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THE ORDER AND ERROR CONSTANT OF A RUNGE-KUTTA TYPE METHOD FOR THE NUMERICAL SOLUTION OF INITIAL VALUE PROBLEM

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ABSTRACT

In this paper, we examine in details how to obtain the order, error constant, consistency and convergence of a Runge-Kutta Type method (RKTM) when the step number $k = 2$. Analysis of the order, error constant, consistency and convergence will help in determining an effective Runge- Kutta Method (RKM) to use. Due to the loss of linearity in Runge –Kutta Methods and the fact that the general Runge –Kutta Method makes no mention of the differential equation makes it impossible to define the order of the method independently of the differential equation.

Keywords: Convergence, initial value problems, step number, differential equation

INTRODUCTION

Mathematical modeling of many engineering and physical system leads to non-linear ordinary and partial differential equations. In general, exact solutions of such equations are unknown and thus numerical integration, perturbation techniques or geometrical methods have applied to obtain approximate solutions. The Runge-Kutta method is one of the most famous and popular method which is used for solving differential equations. The Runge-Kutta method is named for its' creators Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944) (Tamer Abassy, 2000).

Runge-Kutta methods are very popular because of their simple coefficients, efficiency and numerical stability (Agam, 2013). The methods are fairly simple to program, easy to implement and their truncation error can be controlled in a more straight forward manner than multistep methods (Muhammad, R, Y.A Yahaya, A.S Abdulkarim, 2016). The application of Runge-Kutta methods have provided many satisfactory solutions to many problems that have been regarded as insolvable. The popularity and growth of these methods, coupled with the amount of research effort being undertaken, are further evidence that the applications are still the leading source of inspiration for mathematical creativity (Adegboye, 2013).

In the early days of Runge-Kutta methods, the aim seemed to be to find explicit methods of higher and higher order. Later the aim shifted to finding methods that seemed to be optimal in terms of local truncation error and to finding built-in error estimators (John Butcher, 2005).

An ordinary differential equation is a relation between a function, its derivatives and the variable upon which they depend. The most general form of an ordinary differential equation is given by

$$\emptyset(t, y, y', y'', \dots., y^m) = 0 \quad (1)$$

Where m represents the highest order derivative, and y and its derivatives are functions of t. The order of the differential equation is the order of its highest derivative and its degree is the degree of the derivative of the highest order after the equation has been rationalized in derivatives.

The differential equation (1) together with initial conditions (Jain et al., 2003)

$$y^{(v)}(t_0) = \eta_v, v = 0,1,2, \dots\dots, m-1 \quad (2)$$

The initial value problem for first order Ordinary Differential Equation is defined by

$$y' = f(x, y) \qquad y(x_0) = y_0 \qquad x \in [a, b] \quad (3)$$

Butcher defined an s-stage Runge Kutta methods for the first order differential equation in the form

$$y_{n+1} = y_n + h \sum_{i,j=1}^{s} a_{ij} k_i \quad (4)$$

where for $i = 1,2 \dots\dots\dots. s$

$$k_i = f(x_i + \alpha_j h, y_n + h \sum_{i,j=1}^{s-1} a_{ij} k_j) \quad (5)$$

The real parameters α_j, k_j, a_{ij} define the method. The method in Butcher array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $A = a_{ij} = \beta$ (Butcher, 2008)

According to kulikov (2003) if the matrix A is strictly lower triangular (i:e the internal stages can be calculated without depending on later stages), then the method is called an explicit method, otherwise the internal stages depend not only on the previous stages but also on the current stage and later stages, then the method is called an Implicit method. This method is more suitable for solving stiff problems due to its high order of accuracy which makes it more superior to the explicit method (Yahaya and Adegboye, 2011)

DEFINITION OF TERMS

Definition 1: Order and Error Constant of Runge-Kutta Method

The **first** and **second** order Ordinary Differential Equation (ODE) methods are said to be of order $\boldsymbol{p}$ if $\boldsymbol{p}$ is the largest integer for which

$$y(x+h) - y(x) - h\varphi(x, y(x), h) = 0(h^{p+1}) \quad (6)$$

$$y(x+h) - y(x) - h^2\varphi(x, y(x), y'(x), h^2) = 0(h^{p+2}) \quad (7)$$

holds respectively. Where

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) \ldots\ldots\ldots\ldots + \frac{h^s}{s!}y^s(x) \quad (8)$$

$$\varphi(x, y(x), h) = y'(x+h) = f(x, y(x)), \quad (9)$$

$$\varphi(x, y(x), y'(x), h^2) = y''(x+h) = f(x, y(x), y'(x)) \quad (10)$$

in the taylor series expansion about x_0 and compare coefficients of $h^k y^k(x_0), y(x_0)$ is the interval value. The coefficient for which $\boldsymbol{p}$ is the largest integer is known as the **error constant**. (Yahaya and Adegboye 2013).

Definition 2: Consistency of Runge Kutta Methods

The first and second order Ordinary Differential Equation (ODE) methods are said to be consistent if

$$\varphi(x, y(x), 0) \equiv f(x, y(x)) \quad (11)$$

$$\varphi(x, y(x), y'(x), 0) \equiv f(x, y(x), y'(x)) \quad (12)$$

holds respectively.

Note that consistency demands that $\sum_1^s b_s = 1$, and $\sum_1^s b_s = \frac{1}{2}$ = for first and second order respectively. Also $\sum_1^s b_s$ is as shown in the butcher array table.

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2$ $\quad\quad \bar{A} = \bar{a}_{ij} = \beta$ $\quad\quad \beta = \beta e$

Definition 3: Convergence of Runge –Kutta Methods

If $f(x, y(x)), f(x, y(x), y'(x))$ represents first and second order respectively, then for such method consistency is necessary and sufficient for convergence. Hence the methods are said to be convergent if and only if they are consistent. (Yahaya and Adegboye 2013).

METHODOLOGY

Reformulation of Runge - Kutta Type method for Order and Error Constant.

The initial value problem (IVP) for first order Ordinary Differential Equation is defined by

$$y' = f(x, y) \qquad y(x_0) = y \qquad x \in [a, b]$$

The general s-stage Runge Kutta method is defined by

$$y_{n+1} = y_n + h\sum_{i=1}^s a_{ij}k_i \quad (13)$$

where for $i = 1,2 \ldots\ldots\ldots.s$

$$k_i = f(x_i + c_j h, y_n + h\sum_{i=1}^s a_{ij}k_j) \quad (14)$$

The real parameters c_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as

c	β
	W^T

Where $a_{ij} = \beta$

For $c_1, c_2, \ldots., c_s$ and $k_1, k_2, \ldots\ldots k_s$ in (14) we shall let $k_i = f_{c_i}$ implies $k_1 = f_{c_1}$, $k_2 = f_{c_2}$,$k_3 = f_{c_3}$ and $k_s = f_{c_s}$.

RESULTS

Consider the equation for the Block Hybrid Runge-Kutta Type Backward Differentiation Formula for $K = 2$ given as

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + h(0k_1 + \tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4) \\ y_{n+2} &= y_n + h\left(0k_1 + \tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\right) \\ y_{n+1} &= y_n + h\left(0k_1 + \tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{18}k_4\right) \end{aligned}\right\} \quad (15a)$$

Where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + h\{\tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4\}) \\ k_3 &= f(x_n + h, y_n + h\{\tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{18}k_4\}) \\ k_4 &= f(x_n + 2h, y_n + h\{\tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\}) \end{aligned}\right\} \quad (15b)$$

Since $k_i = f_{c_i}$, implies $k_1 = f_{c_1}, k_2 = f_{c_2}, k_3 = f_{c_3}, \; k_4 = f_{c_4}$

Using equation (14), it implies

$c_1 = 0, \; c_2 = \frac{1}{2}, c_3 = 1, c_4 = 2.$

Therefore

$k_1 = f_0, k_2 = f_{\frac{1}{2}}, \; k_3 = f_1, k_4 = f_2$

the equation (13a) now becomes

$$y_{n+\frac{1}{2}} = y_n + h(0f_0 + \tfrac{8}{9}f_{\frac{1}{2}} - \tfrac{11}{24}f_1 + \tfrac{5}{72}f_2)$$

$$y_{n+2} = y_n + h\left(0f_0 + \tfrac{8}{9}f_{\frac{1}{2}} + \tfrac{2}{3}f_1 + \tfrac{4}{9}f_2\right) \quad (16)$$

$$y_{n+1} = y_n + h\left(0f_0 + \tfrac{10}{9}f_{\frac{1}{2}} - \tfrac{1}{6}f_1 + \tfrac{1}{18}f_2\right)$$

Taylor series expansion of

$$y_{n+\frac{1}{2}} = y\left(n + \frac{1}{2}h\right) = y(n) + \frac{1}{2}hy'(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'''(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^s}{s!}y^s(n)$$

$$y_{n+1} = y(n+h) = y(n) + hy'(n) + \tfrac{(h)^2}{2!}y''(n) + \tfrac{(h)^3}{3!}y'''(n) + \tfrac{(h)^4}{4!}y'^v(n) \; \ldots + \tfrac{(h)^s}{s!}y^s(n)$$

$$y_{n+2} = y(n+2h) = y(n) + 2hy'(n) + \tfrac{(2h)^2}{2!}y''(n) + \tfrac{(2h)^3}{3!}y'''(n) + \tfrac{(2h)^4}{4!}y'^v(n) \; \ldots + \tfrac{(2h)^s}{s!}y^s(n)$$

$$f_{\frac{1}{2}} = f\left(n + \tfrac{1}{2}h\right) = y'(n) + \tfrac{1}{2}hy''(n) + \tfrac{\left(\frac{1}{2}h\right)^2}{2!}y'''(n) + \tfrac{\left(\frac{1}{2}h\right)^3}{3!}y'^v(n) + \cdots + \tfrac{\left(\frac{1}{2}h\right)^{(s-1)}}{(s-1)!}y^s(n)$$

$$f_1 = f(n+h) = y'(n) + hy''(n) + \frac{(h)^2}{2!}y'''(n) + \frac{(h)^3}{3!}y'^v(n) + \cdots + \frac{(h)^{(s-1)}}{(s-1)!}y^s(n)$$

$$f_2 = f(n+2h) = y'(n) + 2hy''(n) + \tfrac{(2h)^2}{2!}y'''(n) + \tfrac{(2h)^3}{3!}y'^v(n) + \cdots + \tfrac{(2h)^{(s-1)}}{(s-1)!}y^s(n)`$$

By substituting into the equation (14) above, we have

$y_{n+\frac{1}{2}} - y_n - h\left(\frac{8}{9}f_{\frac{1}{2}} - \frac{11}{24}f_1 + \frac{5}{72}f_2\right) = \frac{-37}{1152}h^4y^4$, the method is of order 3 and the error constant is $\frac{-37}{1152}$.

$y_{n+2} - y_n - h\left(\frac{8}{9}f_{\frac{1}{2}} + \frac{2}{3}f_1 + \frac{4}{9}f_2\right) = \frac{-1}{18}h^4y^4$, the method is of order 3 and the error constant is $\frac{-1}{18}$

Also,

$y_{n+1} - y_n - h\left(\frac{10}{9}f_{\frac{1}{2}} - \frac{1}{6}f_1 + \frac{1}{8}f_2\right) = \frac{-1}{36}h^4y^4$, the method is of order 3 and the error constant is $\frac{-1}{36}$

From definition (2) and (3), the method

0	0	0	0	0
$\frac{1}{2}$	0	$\frac{8}{9}$	$\frac{-11}{24}$	$\frac{5}{72}$
2	0	$\frac{8}{9}$	$\frac{2}{3}$	$\frac{4}{9}$
1	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$
	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$

is consistent since $\sum_1^s b_s = 1$, hence convergent.

Consider this equation for the second derivative of $\boldsymbol{k = 2}$ given as

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \tfrac{37}{108}k_2 - \tfrac{41}{144}k_3 + \tfrac{29}{432}k_4\right)\\
y'_{n+\frac{1}{2}} &= y'_n + h\left(0k_1 + \tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4\right)\\
y_{n+2} &= y_n + 2hy'_n + h^2\left(0k_1 + \tfrac{52}{27}k_2 - \tfrac{2}{9}k_3 + \tfrac{8}{27}k_4\right)\\
y'_{n+2} &= y'_n + h\left(0k_1 + \tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\right)\\
y_{n+1} &= y_n + hy'_n + h^2\left(0k_1 + \tfrac{23}{27}k_2 - \tfrac{4}{9}k_3 + \tfrac{5}{54}k_4\right)\\
y'_{n+1} &= y'_n + h\left(0k_1 + \tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{18}k_4\right)
\end{aligned}\right\} \qquad (17a)$$

where

$$\left.\begin{aligned}
k_1 &= f(x_n, y_n, y'_n)\\
k_2 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \tfrac{37}{108}k_2 - \tfrac{41}{144}k_3 + \tfrac{29}{432}k_4\right),\\
&\quad y'_n + h\left(0k_1 + \tfrac{8}{9}k_2 - \tfrac{11}{24}k_3 + \tfrac{5}{72}k_4\right))\\
k_3 &= f(x_n + h, y_n + hy'_n + h^2\left(0k_1 + \tfrac{23}{27}k_2 - \tfrac{4}{9}k_3 + \tfrac{5}{54}k_4\right),\\
&\quad y'_n + h\left(0k_1 + \tfrac{10}{9}k_2 - \tfrac{1}{6}k_3 + \tfrac{1}{18}k_4\right))\\
k_4 &= f(x_n + 2h,\ y_n + 2hy'_n + h^2\left(0k_1 + \tfrac{52}{27}k_2 - \tfrac{2}{9}k_3 + \tfrac{8}{27}k_4\right),\\
&\quad y'_n + h\left(0k_1 + \tfrac{8}{9}k_2 + \tfrac{2}{3}k_3 + \tfrac{4}{9}k_4\right))
\end{aligned}\right\} \qquad (17b)$$

From equation (14)

$c_1 = 0,\ c_2 = \frac{1}{2}, c_3 = 1, c_4 = 2$

Therefore

$k_1 = f_0, k_2 = f_{\frac{1}{2}},\ k_3 = f_1, k_4 = f_2$

the equation (17a) now becomes

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \tfrac{1}{2}hy'_n + h^2\left(0f_0 + \tfrac{37}{108}f_{\frac{1}{2}} - \tfrac{41}{144}f_1 + \tfrac{29}{432}f_2\right)\\
y'_{n+\frac{1}{2}} &= y'_n + h\left(0f_0 + \tfrac{8}{9}f_{\frac{1}{2}} - \tfrac{11}{24}f_1 + \tfrac{5}{72}f_2\right)\\
y_{n+2} &= y_n + 2hy'_n + h^2\left(0f_0 + \tfrac{52}{27}f_{\frac{1}{2}} - \tfrac{2}{9}f_1 + \tfrac{8}{27}f_2\right)\\
y'_{n+2} &= y'_n + h\left(0f_0 + \tfrac{8}{9}f_{\frac{1}{2}} + \tfrac{2}{3}f_1 + \tfrac{4}{9}f_2\right)\\
y_{n+1} &= y_n + hy'_n + h^2\left(0f_0 + \tfrac{23}{27}f_{\frac{1}{2}} - \tfrac{4}{9}f_1 + \tfrac{5}{54}f_2\right)\\
y'_{n+1} &= y'_n + h\left(0f_0 + \tfrac{10}{9}f_{\frac{1}{2}} - \tfrac{1}{6}f_1 + \tfrac{1}{18}f_2\right)
\end{aligned}\right\} \qquad (18)$$

The taylor series expansion of

$$y_{n+\frac{1}{2}} = y\left(n + \frac{1}{2}h\right) = y(n) + \frac{1}{2}hy'(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'''(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^s}{s!}y^s(n)$$

$$y_{n+1} = y(n+h) = y(n) + hy'(n) + \frac{(h)^2}{2!}y''(n) + \frac{(h)^3}{3!}y'''(n) + \frac{(h)^4}{4!}y'^v(n)\ \ldots + \frac{(h)^s}{s!}y^s(n)$$

$$y_{n+2} = y(n+2h) = y(n) + 2hy'(n) + \frac{(2h)^2}{2!}y''(n) + \frac{(2h)^3}{3!}y'''(n) + \frac{(2h)^4}{4!}y'^v(n)\ \ldots + \frac{(2h)^s}{s!}y^s(n)$$

$$f_{\frac{1}{2}} = f\left(n+\frac{1}{2}h\right) = y'(n) + \frac{1}{2}hy''(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y^{iv}(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^{(s-1)}}{(s-1)!}y^{s}(n)$$

$$f_1 = f(n+h) = y'(n) + hy''(n) + \frac{(h)^2}{2!}y'''(n) + \frac{(h)^3}{3!}y^{iv}(n) + \cdots + \frac{(h)^{(s-1)}}{(s-1)!}y^{s}(n)$$

$$f_2 = f(n+2h) = y'(n) + 2hy''(n) + \frac{(2h)^2}{2!}y'''(n) + \frac{(2h)^3}{3!}y^{iv}(n) + \cdots + \frac{(2h)^{(s-1)}}{(s-1)!}y^{s}(n)$$

Substituting these values above in equation (18), we obtained the Order and Error Constant for the second derivative of $\boldsymbol{k = 2}$ of the Block Hybrid Runge -Kutta Type method as tabulated below.

Method	Order	Error Constant
$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + h^2\left(0f_0 + \frac{37}{108}f_{\frac{1}{2}} - \frac{41}{144}f_1 + \frac{29}{432}f_2\right)$	3	$\frac{-37}{1152}$
$y_{n+2} = y_n + 2hy'_n + h^2\left(0f_0 + \frac{52}{27}f_{\frac{1}{2}} - \frac{2}{9}f_1 + \frac{8}{27}f_2\right)$	3	$\frac{-1}{18}$
$y_{n+1} = y_n + hy'_n + h^2\left(0k_1 + \frac{23}{27}k_2 - \frac{4}{9}k_3 + \frac{5}{54}k_4\right)$	3	$\frac{-1}{36}$

From definition (2) and (3), the methods

0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{8}{9}$	$\frac{-11}{24}$	$\frac{5}{72}$	0	$\frac{37}{108}$	$\frac{-41}{144}$	$\frac{29}{432}$
2	0	$\frac{8}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	0	$\frac{52}{27}$	$\frac{-2}{9}$	$\frac{8}{27}$
1	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$	0	$\frac{23}{27}$	$\frac{-4}{9}$	$\frac{5}{54}$
	0	$\frac{10}{9}$	$\frac{-1}{6}$	$\frac{1}{18}$	0	$\frac{23}{27}$	$\frac{-4}{9}$	$\frac{5}{54}$

are consistent since $\sum_1^s b_s = \frac{1}{2}$, hence convergent.

DISCUSSION

The formula for the first order Block Hybrid Runge-Kutta Type Method (BHRKTM) for $k = 2$ given in equation (15) has each of its stages (i:e $k_i's$ from $i = 1, \ldots 4$) reformulated as a linear multistep with their corresponding values assigned accordingly. The values obtained respectively substituted into equation (15) gave rise to equation (16). The taylor series expansion carried out with values substituted back into equation (16) gave rise to the corresponding order and error constant of each of the methods that form the block. All the tables justified the consistency as well as convergence of the methods. The same procedure was applied for the second derivative of the method and results obtained were tabulated accordingly.

CONCLUSION

For the Block Hybrid Runge-Kutta Type Method (BHRKTM) with step number $k = 2$, each of the stages reformulated into linear multistep method and with the aid of Taylor series expansion gave rise to the uniform order with their error constant. All the methods that formed the block are of uniform order 3 with varying error constants. The procedures highlighted step by step and results obtained tabulated helped to establish the consistency and convergence of the methods. This helps to determine a standard method to adopt at any point in time. The procedure adopted explained a simple approach that speeds up computation and reduces computational effort in determining the order, error constant, consistency and convergence a Runge- Kutta Type Method (RKTM). This will also serve as a guide for researchers on how to determine the order, error constant and convergence of a Block Hybrid Runge-Kutta Type Method (BHRKTM). It will also help to determine a good choice of the method.

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