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# THE ANALYTICAL SOLUTION TO UNSTEADY FLOW OF DUSTY BINGHAM FLUID BETWEEN TWO PARALLEL RIGA PLATES WITH RADIATION EFFECTS

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**ABSTRACT**
The study at hand is to delve into the analytical solution for the unsteady flow of dusty Bingham fluid between two parallel Riga plates with radiation effect. The plates, which includes the upper Riga plate and lower Riga plates are assumed to be immovable hence stationary. The velocity of the fluid is assumed to be identical as that of the velocity of the particle (dusty fluid). The governing equations for the problem is been modeled as dimensionless non-homogeneous PDE (partial differential equation) with non-homogeneous boundary condition. The equations are then solved analytically with the method of Eigenfunction expansion. The impacts of numerous relevant parameters for the velocity and temperature field are scrutinized in details with the use of graphs. Furthermore, graphical explanations are provided on the factors of the friction and the influence of the above parameters on the pattern of the flow together with the shear stress and Nusselt number. Finally, both quantitative and qualitative comparisons are presented.

**Keywords**: Riga plates, Bingham dusty fluid, Non-homogeneous partial differential equation, Analytical method, Radiation effect, Eigenfunction expansion

## INTRODUCTION
Bingham fluid can be best described as a non-Newtonian viscoplastic fluid which exist as a yield strength that must be transcend on before a fluid flow will take place. Bingham fluid is tagged to Bingham (1916) who introduced its mathematical method. One of its best examples is toothpaste which cannot be easily ejected unless a certain force is exerted on its tube or a peanut butter that cannot be removed from its case unless with the help of spoon most times before it can be removed.
Bingham fluid is being utilized in different fields such as mechanical and geological materials aerospace engineering, mechanical engineering and definitely chemical engineering. Also, in the drilling engineering as a mathematical basis of mud flow, slurries, cement, grease, chocolate are all example of Bingham fluid.
Also, dusty fluids can exists in fields like fluidization, environmental pollution, polymer, gas cooling system, combustion, petroleum, polluted soil, polluted air, polluted water, agriculture, purifying of crude oil and systems of dye. Some studies related to heat transfer and heat flow of dusty fluids between parallel plates are extremely useful in improving the design and operation of many industrial and engineering devices.
More so, a Riga plate consists of electrode and permanent magnets i.e. it is a production of electromagnetic plane surface which exhibits hydrodynamic behavior in the flow of fluid instead of polarity and magnetization surface. Gailities and Leilausis (1961) can be said to be the founder of Riga plates, in order to regulate the fluid's flow the plates creates a wall paralleled Lorentz force. The electromagnetic actuator device designed by Gailities and Lielausis to control fluid flow consists of permanent magnets and a series of aligned span wise alternating electrode. The plate is designed for minimizing both the the pressure drag of submarines and friction.
Prasannakumara et. al (2019) made an investigation in the presence of convective boundary condition on the flow problem of MHD viscous two phase dusty flow and heat transfer over a permeable stretching isothermal Riga plate embedded into a porous medium. The problem was solved numerically using Runge-Kutta-Fehlberg-45 arrange strategy and shooting procedure such that each of the non-dimensional amounts are exposed to view graphically for all parameters of the liquid as well as both Nusselt numbers and contact factor which are explained and shown with the aid of graphs. The equation that govern unsteady laminar heat transfer dusty fluid flow passing through two parallel Riga plates was solved using finite difference method by Islam and Nasrin (2020) and the effect with the behavior of the flow properties were analyzed.
Again, Islam and Nasrin (2021) carried out a study on the Unsteady couette flow of laminar heat transferable dusty fluid flow passing through two parallel Riga plates and used the explicit finite difference method to solved the govern equations. In their study, they are able to show the sequel of necessary parameters on both the temperature and velocity distribution together with the Nusselts number and shear stress. The investigation of unsteady viscous incompressible Bingham fluid flow within a parallel plate is solved with explicit forward difference method. This is to explain the dimensionless non-linear coupled PDE (partial differential equations) involved by Muhammad et. al (2019) with the result obtained discussed and illustrated graphically.
Mollah (2019) discussed the EMHD Laminar flow of Bingham fluid flow between two parallel Riga plates with the use of Forward difference method together with MATLAB code whereby the effect of many parameters on the flow pattern, Nusselts number and local shear stress is presented. Gunawan and Van de Ven (2022) derived an asymptotic steady solution while studying a one-dimensional non-steady pressure-driven flow of a Bingham fluid in a channel filled with a uniform high-porosity medium called Darcy-Brinkman medium. The comparison of the semi-analytical and Numerical method to solve the problem of MHD flow of a third grade fluid between two parallel plate with the use of regular perturbation method (RPM) and finite difference method (FDM) was discussed by Lawal et al. (2021) and they discovered that the FDM is more reliable and efficient than HPM from the computational viewpoint.

In alignment with the aforementioned point, it is noted that most of the researcher have not been using analytical methods especially the Eigenfunction expansion method to solve equations involving Riga plates, Bingham fluid and dusty fluid. Our intention is to solve these equations involving the unsteady flow of dusty Bingham fluid between two parallel Riga plates with radiation effects by using method of Eigenfuction expansion. Therefore, the mathematical formulation is discussed below which in turn is solved analytically. The result obtained from the solution is discussed as shown graphically.

## MATERIALS AND METHODS

### Problem formulation

The fluid we considered here is called laminar, incompressible and non-Newtonian Bingham fluid which has an unsteady flow passing through infinite but two parallel Riga plates. The plates are fixed at $y = \pm h$ plane and has its length starting with $x = 0$ to $\infty$ and from $z = 0$ to $\infty$. We should know that $\tilde{v}$ and $\tilde{w}$ are the velocity components and they are zero throughout the plate. Also, for the dusty particles, $\tilde{v}_p$ and $\tilde{w}_p$ are zero throughout the plate. The equation of continuity reduces to zero at the fluid phase $\frac{d\tilde{u}}{dx} = 0$ which implies $\tilde{u} = \tilde{u}(\tilde{y}, \tilde{t})$ and for the dust phase $\frac{\partial \tilde{u}_p}{\partial x} = 0$ implies $\tilde{u}_p = \tilde{u}_p(\tilde{y}, \tilde{t})$. The upper and lower part of the Riga plates are taken to be stationary, also the uniform velocity outside the boundary is $U_o$. The Riga plates, upper part and lower part had their temperature taken at two different constants which are $T_1$ and $T_2$ respectively such that $T_2 > T_1$. The constant pressure gradient $\frac{dp}{dx}$ acts on the Bingham fluid in the $x$ direction. Hence, the fluid velocity vector is given as $\tilde{q} = \tilde{u}\boldsymbol{i} + \tilde{v}\boldsymbol{j}$.

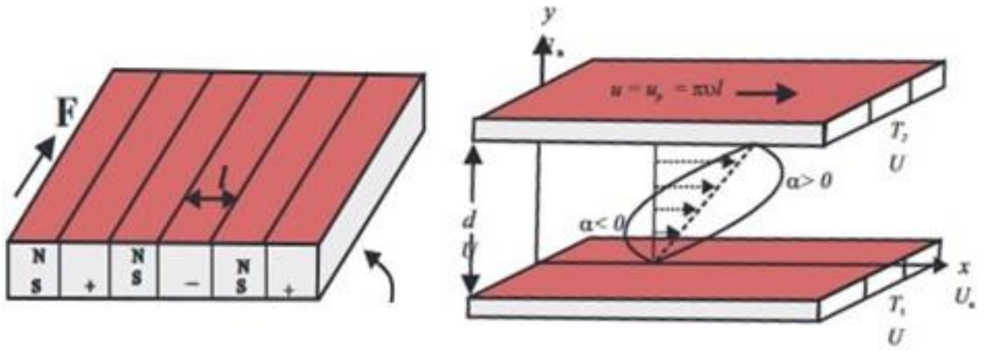


Figure 1: Physical model of the system

The Riga plate has it's volume density of a Lorentz force to be given as $F = J \wedge B$, also by Ohm's law, the current density is defined as $J = \sigma(E + \overline{q} \wedge B)$. The Bingham fluid has a very poor conducting capacity $\left(\sigma = \frac{10^6 S}{\text{m}}\right)$ or smaller then the current density $\sigma(E + \overline{q} \wedge B)$ in small. Therefore, the term $\sigma(E + \overline{q} \wedge B)$ in the equation above will be dismissed. Hence, for finding the flow, the Lorentz force which is the external magnetic field is used along the X – axis which is denoted by: $F = J \wedge B \approx \sigma(\overline{q} \wedge B)$, $F = F_{ex}$ is the density force which conform with Grinberg $\frac{F}{\rho}$. It is an averaged force along Z – axis, which can be equally expressed as an exponential function of y i.e $F = \frac{\pi}{8\rho} J_0 M_0\, e^{-\frac{\pi}{a}y}$ . In light of the aforementioned considerations, the system of the coupled non-linear PDEs governs the equation related to the flow of dusty Bingham fluid between two Riga plates with Radiation effects. The momentum and energy equations for both the clean fluid and dusty particles are furnished as follows:

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\rho}\frac{dp}{d\tilde{x}} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\tilde{\mu}\frac{\partial \tilde{u}}{\partial y}\right) + \frac{\pi}{8\rho} J_0 M_0 \ell^{-\frac{\pi}{a}y} - \frac{1}{\rho} KN\left(\tilde{u} - \tilde{u}_p\right) \quad (1)$$

$$M_p \frac{\partial \tilde{u}_p}{\partial t} = KN\left(\tilde{u} - \tilde{u}_p\right) \quad (2)$$

$$\frac{\partial \tilde{T}}{\partial t} = \frac{k}{\rho c_p}\left(\frac{\partial^2 \tilde{T}}{\partial y^2}\right) + \frac{\tilde{\mu}}{\rho c_p}\left(\frac{\partial \tilde{u}}{\partial y}\right)^2 - \frac{\rho_p c_s}{\rho c_p \gamma T}\left(\tilde{T} - \tilde{T}_p\right) + \frac{1}{\rho c_p}\frac{16\alpha^*}{3k^*} T_2^3 \frac{\partial^2 \tilde{T}}{\partial y^2} \quad (3)$$

$$\frac{\partial \tilde{T}_p}{\partial t} = \frac{1}{\gamma T}\left(\tilde{T} - \tilde{T}_p\right) \quad (4)$$

Obviously, the viscosity of the Bingham fluids is:

$$\mu = K + \frac{\tau_0}{\left(\frac{\partial \tilde{u}}{\partial y}\right)} \quad (5)$$

Where the plastic viscosity of the Bingham fluid is denoted by K and $\tau_0$ denotes the yield stress.

Also, both the initial and boundary conditions corresponding to the equation (1) to (5) is written below:

$t \leq 0, u = u_p = 0, T = T_p = T_1$ for all y ≥ 0

$t > 0\ u = u_p = 0, \tilde{T} = T_p = T_1$ at $y = -h$

$u = u_p = \frac{\pi v}{l}, \tilde{T} = T_p = T_2$ at $y = h$ (6)

where, $\tilde{u}, \tilde{v}, \tilde{w}$ are the clean fluid velocity components, $\tilde{u}_p, \tilde{v}_p, \tilde{w}_p$ denotes the dusty particles components of the velocity, υ denotes the clean fluid's kinematic viscosity J = $\left(J_x, J_y J_z\right)$ denotes the density of the current, B = $(B_x, B_y B_z)$ denotes the vector of the induced magnetic field, N denotes the number of dust particles per unit volume, K denotes the Stokes constant which equals 6πρυ$a$, T denotes the fluid's temperature, $T_P$ denotes the dust particle's temperature, k denotes the fluid's thermal conductivity, $a$ denotes the dust particle's average radius, $m_p$ denotes the dust particle's average mass, $\rho_p$ denotes the material density or mass per unit volume i.e mv$^{-1}$ of dust particles, $c_p$ denotes the specific heat capacity at constant pressure, $c_s$ denotes the particle's specific heat capacity, $\gamma_T$ is the temperature relaxation time which can be defined as: $\gamma_T = \frac{\rho_p c_s}{4k\pi a N}$ or $\frac{3\rho v \rho_p c_s}{2kKN}$ .

Introducing the following non-dimensional parameters,

$$= \frac{\tilde{x}}{l}, y = \frac{\tilde{y}}{l}, u = \frac{\tilde{u}}{u_0}, v = \frac{v}{v_0}, P = \frac{p}{\rho u_0^2}, \tau = \frac{t u_0}{h}, \tilde{\mu} = \frac{\tilde{\mu}}{k}$$

$$\theta = \frac{\tilde{T} - T_1}{T_2 - T_1} \text{ where } v_0 = \frac{\pi v}{a}, h = \frac{u_0 L^2}{v}, L = \frac{a}{\pi} \quad (7)$$

The equations (1) to (6) in non-dimensional form are:

$$\frac{\partial u}{\partial \tau} = \alpha + \frac{1}{R_e}\frac{\partial}{\partial y}\left(\tilde{\mu}\frac{\partial u}{\partial y}\right) + H_r \ell^{-y} - R\left(u - u_p\right) \quad (8)$$

$$\frac{\partial u_p}{\partial \tau} = \frac{1}{G}\left(u - u_p\right) \quad (9)$$

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} + E_c \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{4}{3} R_D \frac{\partial^2 \theta}{\partial y^2} - \frac{2R}{3P_r}\left(\theta - \theta_p\right) \quad (10)$$

$$\frac{\partial \theta_P}{\partial \tau} = L_0\left(\theta - \theta_p\right) \quad (11)$$

$$\tilde{\mu} = 1 + \frac{\tau_D}{\left(\frac{\partial u}{\partial y}\right)} \tag{12}$$

$\tau \leq 0, \quad u(y,0) = u_p(y,0) = 0, \quad \theta(y,0) = \theta_p(y,0) = 0$

$\tau > 0, \quad u(-1,\tau) = u_p(-1,\tau) = 0, \quad u(1,\tau) = u_p(1,\tau) = 0$ (13)

$\theta(-1,\tau) = \theta_p(-1,\tau) = 0 \quad \theta(1,\tau) = \theta_p(1,\tau) = 1$

where;

$\alpha = -\frac{L^2 P}{\rho v^2 \pi^2}$ is Pressure parameter, $R_e = \frac{\rho v_0 l}{K}$ is Remolds number, $H_r = \frac{\alpha^2}{8\rho u_0 v\pi} J_0 M_0$ is Modified Hartman number, $R = \frac{KNL^2}{\rho v u_0}$ is Fluid concentration parameter, $G = \frac{hKN}{m_p u_0^2}$ is Particle mass number, $P_r = \frac{\rho c_p u_0 L^2}{Kh}$ is Prandtl number, $E_c = \frac{u_0 Kh}{p c_p L^2 (T_2 - T_1)}$ is Eckert number, $R_D = \frac{4\sigma^*}{kk^*} T_2^3$ is Radiation parameter, $\tau_D = \frac{\tau_0 h}{K u_0}$ is Bingham number or dimension yield stress, $L_0 = \frac{L^2}{v\pi^2 \gamma T}$ is the temperature relaxation time parameter.

**Theorem**: If the functions $\propto, \beta\ and\ f$ are sufficiently smooth to ensure that $u, \frac{\partial \tilde{u}}{\partial t}, \frac{\partial \tilde{u}}{\partial y}, \frac{\partial^2 \tilde{u}}{\partial y^2}$ are continuous in G and up to the boundary of G $(y,t)$ including two corner point where $\mathrm{G}[(y,t): -L < y < L, t > 0]$ then the equation above with it's initial and boundary conditions has at most one solution.

**Proof:**

If the equation above has two solutions say $u_1\ and\ u_2$ because of linearity we say $u = u_1 - u_2$ is the solution of:

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{\rho} k \frac{\partial^2 \tilde{u}}{\partial y^2} \tag{14}$$

Where $t > 0, \quad u(-L,t) = 0, u(L,t) = 0$

$t \leq 0, \quad u(y,0) = f(y), -L < y < L$ (15)

Multiply equation (14) by $u$ and integrate over $[-L, L]$

$$0 = \int_{-L}^{L} \left( u \frac{\partial u}{\partial t} - \frac{1}{\rho} uk \frac{\partial^2 u}{\partial y^2} \right) dy \tag{16}$$

$$0 = \int_{-L}^{L} \left( \frac{1}{2} \frac{\partial}{\partial t} (u)^2 + \frac{1}{\rho} k \frac{\partial}{\partial y} (u)^2 \right) dy - \frac{k}{\rho} \left[ u \frac{\partial u}{\partial y} \right]_{-L}^{L} \tag{17}$$

Since $\frac{k}{\rho} \left[ u \frac{\partial u}{\partial y} \right]_{-L}^{L} = 0$

$$0 = \int_{-L}^{L} \left( \frac{1}{2} \frac{\partial}{\partial t} (u)^2 + \frac{1}{\rho} k \frac{\partial}{\partial y} (u)^2 \right) dy \tag{18}$$

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{-L}^{L} (u)^2 dy = -\frac{1}{\rho} k \int_{-L}^{L} \left( \frac{\partial}{\partial y} (u)^2 \right) dy \leq 0 \tag{19}$$

Therefore, the function$w(t) = \int_{-L}^{L} u^2(y,t) dy,\ 0 < t < A$ is non-increasing. Since $w(t) \leq 0$ and $w(0) = 0$ (because of the initial condition satisfied by u), this is the only possible integrand in the direction of w above and the arbitraries of A > 0 , we then conclude that $u = 0$ therefore, $u_1 = u_2$.

**Method of solution**

Let's make the fluid's velocity equals the particle's velocity and temperature of the fluid the same as the particle (i.e. $u = u_p, \quad \theta = \theta_p$). With this assumption, equation (8) to (13) becomes

$$\frac{\partial u}{\partial \tau} = \alpha + \frac{1}{R_e} \frac{\partial^2 u}{\partial y^2} + H_r \ell^{-y} \tag{20}$$

$$\frac{\partial \theta}{\partial \tau} = \left( \frac{1}{P_r} + \frac{4}{3} R_D \right) \frac{\partial^2 \theta}{\partial y^2} + E_c \left( \left( \frac{\partial u}{\partial y} \right)^2 + \tau_D \left( \frac{\partial u}{\partial y} \right) \right) \tag{21}$$

$\tau \leq 0, \quad u(y,0) = 0, \quad \theta(y,0) = 0$

$\tau > 0, \quad u(-1,\tau) = 0, \quad u(1,\tau) = 0$ (22)

$\theta(-1,\tau) = 0 \quad \theta(1,\tau) = 1$

Firstly to solve equation (20), we apply the method of Eigenfunction expansion and the procedures are as follows:

Let $q(y,\tau) = \alpha + H_r \ell^{-y}$ so that equation (20) becomes,

$$\frac{\partial u}{\partial \tau} = \frac{1}{R_e} \frac{\partial^2 u}{\partial y^2} + q(y,\tau) \tag{23}$$

which is non-homogeneous PDE with homogeneous initial and boundary conditions (22)

$$\frac{\partial u}{\partial \tau} = \frac{1}{R_e} \frac{\partial^2 u}{\partial y^2} \tag{24}$$

This is homogenous PDE and can be solve by using separation of variable method. The eigenvalues and eigenvectors of this corresponding homogeneous equation (24) are:

$$\lambda_n = (n\pi)^2, Y_n(y) = \sin n\pi y, n = 1,2,3,\ldots \tag{25}$$

Since $[Y_n]_{n=1}^{\infty}$ is a complete set, we considered the solution of the form

$$u(y,\tau) = \sum_{n-1}^{\infty} C_n(\tau) Y_n(y) \tag{26}$$

Substituting equation (26) into equation (23) gives

$$\sum_{n-1}^{\infty} \left[ C_n'(\tau) + \frac{1}{R_e} \lambda_n C_n(\tau) \right] Y_n(y) = q(y,\tau) \tag{27}$$

Multiplying equation (27) by $Y_m(y)$, integrating from -1 to 1 and consider the orthogonality of $Y_n(y)$ to obtain

$$\left[ C_m'(\tau) + k\lambda_m C_m(\tau) \right] \int_{-1}^{1} Y_m^2(y) dy = \int_{-1}^{1} q(y,\tau) Y_m(y) dy \tag{28}$$

Replacing $m$ by $n$ in equation (28) to yield

$$\left[ C_n'(\tau) + \frac{1}{R_e} \lambda_n C_n(\tau) \right] = \frac{\int_{-1}^{1} q(y,\tau) Y_n(y) dy}{\int_{-1}^{1} Y_n^2(y) dy} \tau > 0, n = 1,2,3,\ldots \tag{29}$$

The boundary conditions are automatically satisfied since each of the $Y_n$ in equation (26) satisfies them,

From equation (26) and the initial condition in (22), we realize that,

$$u(y,0) = 0 = \sum_{n-1}^{\infty} C_n(0) Y_n(y) \tag{30}$$

the initial condition becomes

$$[C_n(0)] = \frac{\int_{-1}^{1} 0.Y_n(y) dy}{\int_{-1}^{1} Y_n^2(y) dy} \quad , n = 1,2,3,\ldots \tag{31}$$

Clearly, $q(y,\tau) = \sum_{n-1}^{\infty} q_n(\tau) Y_n(y),\ 0 = \sum_{n-1}^{\infty} f_n Y_n(y),$ (32)

where the $q_n(\tau)\ and\ f_n$ are given by equation (29) and (31) respectively.

$$C_n'(\tau) + \frac{1}{R_e} (n\pi)^2 C_n(\tau) = q_n(\tau) = \frac{2}{L} \int_{-1}^{1} q_n \sin n\pi y dy \tag{33}$$

Solving equation (33) with initial condition we get (34)

$$C_n(0) = f_n = 2 \int_{-1}^{1} (0) \sin n\pi y dy = 0 \tag{34}$$

Solving equation (33) normally we get,

$$C_n(\tau) = \frac{2\ell^{-\frac{n^2\pi^2\tau}{R_e}} R_e H_r}{n^2\pi^2(n^2\pi^2+1)} \begin{pmatrix} \begin{pmatrix} \cos(n\pi)\,\ell^2 n\pi - n\cos(n\pi)\,\pi \\ -\sin(n\pi) - \sin(n\pi) \end{pmatrix} \ell^{\frac{n^2\pi^2\tau - R_e}{R_e}} \\ -\begin{pmatrix} \cos(n\pi)\,\ell^2 n\pi - n\cos(n\pi)\,\pi \\ -\sin(n\pi)\,\ell^2 - \sin(n\pi)\,\ell^{-1} \end{pmatrix} \end{pmatrix} \tag{35}$$

Therefore equation (26) becomes

$$u(y,\tau) = \sum_{n=1}^{\infty} \left( \frac{2\ell^{-\frac{n^2\pi^2\tau}{R_e}} R_e H_r}{n^2\pi^2(n^2\pi^2+1)} \begin{pmatrix} \left((-1)^n \ell^2 n\pi - n(-1)^n \pi\right) \ell^{\frac{n^2\pi^2\tau - R_e}{R_e}} \\ -\left((-1)^n \ell^2 n\pi - n(-1)^n \pi\right) \ell^{-1} \end{pmatrix} \right) \sin n\pi y \tag{36}$$

which gives the general solution of equation (20) with corresponding initial and boundary condition in (22).

To solve non-homogeneous energy equation (21) with corresponding initial and non-homogeneous boundary condition in (22). We substitute equation (36) at n = 1 into equation (21) and assume a solution of the form:

$$\theta(y,\tau) = v(y,\tau) + w(y,\tau) \tag{37}$$

Also, let $w(y,\tau) = a_1(\tau) + a_2(\tau) y$ is a linear polynomial that satisfy the boundary condition in (22).

Because of this,

$$w(y,\tau) = \frac{1}{2} + \frac{1}{2} y \tag{38}$$

Which made equation (37) to become

$$\theta(y,\tau) = v(y,\tau) + \frac{1}{2} + \frac{1}{2}y \quad (39)$$

Substituting equation (39) into equation (21) to gives

$$\frac{\partial v}{\partial \tau} = P\frac{\partial^2 v}{\partial y^2} + q_T(y,\tau) \quad (40)$$

By initial and boundary conditions in (22) becomes

$$v(y,0) = -\frac{1}{2} - \frac{1}{2}y, v(-1,\tau) = 0, v(1,\tau) = 0 \quad (41)$$

With the above procedure, it is noted that the non-homogenous boundary condition has turned to homogenous boundary condition.

Therefore, equation (40) with (41) are solve with method of eigenfunction expansion to arrives at;

$$v(y,t) = \sum_{n-1}^{\infty} \frac{2((-1)^n)e^{-Pn^2\pi^2\tau} sin(n\pi y)}{n\pi} \quad (42)$$

Putting equation (42) into equation (39) to obtain

Then, $$\theta = \frac{1}{2} + \frac{1}{2}y + \sum_{n-1}^{\infty} \frac{2((-1)^n)e^{-Pn^2\pi^2\tau} sin(n\pi y)}{n\pi} \quad (43)$$

**Shear stress and Nusselt number**

According to the investigation of the study, the velocity of the clean particle is the same as the velocity of the dusty particle, so the shear stress of the fluid will only be investigated without the shear stress of the dusty particle. Also, the shear stress of both upper velocity profile and lower velocity profile is the same we are to just calculate one of it. Therefore, the local shear stress of the fluid at the lower plate is then given as the relation $\tau_L = \mu\left(\frac{\partial U}{\partial Y}\right)_{Y=-1}$. Furthermore, since the fluid's temperature is the same as the particle's temperature, also, the Nusselt number at both upper and lower temperature profile are the same, we therefore investigate the Local Nusselt number of the fluid and the shear stress at lower plate. The dimension is also given as the relation $Nu_L = -\mu\left(\frac{\partial \theta}{\partial Y}\right)_{Y=-1}$.

**Validation of results for Local shear stress and Local Nusselts number**

The table below shows the impact of different variables on Local shear stress and Local Nusselts number at the lower Riga plate. $\alpha = -2, R_D = 0.05, \tau_D = 0.001, E_C = 0.01, \tau = 0.5$

**Table 1: Impact of different variables on Local shear stress and Local Nusselts number at the lower Riga plate**

| Effect of parameters | | | | Profiles | |
|---|---|---|---|---|---|
| $H_r$ | $P_r$ | $\mu$ | $R_e$ | $\tau_L$ | $Nu_L$ |
| **1** | 2.5 | 0.1 | 2 | 0.139815 | -0.070014 |
| **2** | 2.5 | 0.1 | 2 | 0.279631 | -0.070014 |
| **3** | 2.5 | 0.1 | 2 | 0.419446 | -0.070014 |
| **4** | 2.5 | 0.1 | 2 | 0.559262 | -0.070014 |
| 2 | **1.5** | 0.1 | 2 | 0.279631 | -0.055363 |
| 2 | **2.5** | 0.1 | 2 | 0.279631 | -0.070014 |
| 2 | **5.0** | 0.1 | 2 | 0.279631 | -0.104681 |
| 2 | **7.5** | 0.1 | 2 | 0.279631 | -0.128429 |
| 2 | 2.5 | **0.1** | 2 | 0.279631 | -0.070014 |
| 2 | 2.5 | **0.2** | 2 | 0.559262 | -0.140027 |
| 2 | 2.5 | **0.3** | 2 | 0.838892 | -0.210041 |
| 2 | 2.5 | **0.4** | 2 | 1.118523 | -0.280054 |
| 2 | 2.5 | 0.1 | **1** | 0.146529 | -0.070014 |
| 2 | 2.5 | 0.1 | **2** | 0.279631 | -0.070014 |
| 2 | 2.5 | 0.1 | **5** | 0.572325 | -0.070014 |
| 2 | 2.5 | 0.1 | **10** | 0.909952 | -0.070014 |

**RESULTS AND DISCUSSION**

Due to the developed mathematical model, in order to investigate the physical characteristics of the unsteady-state of the fluid, using the method of Eigenfunction expansion, analytical solution has been established after the primary velocity and the temperature field between the equation's boundary layer. The influences of different parameter are being considered to validate our solution. Examples of such parameters are Bingham number($\tau_D$), Prandtl number ($P_r$), Eckert number ($E_c$), Reynolds number ($R_e$), Radiation number ($R_D$) and Modified Hartman number ($H_r$).

Reynolds number is a dimensionless quantity which is used to ascertain the flow pattern as either laminar flow or turbulent flow while flowing through a region. In figure 2 increase in Reynolds number increases the velocity of the flow at the lower plate but decreases the flow towards the upper plate. This is due to the fact that the higher the Reynolds number results in the sluggishness of viscous forces which is the force that is known to oppose the motion of the fluid. Also, when the Reynolds number is reducing, it will happen that the influence of the viscous forces will surpass the inertial forces, hence it causes the fluid to be sluggish while reducing the velocity of the flow. Also the higher the Reynolds number causes the primary velocity profile to increase. Despite the fluid being unsteady and considering the range at which it is moving between the two Riga plates, the flow still moves in an increasing manner.

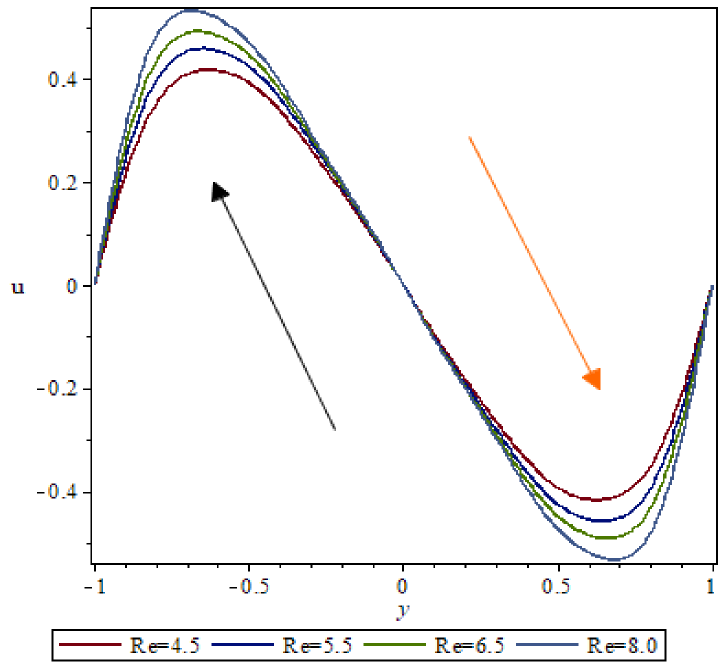


Figure 2: Effects of various values of $Re$on the velocity profile when $\alpha = -2, H_r = 1.0,$ $P_r = 2.5, R_D = 0.05, \tau_D = 0.001, E_c = 0.01, \tau = 0.5$

In Figure 3, it has been seen that a higher Reynolds number causes the temperature profile to rise. An increase in velocity causes more fluid particles to collide, which causes heat to dissolve in the boundary layer region hence raises the temperature as a result. Reynolds number is a measure of the ratio of the inertial forces to the viscosity forces; therefore, an increase in Reynolds number causes the viscous force to decrease. Viscous forces, which are known to oppose the fluid's motion in turn leads to the increase in both velocity and temperature.

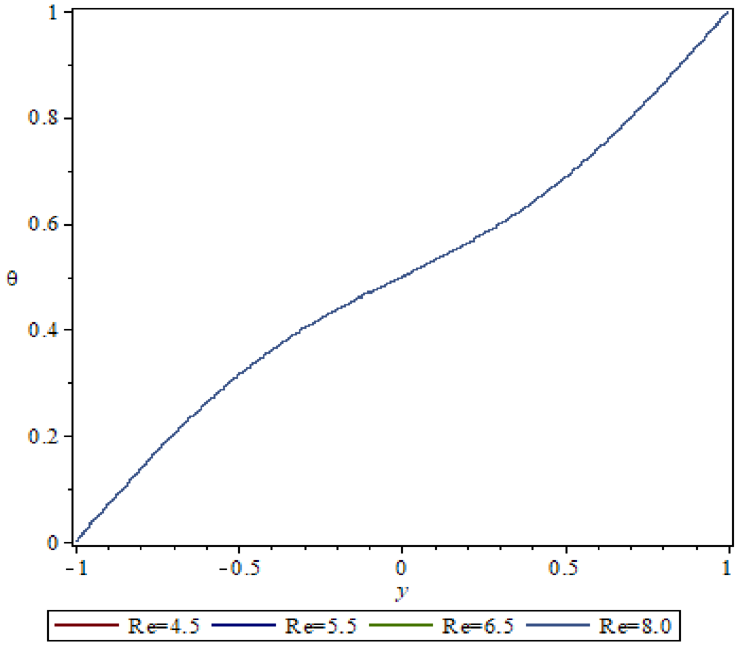


Figure 3: Effects of various values of $Re$ on the temperature profile when $\alpha =$ $-2,\ H_r = 1.0, P_r = 2.5, R_D = 0.05, \tau_D = 0.001, E_c = 0.01, \tau = 0.5$

In figure 4, it has been seen that a higher Prandtl number causes the temperature profile to rise at the two plates, this is because the Prandtl number's intrinsic feature of the fluid is due to the fact that it is a dimensionless quantity. It correlates a fluid's viscosity with it's thermal conductivity. Therefore, the link between the momentum and thermal conductivity can be accessed by it. Free-flowing liquids that has it's thermal conductivity to be high are those with a low Prandtl number. The ratio of viscous diffusion rate to thermal diffusion is known as Prandtl number. A rise in Prandtl number indicates that the thermal diffusivity of the fluid will decrease causing the fluid to expand and ha it's molecule to separate with rise in temperature. Since it is an unsteady case, the fluid tends to change slightly, yet increasing with time.

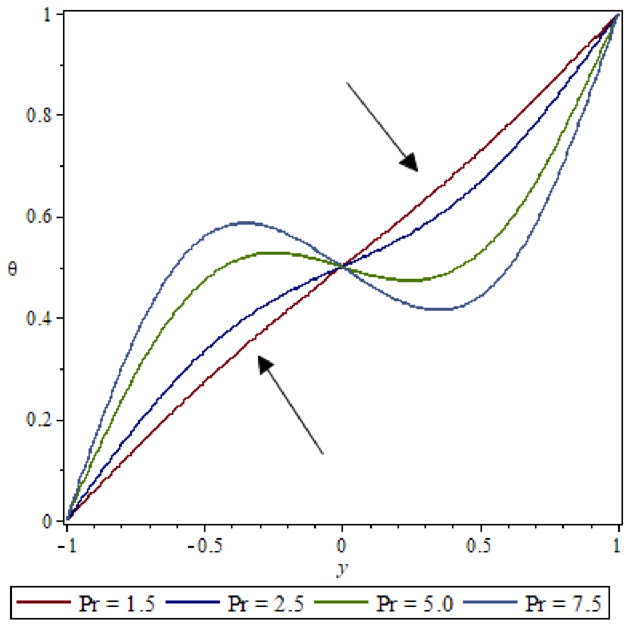


Figure 4: Effects of various values of $P_r$ on the temperature profile when $\alpha = -2, H_r = 1.0, R_D = 0.05, \tau_D = 0.001, E_c = 0.01, \tau = 0.5, R_e = 2$

In figure 5, it is seen that the fluid converges slowly to its starting point with the increase in Hartman number but the rise in Hartman number causes velocity profile at the lower plate to rise but decrease in temperature at the upper plate. Yet, at the upper plate, Hartman layers are formed at the interface with different electrical properties in the presence of magnetic fields. It is actually the ratio of electromagnetic force to that of the viscous force. In a Hartman layer of thickness, the Lorentz forces are in balance with the viscous forces.

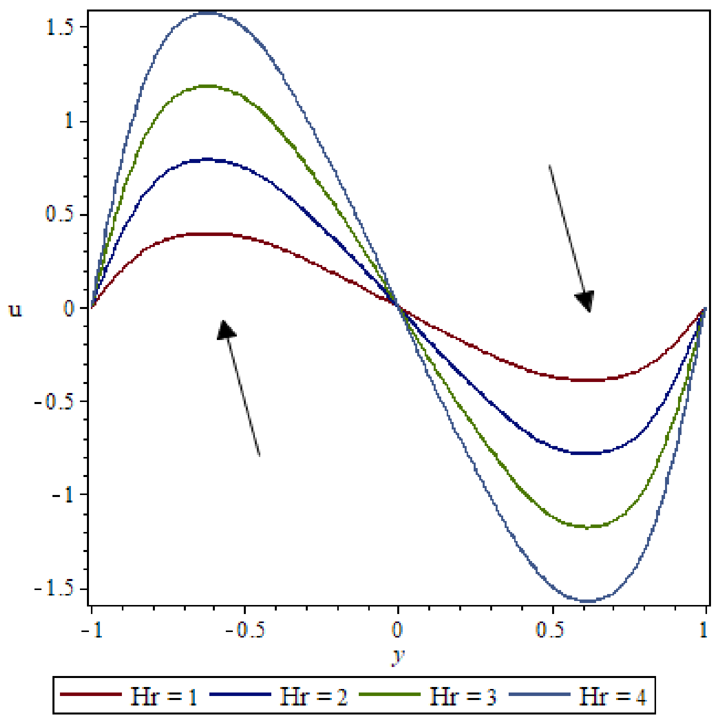


Figure 5: Effects of various values of $H_r$ on the velocity profile when $\alpha = -2, P_r = 2.5, R_D = 0.05, \tau_D = 0.001, E_c = 0.01, \tau = 0.5, R_e = 2$

In figure 6 below, we see that the fluid converges with no Hartman number, Hartman number is the ratio of electromagnetic force to that of the viscous force, it occur mostly in fluid flow through magnetic fields. It is observed that the surface temperature induces surface flow and the temperature converges to an insignificant effect but it makes the viscosity of the fluid increases as the Hartman number increases with temperature.

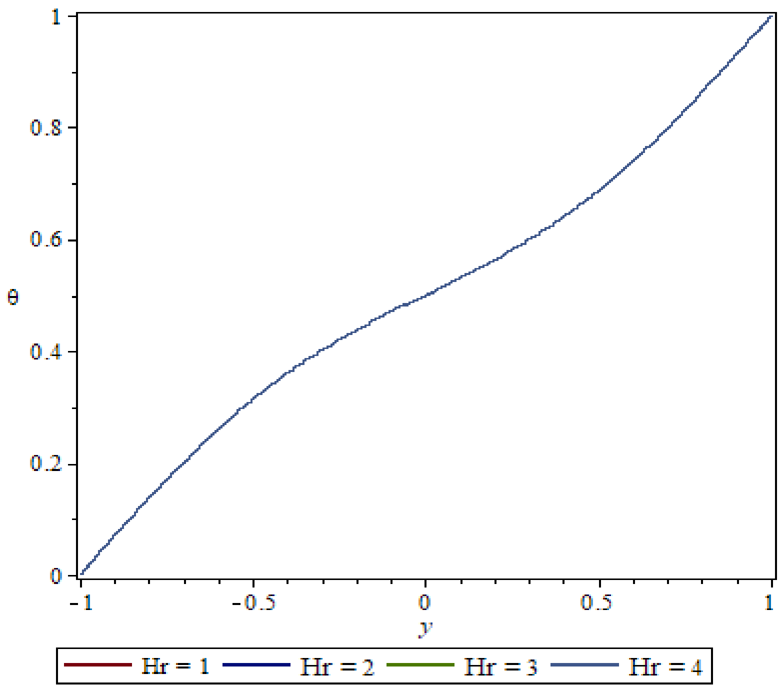


Figure 6: Effects of various values of $H_r$ on the temperature profile when $\alpha = -2, P_r = 2.5, R_D = 0.05, \tau_D = 0.001, E_c = 0.01, \tau = 0.5, R_e = 2$

In figure 7, it is observed that the Radiation parameter exists in the equation of the temperature profile. It defines relatively the contribution of conduction heat transfer to thermal heat transfer. It is noted that the temperature rises as the Radiation parameter rise.

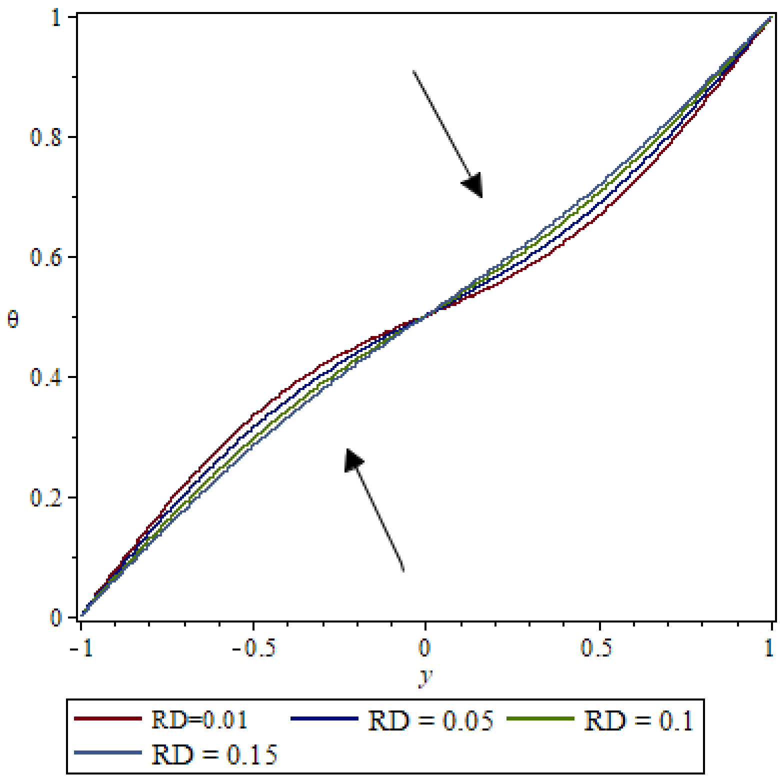


Figure 7: Effects of various values of $R_D$ on the temperature profile when $\alpha = -2, H_r = 2.0, P_r = 2.5, \tau_D = 0.001, E_c = 0.01, \tau = 0.5, R_e = 2$

## CONCLUSION

In this study, the analytical solution of unsteady flow of dusty Bingham fluid between two parallel riga plate with radiation effect has been established by using the Eigenfunction expansion method. The results were discussed graphically on the velocity and temperature of the fluid, noting that the velocity of the clean fluid is same as the velocity of the dusty fluid and it is same as their temperature.

The important discoveries on the investigation in this study are as follows:

i. Reynolds number: A rise in Reynolds number leads to the rise of both velocity profile and temperature profile.
ii. Prandtl number: As prandtl number has no visible effect on velocity, we saw that the rise in Prandtl number causes the temperature to rise.
iii. Hartman number: The rise in Hartman number causes both velocity as well as the temperature to rise.
iv. Radiation parameter: The increase in Radiation effect has a more effect on the temperature as it deals more with heat transfer and thermal conduction. It exists only in the temperature equation so it has no effect in the velocity profile.

Note: The wake-like diagram of the graph is due to the boundary condition i.e from negative to positive. The increased flow at the negative part is the decreased flow at the positive part.

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