. The sets $A_{[n]}$ and $A_{(n)}$ are defined accordingly, $A_{[n]} = \{x \in X |\ m_A(x) \leq n, n \in \mathbb{N}\}$ and $A_{(n)} = \{x \in X |\ m_A(x) < n, n \in \mathbb{N}\}$ are referred as strong and weak upper cuts of A.

Example 28: Let $X = \{e, a, b, c\}$ be a group, then $A = \{e^2, a^5, b^4, c^5\}$ is an antimultigroup of X. Thus,
$A_{[1]} = \emptyset$
$A_{[2]} = \{e\}$
$A_{[3]} = \{e\}$
$A_{[4]} = \{e, b\}$
$A_{[5]} = \{e, a, b, c\}$
and
$A_{(1)} = \emptyset$
$A_{(2)} = \emptyset$
$A_{(3)} = \{e\}$
$A_{(4)} = \{e\}$
$A_{(5)} = \{e, b\}$

Definition 29: Let $A \in AMG(X)$. The sets $A^{[n]}$ and $A^{(n)}$ are defined accordingly, $A^{[n]} = \{x \in X \mid m_A(x) \geq n, n \in \mathbb{N}\}$ and $A^{(n)} = \{x \in X \mid m_A(x) > n, n \in \mathbb{N}\}$ are referred as strong and weak lower cuts of A.

Proposition 30: Suppose $A, B \in AMG(X) \ni m, n \in \mathbb{N}$. The assertions below are valid.

i. $A_{[n]} \subseteq A_{[m]}$ precisely if $n \leq m$.
ii. $A \subseteq B$ precisely if $A_{[n]} \subseteq B_{[n]}$.

Proof.
(i) Let $x \in A_{[n]} \Rightarrow m_A(x) \leq n$. Since $n \leq m \Rightarrow m_A(x) \leq n \leq m$. Hence, $A_{[n]} \subseteq A_{[m]}$.
Conversely, if $A_{[n]} \subseteq A_{[m]}$, we have $n \leq m$. Hence the proof.
(ii) Suppose $A \subseteq B$, then $m_A(x) \leq m_B(x)\ \forall x \in X$. Since $x \in A_{[n]}$ and $x \in B_{[n]} \Rightarrow m_A(x) \leq m_B(x) \leq n$. This implies that $A_{[n]} \subseteq B_{[n]}$.
The converse follows since $A_{[n]} \subseteq B_{[n]}$, then it implies that $A \subseteq B$.

Remark 31: Let $A, B \in AMG(X) \ni m, n \in \mathbb{N}$. The assertions below are valid.

i. $A^{[n]} \subseteq A^{[m]}$ precisely if $n \leq m$.
ii. $A \subseteq B$ precisely if $A^{[n]} \subseteq A^{[m]}$.

Theorem 32: Let $A \in AMG(X)$. Then X contains $A^{[n]}, n \in \mathbb{N}$ as a subgroup such that $n \leq m_A(e)$, where X has e as its identity.

Proof: Let $x, y \in A^{[n]}$, then $m_A(x) \geq n$ and $m_A(y) \geq n$. Since $A \in AMG(X)$, we have
$m_A(xy^{-1}) \leq \left(m_A(x) \vee m_A(y)\right) \geq n$
$= m_A(x) \geq n \vee m_A(y) \geq n$
Thus, $xy^{-1} \in A^{[n]}$. Hence, X contains $A^{[n]}, n \in \mathbb{N}$ as a subgroup such that $n \leq m_A(e)$. Also X contains $A_{(n)}, n \in \mathbb{N}$ as a subgroup for all $n < m_A(e)$.

Theorem 33: Suppose $\{A_i\}_{i \in I} \in AMG(X)$, the following holds

i. $(\bigcap_{i \in I} A_i)_{[n]} = \bigcap_{i \in I}(A_i)_{[n]}$
ii. $(\bigcup_{i \in I} A_i)_{[n]} = \bigcup_{i \in I}(A_i)_{[n]}$
iii. $(\bigcap_{i \in I} A_i)^{[n]} = \bigcap_{i \in I}(A_i)^{[n]}$
iv. $(\bigcup_{i \in I} A_i)^{[n]} = \bigcup_{i \in I}(A_i)^{[n]}$

Proof. (i) Suppose $D = \bigcap_{i \in I} A_i$, we have $m_D(x) = \bigwedge_{i \in I} m_{A_i}(x)$. Then,
$D_{[n]} = \{x \in X \mid m_D(x) \leq n\}$
$= \left\{x \in X \mid (\bigwedge_{i \in I} m_{A_i}(x)) \leq n\right\}$
$= \left\{x \in X \mid \bigwedge_{i \in I}\left(m_{A_i}(x)\right) \leq n\right\} = \bigcap(A_i)_{[n]_{i \in I}}$
Hence, $(\bigcap_{i \in I} A_i)_{[n]} = \bigcap_{i \in I}(A_i)_{[n]}$.
(ii) – (iv) follows similarly.

Theorem 34: Let $\{A_i\}_{i \in I}$ be a class of antimultigroup of X. For $n \geq m_{A_i}(e)$

i. X contains $\bigcap_{i \in I}(A_i)_{[n]}$ as subgroup.
ii. X contains $\bigcup_{i \in I}(A_i)_{[n]}$ as subgroup if $\{A_i\}_{i \in I}$ have sup/inf assuming chain.

Proof. (i) Suppose $D = \bigcap_{i \in I}(A_i)$, then $m_D(x) = \bigwedge_{i \in I} m_{A_i}(x)$. Let $e \in D_{[n]}$ since $D_{[n]} \neq \emptyset$, it follows that $m_D(e) = m_D(xx^{-1}) = \bigwedge_{i \in I} m_{A_i}(xx^{-1}) \leq \bigwedge_{i \in I} m_{A_i}(x) \leq n$.
Let $x, y \in X$, then we have
$m_D(xy) = \bigwedge_{i \in I} m_{A_i}(xy) \leq n$
$\leq \bigwedge_{i \in I}\left(m_{A_i}(x) \vee m_{A_i}(y)\right) \leq n$
$= \bigwedge_{i \in I} m_{A_i}(x) \leq n \vee \bigwedge_{i \in I} m_{A_i}(y) \leq n$
$= \left(m_D(x) \vee m_D(y)\right) \leq n$
implies that $m_D(x) \leq n$ and $m_D(y) \leq n$. So $xy \in (\bigcap_{i \in I} A_i)_{[n]}$.
Consequently,
$m_D(xy^{-1}) = \bigwedge_{i \in I} m_{A_i}(xy^{-1}) \leq n$
$\leq \bigwedge_{i \in I}\left(m_{A_i}(x) \vee m_{A_i}(y)\right) \leq n$
$= \left(m_D(x) \vee m_D(y)\right) \leq n$
So, $xy^{-1} \in (\bigcap_{i \in I} A_i)_{[n]}$. Hence, X contains $(\bigcap_{i \in I} A_i)_{[n]}$ as subgroup which follows that X contains $\bigcap_{i \in I}(A_i)_{[n]}$ as subgroup of by proposition 5.6.
(ii) Proof follows from (i)

Corollary 35: Let $\{A_i\}_{i \in I}$ be a class of antimultigroup of X. For $n \leq m_{A_i}(e)$

i. X contains $\bigcup_{i \in I}(A_i)^{[n]}$ as subgroup.
ii. X contains $\bigcup_{i \in I}(A_i)^{[n]}$ as subgroup of if $\{A_i\}_{i \in I}$ have sup/inf assuming chain.

Proposition 36: Let $A, B \in AMG(X)$. Thus X contains $(A + B)_{[n]}, n \in \mathbb{N}$ as subgroup $\forall\ n \geq m_A(e)$, where X has e as its identity element.

Proof: Let $x, y \in (A + B)_{[n]} \Rightarrow m_{A+B}(x) \leq n$ and $m_{A+B}(y) \leq n$. If $A, B \in AMG(X)$ then
$m_{A+B}(xy^{-1}) = [m_A(xy^{-1}) + m_B(xy^{-1})]$
$\leq \left[\left(m_A(x) \vee m_A(y)\right) + \left(m_B(x) \vee m_B(y)\right)\right]$
$\leq \left[\left(m_A(x) + m_B(x)\right) \vee \left(m_A(y) + m_B(y)\right)\right]$
$\leq \left[\left(m_{A+B}(x)\right) \vee \left(m_{A+B}(y)\right)\right] \leq n$
$= m_{A+B}(x) \leq n \vee m_{A+B}(y) \leq n$
Thus, $x, y \in (A + B)_{[n]} \Rightarrow xy^{-1} \in (A + B)_{[n]}$. Hence X contains $(A + B)_{[n]}$ as subgroup.

Corollary 37: Let $A, B \in AMG(X)$. Thus X contains $(A + B)^{[n]}, n \in \mathbb{N}$ as subgroup $\forall\ n \leq m_A(e)$, where X has e as its identity element.

Proposition 38: Consider $A, B \in AMG(X)$ such that $B \subseteq A$. Then X contains $(A - B)_{[n]}, n \in \mathbb{N}$ as subgroup $\forall\ n \geq m_A(e)$, where X has e as its identity element.

Proof: Let $x, y \in (A - B)_{[n]} \Rightarrow m_{A-B}(x) \leq n$ and $m_{A-B}(y) \leq n$. If $A, B \in AMG(X)$ then
$m_{A-B}(xy^{-1}) = [m_A(xy^{-1}) - m_B(xy^{-1})]$
$\leq \left[\left(m_A(x) \vee m_A(y)\right) - \left(m_B(x) \vee m_B(y)\right)\right]$
$\leq \left[\left(m_A(x) - m_B(x)\right) \vee \left(m_A(y) - m_B(y)\right)\right]$
$\leq \left[\left(m_{A-B}(x)\right) \vee \left(m_{A-B}(y)\right)\right] \leq n$
$= m_{A-B}(x) \leq n \vee m_{A-B}(y) \leq n$
Thus, $x, y \in (A - B)_{[n]} \Rightarrow xy^{-1} \in (A - B)_{[n]}$. Hence, X contains $(A - B)_{[n]}$ as subgroup.

Corollary 39: Let $A, B \in AMG(X)$. Then X contains $(A - B)^{[n]}, n \in \mathbb{N}$ as subgroup $\forall\ n \leq m_A(e)$, where X has e as its identity element.

CONCLUSION

This paper pushes the boundaries of antimultigroup by introducing a groundbreaking extension of comultiset to antimultigroup, unlocking new avenues for research. We delve into the uncharted territory of cuts in antimultigroup, uncovering novel insights and paving way for future exploration. Moreover we identify a promising direction for further investigation: the integration of normal submultigroup concept into antimultigroup, holding potential for revolutionary breakthrough.

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