FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 9 No. 2, February, 2025, pp 75 - 82
DOI: https://doi.org/10.33003/fjs-2025-0902-2676

EXPANSION METHOD FOR SOLVING FRACTIONAL ORDER HIV/AIDS MODEL

***Odetunde, O. and Ibrahim, M. O.**

Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Kwara State, Nigeria.

*Corresponding authors' email: odetunde.o@unilorin.edu.ng

ABSTRACT

This work focused on application of non-integer order derivative to solve a system of differential equations governing a nonlinear HIV/AIDS model. The model was formulated based on the efficacy of the administered anti-retrovirus therapy (ART) in abating the destructive effect of the exponential growth of the causative virus, together with time effect analysis of the impact of the drug on both healthy and infected cells within the host. Two of the most important antibodies were considered in the development of the model, the Clusters of differentiation 4 (CD4) cells and Macrophages. Their impact on the overall metabolic system of healthy and HIV infected human are modeled mathematically as a system of nonlinear differential equation. Some basic definitions of known fractional operators were adopted and applied to the model equation. The Caputo fractional derivative equivalent of the model equations was methodically solved using the expansion method of solution, numerically simulated using the Maple 18 in-built standard Runge-Kutta order 4 method, and the graph was plotted for various values of α from $0.5 - 1$. The result revealed the pattern of each compartments within the time frame, and it can be deduced from the graph that early discovery of the infection together with therapy can significantly lower the exponential growth rate of the virus, which in turn will culminate in healthy lifestyle for the carrier.

Keywords: ART, Macrophages, Caputo, Fractional derivative, Simulate, Runge-Kutta

INTRODUCTION

Human Immunodeficiency Virus (HIV) spreads within host by incorporating its gene into host's Deoxyribonucleic acid (DNA) after contact. HIV has several modes of transmission, which include, unprotected sexual intercourse, unscreened blood transfusion, vertical and perinatal transmission among others. The latter stage of HIV infection is Acquired Immune Deficiency Syndrome (AIDS), a stage where the immune system is already overpowered by the virus and rendered almost inactive. Hence, the term HIV/AIDS is mostly used to describe the infection and the resulting disease from this class of highly infectious retrovirus class. It was stated that since the emergence of HIV in the late 1970s, it remains among the top diseases with high fatality cases yearly (Burg *et al.*, 2009). AIDS constitutes a major disease with zero probability of medical cure according to Center for Disease Control and Prevention (CDC, 2015). The virus attacks the defense system of the body by fighting the immune system, such as the CD4 − *cells*, macrophages etc. Its constant and continuous attack on the immune system weakens the defense cells, thereby leaving the body system vulnerable to external attacks of opportunistic infections. Research shows that the quality of life in individual depends on the concentration these antibodies, and a healthy individual has between 500 - 1500 CD4 cells. Similarly, another important defense cells that receives same attack from this virus is macrophages. Macrophages are large white blood that exists in nearly all tissues and are produced by monocyte differentiation (Science Daily, 2010)

HIV/AIDS remains among the infections without curative measures. With several years of extensive research, there is no prescribed therapy that can effectively remove the virus from the system of the infectious, although there are medicines that can control HIV and prevent complications. World Health Organization (WHO, 2024) established that HIV infection currently has no effective treatment strategies, but, with early adoption of anti-retrovial therapy (ART) and adequate preventive measures against further opportunistic infection by the infected, it can be properly managed with normal quality of life guaranteed. Further research is ongoing from all quarters, including epidemiologist, biologist, pharmacist and medical practitioners on making HIV a treatable infection (Mukandavire *et al.* (2009), Elaiw et al. (2013)). Mathematical epidemiologist are also involved in the pursuit of this common goal to eradicate the menace of HIV, through model building, formulation and analysis to study transmission pathway and possible means of curbing further spread. Several mathematical model on the dynamics of HIV/AIDS were formulated and analyzed with each targeting some specific objectives. Akinyemi *et al.*, (2016) formulated a staged-progression HIV/AIDS model with screening and drug resistance. They obtained the qualitative properties of their model. Furthermore, their analysis of the equilibrium points established the region of stability for each of the obtained equilibrium points. Odetunde *et al.*, (2018) presented the optimal control analysis of ART treatment therapy on the immune cells. Their result established that early detection of the infection and effective therapeutic treatment can affords the carrier a healthy long and normal life. Ogunniran and Ibrahim (2023) analyzed a superinfection HIV model by considering two strains of the causative virus. They applied the Elzaki transform method to obtain the semi-analytic solution of the model. Their findings showed that the concentration of the helper cells decreased with time in the presence of HIV strain and much faster when two strains are super-imposed. Other models on dynamics of HIV/AIDS exist in literature, with each model having some specific targets (Check Samuel and Carsten (2012), Elaiw (2013), Haas et al. (1997), Michele (2011), Miquel (2014), Mukandavire et al. (2009), Mushayabasa and Bhunu (2011), Naresh et al. (2009), Tripathi et al. (2007)), Yong *et al.*,(2022) etc.). However, most of these models were depicted as an integer order derivative. Further extension to these, is a model with fractional derivative, which has an advantage over the integer counterpart, to consider time fragment (or memory effect) associated with the infection within the continuous system contest. Obtaining analytic solution to a mathematical model in epidemiology is somewhat tedious or seemingly impossible in some cases due to the nonlinearity in the system of equations. Hence,

mathematical epidemiologist most times relies on the qualitative properties of the model and numerical solution to draw their inference on the dynamics of the model. Usman *et al.,* (2023) adopted non-standard finite difference (NSFD) method to solve a mathematical model on COVID-19 with fear of infection. They compared their numerical scheme with standard Runge-Kutta 4 (RK4) method and concluded that their scheme perform as good as RK4.

Fractional order derivative is more generalized and has several advantages over the integer order. Some researchers have analyzed an HIV/AIDS model with fractional derivative. Aslam *et al.,* (2021) considered a HIV/AIDS model by stratifying human population into four epidemiological classes using fractional derivative. They applied ABC-fractional derivative on their model and used Lagrange interpolation to simulate their model numerically. In their own work, Tanvi *et al.*,(2021) applied fractional order using Caputo definition to an HIV-TB coinfection model. By distinct analysis of both infections, their study established the crucial role of memory effects on the dynamics of the coinfection model, and fractional order derivative is significant in modeling the spread of the infection. Yong *et al.*,(2022) also considered the influence of uncertainty in the initial data for HIV-1 infection using a non-integer derivative model. They applied the fuzzy Laplace Adomian decomposition method, and they concluded that fuzziness with non-integral operators establishes the global properties of the model. Salah *et al.*,(2024) considered the analysis of a HIV/AIDS dynamics with conformable fractional order model, their results established that conformable non-integer operator models have a physiological advantage over the conventional integer derivative.

MATERIALS AND METHODS

Model Formulation

In this work, the focus of our research is on the dynamics of HIV infection within the immune system of the host using fractional order derivative. The aim is to consider the memory effect on the multiplicative tendency of the virus. We consider impact of HIV on two of the major antibodies of human, CD4-T cells and the Macrophages. The population of healthy CD4 cells present at any time t is represented as $x(t)$ and its rate of change with time is given as $\frac{dx}{dt}$. The infected CD4 cells population is denoted as x_1. Similarly, the population of healthy macrophages present at any time t is represented as $y(t)$, its rate of change with time is given as $\frac{dy}{dt}$ and the infected macrophages population is denoted as y_1. The viral load present at any time t is denoted as $v(t)$. Normal CD4 cells and Macrophages are produced at the rate π_1, π_2 respectively. We assume that ART helps the body to produce more of these immune cells at the rate γ_1, γ_2 respectively. All cells dies normally at the rate d_1, d_2 for CD4 and Macrophages respectively. Healthy cells becomes infected when they come in contact with the virus at the rate β_1, β_2 respectively. Virus induce death of both cells occurs at the rate μ_1

Hence, we seek to model ART significance in cleaning off a virus contaminated CD4 cells and macrophages, its effectiveness in combating the virus and its potency in helping the immune system and the antibodies by replenishing the lost cell and macrophages with new ones.

Some of the basic assumptions made while developing this model include:

Aside from normal production of the immune cells through healthy living, ART also boost their production rate within the body of the infected

Increase in viral count in an infected individual resulted from increase in infected immune cells, that is, infected cells contribute greatly to viral count

Among other things, ART is assumed to be efficient in removing the infected cells, reducing the spread of the virus and bolster the immune system.

Quick discovery of HIV status and early commencement of ART have a positive effect on defense cells abilities to fight off other infections.

All model variables and parameters have positive initial values.

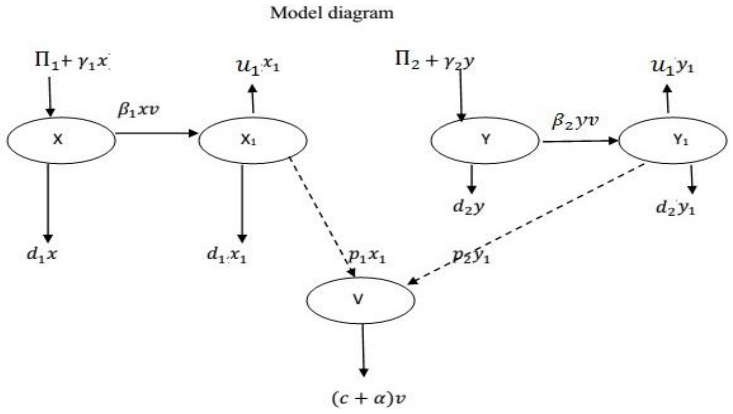

Figure 1: Model Flow Diagram

The model governing equations is:

$$\frac{dx}{dt} = \Pi_1 - d_1 x - \beta_1 xv + \gamma_1 x$$
$$\frac{dx_1}{dt} = \beta_1 xv - (d_1 + u_1)x_1$$
$$\frac{dy}{dt} = \Pi_2 - d_2 y - \beta_2 yv + \gamma_2 y$$
$$\frac{dy_1}{dt} = \beta_2 yv - (d_2 + u_1)y_1$$
$$\frac{dv}{dt} = p_1 x_1 + p_2 y_1 - (c + \alpha)v \quad (1)$$

Table 1: Table of parameter value for Model 1

Symbol	Description	Value	Source
Π_1	Normal Healthy CD4 T-cell production Rate	$0.17\mu L^{-1}$	Samuel & Carsten (2012)
Π_2	Normal Healthy Macrophages production Rate	$0.17\mu L^{-1}$	Assumed
d_1	CD4 T-cell death rate	0.01/day	Samuel & Carsten (2012)
d_2	Macrophages death rate	0.01/day	Samuel & Carsten (2012)
β_1	Healthy CD4 T-cell infection rate	$6.5 \times 10^{-6} \frac{\mu}{LV} day$	Samuel & Carsten (2012)
β_2	Healthy Macrophages infection rate	0.04	Assumed
γ_1	HAART efficacy to boost production of CD4-cell	$0.1\mu L^{-1}$	Michael et al. (1998)
γ_2	HAART efficacy to boost production of Macrophages	$0.1\mu L^{-1}$	Michael et al. (1998)
p_1	Contribution of infected T-Cells to viral load	2	Michael et al. (1998)
p_2	Contribution of infected Macrophages to viral load	3	Assumed
u_1	HAART removal rate of infected Cells & Macrophages	0.05	Assumed
c	Clearance rate of virus due to body immunity	0.034/day	Michael et al. (1998)
α	ART clearance rate of virus	0.5	Assumed

Methodology

Fractional Calculus (FC) is a generalization of integration and differentiation to non-integer order.

Among several known definitions used for general fractional differential and integral are the: Riemann-Liouville (RL), Caputo, Grunwald-Letnikov (GL), Laplace, Weyl, Fourier, Cauchy, Abel, etc. However, this study shall be restricted to the first two mentioned definitions in which emphasis shall be placed on the Caputo definition for its advantages on the initial value problem.

FC is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order (Gorenflo and Mainardi, 1997). Leibnitz, Euler, Laplace, Riemann, Liouville, Fourier, etc were the earliest Mathematician that used fractional calculus with derivatives and integrals to solve real life problem (Atanackovic and Stankovic, 2004). Fractional order differential equation has become a subject of intense study in epidemiology because it is a generalization of integer order differential equation, possesses memory term and also helps to reduce errors arising from the neglected parameters in real life modeling.

Definition 1. (Mohammed and Nemat (2013)). The RL Fractional Integral (FI) of order $\omega > 0$ for a function $f: <^{+} \rightarrow <$ is defined by

$$I_0^{\omega} f(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\phi)^{\omega-1} f(\phi) d\phi \qquad (2)$$

where $\Gamma(\omega)$ is the Euler Gamma function.

For fractional derivatives (FD), the RL definition is given by

$$\mathrm{D}_0^{\omega} f(t) = \frac{1}{\Gamma(k-\omega)} \left(\frac{d}{dt}\right)^k \int_0^t (t-\phi)^{k-\omega-1} f(\phi) d\phi \quad (3)$$

for $\omega > 0$, $k-1 < \omega < k$ and $k \in Z^{+}$, (El-Shahed and AlSaedi, 2011).

Definition 2. (El-Shahed and AlSaedi, 2011). The Caputo Fractional Derivatives (CFD), which is a sort of regularization of the RL fractional derivative is defined as:

$$\mathrm{D}_t^{\omega} f(t) = \frac{1}{\Gamma(k-\omega)} \int_0^t \frac{f^{(k)}(\phi)}{(t-\phi)^{\omega+1-k}} d\phi$$

for $\omega > 0$, $k-1 < \omega < k$ and $k \epsilon Z^{+}$ (4)

Where the function $f \in AC^{k-1}$. The initial value problem related to the above definition is

$$\left.\begin{aligned} D^{\omega} x(t) &= f(t, x(t)), \\ x(t)|_{t=0^{+}=x_0}, \end{aligned}\right\} \qquad (5)$$

where $0 < \omega < 1$ and $D^{\omega} x(t) = D_0^{\omega}$, (Javidi and Ahmad, 2014). In what follows, system (1) is re-modeled using the approach adopted by Ayoade *et al.*,(2018). The transformation is basically done by considering a discretization effect of time, among other factors, on the continuous dynamics of each equations in (1). This enables us to check and numerically compare any slight change within the system over a certain define interval.

Introducing fractional order to the system (1), using the definition of CFD of order α, $0 < \alpha < 1$, the system of equations above becomes:

$$\begin{aligned} D_t^{\alpha} x(t) &= \Pi_1 - (d_1 - \gamma_1)x - \beta_1 xv \\ D_t^{\alpha} x_1(t) &= \beta_1 xv - (d_1 + u_1)\, x_1 \\ D_t^{\alpha} y(t) &= \Pi_2 - (d_2 - \gamma_2)y - \beta_2 yv \\ D_t^{\alpha} y_1(t) &= \beta_2 yv - (d_2 + u_1)\, y_1 \\ D_t^{\alpha} v(t) &= p_1 x_1 + p_2 y_1 - (c + \alpha)v \end{aligned} \qquad (6)$$

With initial conditions $x(0) = x_0$, $x_1(0) = x_{1,0}$, $y(0) = y_0$, $y_1(\delta) = y_{1,0}$, $v(0) = v_0$

RESULTS AND DISCUSSION

The approach of Atanackovic and Stankovic (2004) which was adopted by Mohammad and Nemat (2013), Ayoade *et al.*,(2018) will be adopted for system (6). This approach is straight-forward and well-suited to both linear and nonlinear FD.

We recall (4), replacing ω with α and using $k = 1$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\phi)^{-\alpha} f'(\phi) d\phi \qquad (7)$$

Integrating the resulting equation by part using,

$u = f^{1}(\varphi)$; $du = f^{11}(\varphi)d\varphi$

$dv = (t-\varphi)^{-\alpha}$; $v = -(t-\varphi)^{1-\alpha}$

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left\{ f'(\phi)(-(t-\phi)^{1-\alpha})|_0^t - \int_0^t -(t-\phi)^{1-\alpha} f''(\phi) d\phi \right\} \qquad (8)$$

$$= \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{f'(0)}{t^{\alpha-1}} + \int_0^t (t-\phi)^{1-\alpha} f''(\phi) d\phi \right\} \qquad (9)$$

Using binomial expansion for

$$(t-\phi)^{1-\alpha} = t^{1-\alpha}\left(1-\frac{\phi}{t}\right)^{1-\alpha}$$
$$= t^{1-\alpha}\sum_{p=0}^{\infty}(-1)^p\begin{pmatrix}1-\alpha\\ p\end{pmatrix}\left(\frac{\phi}{t}\right)^p$$
$$= t^{1-\alpha}\sum_{p=0}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\left(\frac{\phi}{t}\right)^p \tag{10}$$

where $\left|\frac{\phi}{t}\right| < 1$ and

$$\begin{pmatrix}\gamma\\ k\end{pmatrix}(-1)^k = \frac{\Gamma(k-\gamma)}{\Gamma(-\gamma)k!}$$

Substituting (10) into (9) with some algebraic simplifications to obtain;

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left\{\frac{f'(0)}{t^{\alpha-1}} + \frac{1}{t^{\alpha-1}}\int_0^t\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\left(\frac{\phi}{t}\right)^p f''(\phi)d\phi\right\}$$

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left\{\frac{f'(0)}{t^{\alpha-1}} + \frac{1}{t^{\alpha-1}}\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!t^p}\int_0^t\phi^p f''(\phi)d\phi\right\} \tag{11}$$

Further simplification of the integral part of (11) by techniques of integration in elementary calculus gives

$$\int_0^t\phi^p f''(\phi)d\phi = \left[\phi^p f'(\phi)|_0^t - \int_0^t p\phi^{p-1}f'(\phi)d\phi\right]$$

$$= t^p f'(t) - p\left\{\phi^{p-1}f(\phi)|_0^t - \int_0^t(p-1)\phi^{p-2}f(\phi)d\phi\right\}$$

$$= t^p f'(t) - pt^{p-1}f(t) + p(p-1)\int_0^t\phi^{p-2}f(\phi)d\phi, \qquad p\geq 2 \tag{12}$$

Substituting (12) into (11)

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left\{\frac{f'(0)}{t^{\alpha-1}} + \frac{1}{t^{\alpha-1}}\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!t^p}\left[t^p f'(t) - pt^{p-1}f(t) + p(p-1)\int_0^t\phi^{p-2}f(\phi)d\phi\right]\right\}$$

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left\{\frac{f'(t)}{t^{\alpha-1}}\left[1+\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\right]\right.$$
$$\left. - \left[\frac{\alpha-1}{t^{\alpha}}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\left(\frac{f(t)}{t^{\alpha}} + \frac{V_p(f)(t)}{t^{p-1+\alpha}}\right)\right]\right\}$$

$$V_p(f)(t) = -(p-1)\int_0^t\tau^{p-2}f(\tau)d\tau, \quad p=2,3,\cdots \tag{13}$$

with the following properties

$$\frac{d}{dt}V_p(f) = -(p-1)t^{p-2}f(t), \quad p=2,3,\cdots \tag{14}$$

we approximate $D^{\alpha}f(t)$ by using m terms in sums appearing in equation (13) as follows

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left\{\frac{f'(t)}{t^{\alpha-1}}\left[1+\sum_{p=1}^{m}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\right]\right.$$
$$\left. - \left[\frac{\alpha-1}{t^{\alpha}}f(t) + \sum_{p=2}^{m}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\left(\frac{f(t)}{t^{\alpha}} + \frac{V_p(f)(t)}{t^{p-1+\alpha}}\right)\right]\right\} \tag{15}$$

(15) can compactly be written as:

$$D^{\alpha}f(t) = \Upsilon(\alpha,t,m)f'(t) + \Phi(\alpha,t,m)f(t) + \sum_{p=2}^{m}A(\alpha,t,m)\frac{V_p(f)(t)}{t^{p-1+\alpha}} \tag{16}$$

where

$$\Upsilon(\alpha,t,m) = \frac{1+\sum_{p=1}^{m}\frac{\Gamma(p-1+\alpha)}{\Gamma(1-\alpha)t^{\alpha-1}}}{\Gamma(1-\alpha)t^{\alpha-1}}$$

$$\Phi(\alpha,t,m) = R(\alpha,t) + \sum_{p=2}^{m}\frac{A(\alpha,t,p)}{t^{\alpha}}$$

$$A(\alpha,t,p) = \frac{\Gamma(p-1+\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha-1)p!}$$

$$R(\alpha,t) = \frac{1-\alpha}{t^{\alpha}\Gamma(1-\alpha)}$$

By setting

$$\left.\begin{aligned}
&\Psi_1(t) = x(t), \quad \Psi_p(t) = V_p(x)(t); \quad p = 2, 3, \cdots, m \\
&\Psi_{m+1}(t) = x_1(t), \quad \Psi_{m+p}(t) = V_p(x_1)(t); \quad p = 2, 3, \cdots, m \\
&\Psi_{2m+1}(t) = y(t), \quad \Psi_{2m+p}(t) = V_p(y)(t); \quad p = 2, 3, \cdots, m \\
&\Psi_{3m+1}(t) = y_1(t), \quad \Psi_{3m+p}(t) = V_p(y_1)(t); \quad p = 2, 3, \cdots, m \\
&\Psi_{4m+1}(t) = v(t), \quad \Psi_{4m+p}(t) = V_p(v)(t); \quad p = 2, 3, \cdots, m
\end{aligned}\right\} \tag{17}$$

system of equations in (10) can be re-written using (16) and (17) as

$$\left.\begin{aligned}
&\Upsilon(\alpha,t,m)\Psi_1'(t) + \Phi(\alpha,t,m)\Psi_1(t) + \sum_{p=2}^{m} A(\alpha,t,m)\frac{\Psi_p(t)}{t^{p-1+\alpha}} \\
&= \Pi_1 - (d_1 - \gamma_1)\Psi_1(t) \not\beta \ {}_1\Psi_1(t)\Psi_{4m+1}(t) \\
&\Upsilon(\alpha,t,m)\Psi_{m+1}'(t) + \Phi(\alpha,t,m)\Psi_{m+1}(t) + \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{m+p}(t)}{t^{p-1+\alpha}} \\
&\not\beta \ {}_1\Psi_1(t)\Psi_{4m+1}(t) - (d_1 + u_1)\Psi_{m+1}(t) \\
&\Upsilon(\alpha,t,m)\Psi_{2m+1}'(t) + \Phi(\alpha,t,m)\Psi_{2m+1}(t) + \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{2m+p}(t)}{t^{p-1+\alpha}} \\
&= \Pi_2 - (\delta - \gamma_2)\Psi_{2m+1}(t) \not\beta \ {}_2\Psi_{2m+1}(t)\Psi_{4m+1}(t) \\
&\Upsilon(\alpha,t,m)\Psi_{3m+1}'(t) + \Phi(\alpha,t,m)\Psi_{3m+1}(t) + \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{3m+p}(t)}{t^{p-1+\alpha}} \\
&\not\beta \ {}_2\Psi_{2m+1}(t)\Psi_{4m+1}(t) - (d_1 + u_1)\Psi_{3m+1}(t) \\
&\Upsilon(\alpha,t,m)\Psi_{4m+1}'(t) + \Phi(\alpha,t,m)\Psi_{4m+1}(t) + \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{4m+p}(t)}{t^{p-1+\alpha}} \\
&= p_1\Psi_1(t) + p_2\Psi_{2m+1}(t) - (c + \alpha)\Psi_{4m+1}(t)
\end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned}
\Psi_p(t) &= -(p-1)\int_0^t \tau^{p-2}\Psi_1(\tau)d\tau \\
\Psi_{m+p}(t) &= -(p-1)\int_0^t \tau^{p-2}\Psi_{m+1}(\tau)d\tau \\
\Psi_{2m+p}(t) &= -(p-1)\int_0^t \tau^{p-2}\Psi_{2m+1}(\tau)d\tau \\
\Psi_{3m+p}(t) &= -(p-1)\int_0^t \tau^{p-2}\Psi_{3m+1}(\tau)d\tau \\
\Psi_{4m+p}(t) &= -(p-1)\int_0^t \tau^{p-2}\Psi_{4m+1}(\tau)d\tau
\end{aligned}\right\} \tag{19}$$

Using the above, we re-write system of equations (18) as:

$$\left.\begin{aligned}
\Psi_1'(t) &= \frac{1}{\Upsilon(\alpha,t,m)}\left\{\Pi_1 - (d_1 - \gamma_1)\Psi_1(t) \not\beta \ {}_1\Psi_1(t)\Psi_{4m+1}(t) - \Phi(\alpha,t,m)\Psi_1(t) - \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_p(t)}{t^{p-1+\alpha}}\right\} \\
\Psi_{m+1}'(t) &= \frac{1}{\Upsilon(\alpha,t,m)}\left\{\not\beta \ {}_1\Psi_1(t)\Psi_{4m+1}(t) - (d_1 + u_1)\Psi_{m+1}(t) - \Phi(\alpha,t,m)\Psi_{m+1}(t) - \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{m+p}(t)}{t^{p-1+\alpha}}\right\} \\
\Psi_{2m+1}'(t) &= \frac{1}{\Upsilon(\alpha,t,m)}\left\{\Pi_2 - (\delta - \gamma_2)\Psi_{2m+1}(t) \not\beta \ {}_2\Psi_{2m+1}(t)\Psi_{4m+1}(t) - \Phi(\alpha,t,m)\Psi_{2m+1}(t) - \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{2m+p}(t)}{t^{p-1+\alpha}}\right\} \\
\Psi_{3m+1}'(t) &= \frac{1}{\Upsilon(\alpha,t,m)}\left\{\not\beta \ {}_2\Psi_{2m+1}(t)\Psi_{4m+1}(t) - (d_1 + u_1)\Psi_{3m+1}(t) - \Phi(\alpha,t,m)\Psi_{3m+1}(t) - \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{3m+p}(t)}{t^{p-1+\alpha}}\right\} \\
\Psi_{4m+1}'(t) &= \frac{1}{\Upsilon(\alpha,t,m)}\left\{p_1\Psi_1(t) + p_2\Psi_{2m+1}(t) - (c + \alpha)\Psi_{4m+1}(t) - \Phi(\alpha,t,m)\Psi_{4m+1}(t) - \sum_{p=2}^{\infty} A(\alpha,t,m)\frac{\Psi_{4m+p}(t)}{t^{p-1+\alpha}}\right\}
\end{aligned}\right\} \tag{20}$$

with initial conditions

$$\Psi_1(\delta) = x_0, \quad \Psi_p(\delta) = -\frac{p-2}{2}\delta t^{t-1}x_0$$
$$\Psi_{m+1}(\delta) = x_{1,0}, \quad \Psi_{m+p}(\delta) = -\frac{p-2}{2}\delta t^{t-1}x_{1,0}$$
$$\Psi_{2m+1}(\delta) = y_0, \quad \Psi_{2m+p}(\delta) = -\frac{p-2}{2}\delta t^{t-1}y_0$$
$$\Psi_{3m+1}(\delta) = y_{1,0}, \quad \Psi_{3m+p}(\delta) = -\frac{p-2}{2}\delta t^{t-1}y_{1,0}$$
$$\Psi_{4m+1}(\delta) = v_0, \quad \Psi_{4m+p}(\delta) = -\frac{p-2}{2}\delta t^{t-1}v_0$$

Numerical Examples

Using parameter values in Table 1 for different values of fractional order *α,* and the $M-term$ set as 10, we obtain the numerical solution of the fractional order HIV/AIDS model 2 as follows:

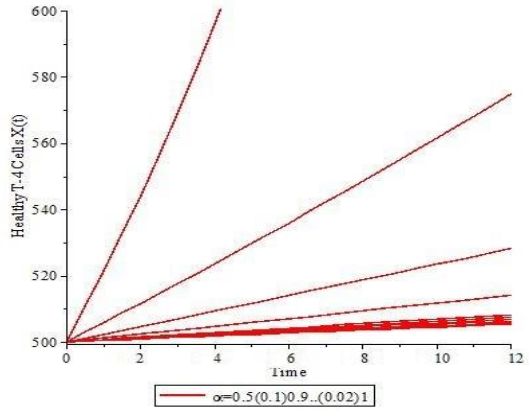

Figure 2: Simulation for Healthy CD4 cells

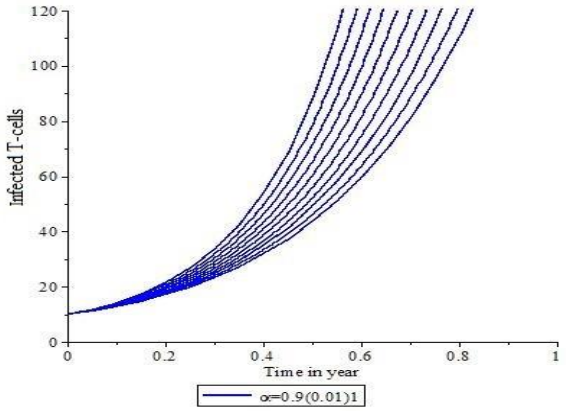

Figure 3: Simulation for Infected CD4 cells

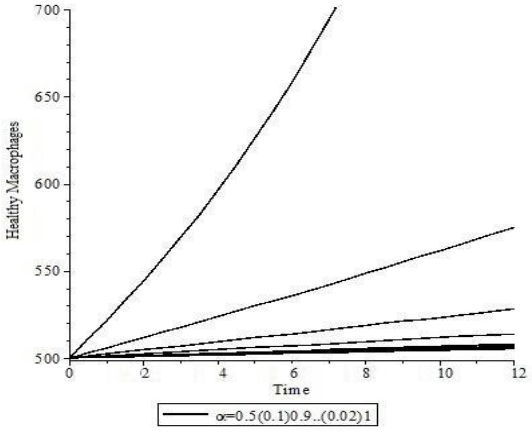

Figure 4: Simulation for Healthy Macrophages

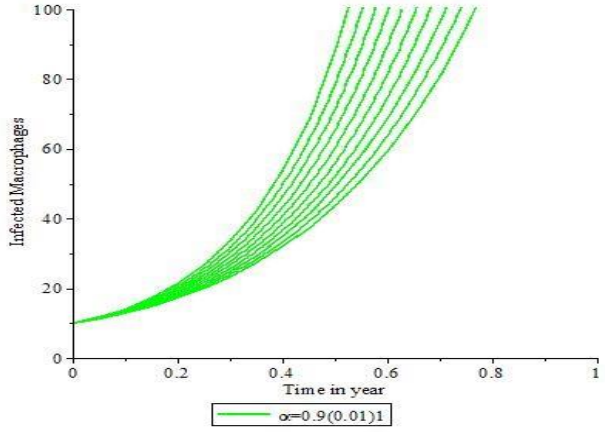

Figure 5: Simulation for Infected Macrophages

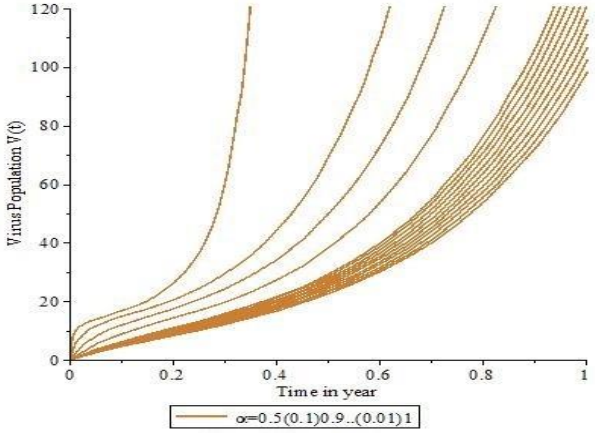

Figure 6: Simulation for Virus Population

Discussion

The fractional order differential equation for the model was numerically simulated for different values of α. Figure 2 showcased the graph of healthy CD4 cells with several fractional values. Regardless of the initial starting value, the obtained result showed a similar behaviour for the model. This establishes the uniqueness of solution of the fractional order model. The graph also showed similarities with the integer order (that is case $\alpha = 1$). In the same vein, Figure 3, 4 and 5 depict the dynamics of infected CD4 cells, healthy and infected macrophages population changes respectively. The graph obtained showed that infected cells increased but not at the expected exponential rate. This is not unconnected with the early detection and quick intervention of ART in removing the infected cells, thereby reducing the viral load within the cells. Similarly, healthy macrophages depleted slowly for certain values of α while it approaches common point for other $\alpha-$values. Infected macrophages has similar behaviour compared to x_2 compartment. This is because their dynamics looks similar, and the impact of ART is felt in the same manner. Virus population increases in all graphs, which shows the powerful multiplicative tendency of the virus. However, in the presence of ART (when c impact is increased), it is expected that the growth rate of the virus should reduce.

CONCLUSION

In this work, a mathematical model for the dynamics of viral load multiplication within the host immune system was formulated using fractional order nonlinear model. Caputo fractional order was implemented on the model and the result was qualitatively analyzed. The fractional order solutions for the models expatiate on the effect of time lapse (time delay) in the transmission dynamics of the infection. The graphical solutions ascertain that early discovery and treatment helps in: (i) altering the exponential growth of the virus and its effect on the antibodies (ii) reducing the spread of the infection within the cells if ART is administered timely, and; (iii) prolonging the life span of the defense cells. The variation in the graph as the value of the Caputo derivative α is slightly altered showed the impact of untimely discovery in the transmission dynamics of the infection

REFERENCES

Akinyemi, S. T., Ibrahim, M. O., Moses, M. O., Edogbanya, H. O. and Odetunde, O. (2016). Mathematical analysis of a staged progression HIV/AIDS model with screening and drug resistance ABACUS. 43(2) 162 179. Published by the Mathematical Association of Nigeria

Aslam, M., Murtaza, R., Abdeljawad, T., Rahman, G. U., Khan, A., Khan, H. and Gulzar, H. (2021). A fractional order HIV/AIDS epidemic model with Mittag-Leffler kernel. Adv Differ Equ 2021, 107 (2021). https://doi.org/10.1186/s13662-021-03264-5

Atanackovic, T. M. and Stankovic, B. (2004). An Expansion Formula for Fractional Derivatives and its Application. Fractional Calculus and Applied Analysis, an International Journal for Theory and Applications 7(3): 366-378 ISSN 1311-0454

Ayoade, A.A., Ibrahim, M.O. and Odetunde, O. (2018): Analytical Solution of a Fractional Order Cholera Model. Nigerian Journal of Scientific Research, 17 (2) 158 164. Published by Mathematics Department, Ahmadu Bello University, Zaria

Burg, D., Rong, L. Neumann, A. U. and Dahari, H. (2009). Mathematical modeling of viral kinetics under immune control during primary HIV-1 infection. Journal of Theoretical Biology, 259:751-759

Center for Disease Control, CDC (2016). HIV/AIDS; Myths, origin and Prevention. Retrieved from www.cdc.gov>CDC>HIV/AIDS;>Myths, origin and & Prevention.

Centers for Disease Control and Prevention (2015). HIV Among Gay and Bisexual Men. Retrieved from www.cdc.gov>HIV Among Gay and Bisexual Men

El-Shahed, M. and Alsaedi, A. (2011). The Fractional SIRC Model and Influenza A. *Mathematical Problems in Engineering*, 9 pages, Article ID 480378. http://dx.doi.org/10.1155/2011/480378

Elaiw A. M., Abukwaik R. M. and Alzahrani E. O. (2013). Global Stability of HIV infection of CD4+ T-Cells and Macrophages with CTL Immnune Response and Distributed Delay. Computational and Mathematical Methods in Medicine. Article ID 653204, http://dx.doi.org/10.1155/2013/653204 Hindawi Publishing Cooperation

Gorenflo, R., and Mainardi, F. (1997). Fractional Calculus: Integral and Differential Equations of Fractional Order in Mainardi F. and Carpinteri A. (Eds), *Fractals and Fractional Calculus in Continuum Mechanics*, volume 378 of *CISM Courses and Lectures*, pages 223-276. Springer-Verlag, Wien and New York.

Haas, G., Hosmalin, A., Hadida, F., Duntze, J., Debre, P. and Autran, B. (1997). Dynamics of HIV variants and specific cytotoxic T-cell recognition in nonprogressors and progressors. Immunol. Lett. 57:63-68

Javidi, M. and Ahmad, B. (2014). A Study of a Fractional-Order Cholera Model. *Applied Mathematics & Information Sciences*, 8(5): 2195-2206. http://dx.doi.org/10.12785/amis/080513

Michele F. Karim D., and Angelita R. (2011). CD4+ T Cell Depletion in Human Immunodeficiency Virus (HIV) Infection: Role of Apoptosis. Viruses 2011, 3, 586-612; https://doi.org/10.3390/v3050586

Michael R., Maryann D., David M. H., Stuart F., Rakesh K. J. and PAUL J. R. (1998). Increased rates of CD4+ and CD8+ T lymphocyte turnover in simian immunodeficiency virus-infected macaques. Proc. Natl. Acad. Sci. USA 95, 6388-6393, Medical Sciences

Miquel Porta (2014). *A Dictionary of Epidemiology*. Oxford University Press, Sixth Edition

Mohammad Javidi and Nemat Nyamoradi (2013). Numerical Behavior of a Fractional Order HIV/AIDS Epidemic Model. World Journal of Modelling and Simulation, 9: 139-149 ISSN 1 746-7233, England, UK

Mukandavire Z., Garira W., and Tchuenche J.M. (2009). Modelling effects of public health educational campaigns on HIV/AIDS transmission dynamics. Applied Mathematical

Modelling 33: 2084-2095 journal homepage: www.elsevier.com/locate/apm

Mushayabasa S., and C. P. Bhunu (2011). Modeling HIV Transmission Dynamics among Male prisoners in Sub-Saharan Africa. IAENG International Journal of Applied Mathematics, 41(1), IJAM41107

Naresh Ram, Agraj Tripathi and Dileep Sharma (2009). Modelling and analysis of the spread of AIDS epidemic with immigration of HIV infectives. Mathematical and Computer Modelling, 49(56), 880-892 http://www.sciencedirect.com/science/article/pii/S0895717708003336

Odetunde, O., Lawal, J., Edogbanya, H.O., and Ibrahim, M. O. (2018). Optimal control analysis of effect of HAART on immune cells against HIV infection. Advances in Mathematics: Scientific Journal 7 (2018), no.2, 89107

Ogunniran, M. & Ibrahim, M. O. (2023). Application of Elzaki Transform Method for Solving and Interpreting HIV Superinfection Model. International Journal of Applied Mathematics and Informatics. 17. 1-5. https://doi/org/10.46300/91014.2023.17.1

Tanvi, A,. Aggarwal R. and Raj Y.A. (2021). A fractional order HIV-TB co-infection model in the presence of exogenous reinfection and recurrent TB. Nonlinear Dyn. 2021;104(4):4701-4725. https://doi/org/10.1007/s11071-021-06518-9 . Epub 2021 May 28. PMID: 34075277; PMCID: PMC8159726

Tripathi, A., Naresh, R. and Sharma,D.(2007). Modelling the effect of screening of unaware infectives on the spread of HIV infection. Applied Mathematics and Computation, 184: 1053-1068

Salah, E. Y., Sontakke, B., Abdo, M. S., Shatanawi, W., Abodayeh, K. and Albalwi, M. D.(2024). Conformable Fractional-Order Modeling and Analysis of HIV/AIDS Transmission Dynamics. International Journal of Differential Equations, 2024, 1958622, 16 pages, 2024. https://doi.org/10.1155/2024/195862

Samuel A. and Carsten M. (2012). Modelling the Course of an HIV Infection: Insights from Ecology and Evolution. Viruses 2012, 4, 1984-2013; https://doi/org/10.3390/v4101984

Science daily (2010). American Physiological Society. http://www.sciencedaily.com/releases/2010/08/100826141232.htm

Usman I. G., Ibrahim M. O., Isah B. Y., Lawal N., & Akinyemi S. T. (2023). Application of non-standard finite difference method on covid-19 mathematical model with fear of infection. *FUDMA JOURNAL OF SCIENCES*, *7*(4), 357 - 368. https://doi.org/10.33003/fjs-2023-0704-1933

Yong, W., Shabir, A., Aman U. and Kamal S. (2022). Study of the Fractional-Order HIV-1 Infection Model with Uncertainty in Initial Data. Hindawi: Mathematical Problems in Engineering Volume 2022, Article ID 7286460, 16 pages https://doi.org/10.1155/2022/7286460

World Health Organization (2024). HIV and AIDS, Key facts. Last Updated, 13th July, 2023, accessed on 22nd July, 2024. https://www.who.int/news-room/fact-sheets/detail/hiv-aids