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SOME EFFICIENT MODIFIED HOUSEHOLDER ITERATIVE SCHEMES FOR NONLINEAR EQUATIONS

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ABSTRACT

The Householder scheme is a well-known iterative scheme for estimating the solution of nonlinear equation. Its major setbacks includes low convergence order, non-optimal and require high function derivative evaluation assessment in an iteration cycle. In this manuscript, we offered an iterative scheme that is an extension of the Householder scheme. In order to circumvent the highlighted setbacks of the Householder scheme, its modified form that is of convergence order four, optimal and require no second function derivative is put forward. The scheme was further modified via the use of weight functions to scale its convergence order from four to eight at the expense of one additional function evaluation and still retains its optimality property. The performance measures on the developed schemes was carried out by comparing their computational performance with that of some existing robust schemes.

Keywords: Iterative schemes, Nonlinear equations, Householder scheme, Optimal order of convergence

INTRODUCTION

Nonlinear problems of the form $f(w)=0$, $w \in R$ and $f: R \rightarrow R$ are continually encountered in many branches of science and engineering. This is because, real life situations are modelled into nonlinear equations and in most cases, their solutions are desired to be obtained for further study of the problems. Because of that fact that, most of these nonlinear equations defy the analytic procedures of obtaining their solutions, alternative procedures are being employed. For this reason, numerical analysis have considered the problem of obtaining solution to nonlinear equation as an important problem. In numerical analysis, the iterative procedures are being utilized to deal with the solutions of nonlinear equations. The procedures involves guessing the true solution of the nonlinear equation and then use it to perform a repetitive computation that produces sequence of approximations that eventually converge to the true solution of the nonlinear equation. This procedure is referred to as iterative process. An old and widely used iterative process for obtaining the solution of nonlinear equations is the Newton method (NM) (Traub, 1974) presented as

$$y_{k+1} = w_k - \frac{f(w_k)}{f'(w_k)}, \quad k = 0,1,2,\ldots \tag{1}$$

The NM determines the solution of the nonlinear equation with convergence order (CO) two provided $f'(w_k) \neq 0$. Its efficiency is measured by the efficiency index due to Traub (1974) and given by $E = \sqrt[\beta]{\mu}$, where β is number of distinct functions assessment per iteration cycle and μ is the iterative scheme CO. Consequently, NM efficiency index is 1.4142, because it require the assessment of two functions $f(w_k)$ and $f'(w_k)$ in one cycle of iteration process.

Many diverse modifications of the NM with better CO and efficiency are available in literature. They includes the one point and multipoint methods (see Householder,1970; Nazeer *et al*., 2016; Nadee *et al*.,2023, Sarima *et al*.,2020; Obadah,2020; Ogbereyivwe and Umar,2023a; Ogbereyivwe and Umar,2023b; Ogbereyivwe *et al*.,2023). These modifications were made possible via the use of one or combinations of Taylor expansion, composition, quadrature, variational iteration, weight function or divided difference techniques. The fundamental motivation behind the development of modified NM in most literature, were mainly hinged on providing high CO and efficiency schemes.

Consequently, we are motivated to develop a new and optimal family of iterative scheme that is an extension of the NM and the Householder method via the Taylor series expansion technique in this paper. The weight function and function derivative approximation technique was further fused into the developed method with the aim of scaling up its CO and efficiency. This resulted to the development of a new optimal CO eight method.

MATERIALS AND METHODS

The Schemes

Suppose w_k is the kth iteration estimation of the zero w^* of $f(w)=0$ using an iterative scheme, then

$$w_* = w_k + E_k \tag{2}$$

where E_k is the kth iteration error. Consequently,

$$f(w_*) = 0 \tag{3}$$

The Taylor's series expression of $f(w)$ around w_k when w is set as $w = w_*$, is

$$f(w^*) = f(w_k) + \sum_{j=1}^{\infty} \frac{(w_* - w_k)^j}{j!} f^{(j)}(w_k),$$

$$= f(w_k) + \sum_{j=1}^{\infty} \frac{E_k^j}{j!} f^{(j)}(w_k), \tag{4}$$

where $f^{(j)}(w_k)$ is jth derivative of the function $f(w)$ and evaluated at $w = w_k$. By disposing the higher error terms E_k^j, $j \geq 3$, in (4) and then substituting it in (3), we have

$$2f(w_k) + 2E_k f'(w_k) + E_k^2 f''(w_k) = 0. \tag{5}$$

The solution of the equation in (5) with respect to E_k is

$$E_k = \frac{f(w_k)}{f'(w_k)}\left[1 + \frac{1}{2}\eta + \frac{1}{2}\eta^2\right] \tag{6}$$

where $\eta = \frac{f''(w_k) f(w_k)}{\left(f'(w_k)\right)^2}$.

Consequent upon the equations in (2) and (6), a scheme for the estimation of the exact solution of nonlinear equation can be obtained by implementing the following scheme.

Scheme 1

For an initial guess w_0 in the neigbourhood of w_* where $w_0, w_* \in R$, the $(k+1)$th iteration estimation of the solution w_* of $f(w)=0$ is obtained using:

$$w_{k+1} = w_k - \frac{f(w_k)}{f'(w_k)}\left[1 + \frac{1}{2}\eta + \frac{1}{2}\eta^2\right]. \tag{7}$$

Remark 1: We note that, when both $\eta = 0$ and $\eta^2 = 0$ in (7), the famous NM is obtained. Again, for η not vanishing and $\eta^2 = 0$, the scheme in (7) reduces to the CO three Householder scheme (HS) presented in Householder (1970). Consequently, in the case where both $\eta \neq 0$ and $\eta^2 \neq 0$ in (7), we refer to it as the Extended Householder scheme (EHS) and claim that it retains the HS CO but with better precision. This claim will be substantiated in the proof of its convergence theorem and numerical implementation in the next sections.
The both of HS and EHS have convergence order three and require second derivative evaluation inη which makes it computationally expensive in implementation. Again, the schemes are not optimized in the view of Kung and Traub as reported in Kung and Traub (1974). Kung and Traub posited that an iterative scheme is optimized if it utilizes all of its β number of distinct function evaluation to attain a maximum convergence order of $2^{\beta-1}$. The EH and EHS both require three assessment of the distinct functions $f(w_k), f'(w_k)$ and $f''(w_k)$ at each iteration cycle. For the HS and EHS to satisfy the Kung and Traub optimal condition, they must be made to attain CO four without requiring additional new function evaluation. Again, not all functions are easily differentiable, let alone two times differentiable. To eliminate the presence of high derivativesand optimize the EHS, the following estimations of $f'(y_k)$ given as:

$$f'(y_k) \approx \frac{f'(w_k)(f(w_k)+(\theta-2)f(w_k))}{f(w_k)+\theta f(y_k)} = \Psi[f'(w_k), f(w_k)], \quad \theta \in R \tag{8}$$

was utilized to suggest the next scheme.

Scheme 2

For w_0 close to w_*, compute w_{k+1} such that

$$w_{k+1} = y_k - G(\varphi)\frac{f(y_k)}{\Psi[f'(w_k), f(w_k)]}\left[1 + \frac{1}{2}\phi + \frac{1}{2}\phi^2\right], \tag{9}$$

$$\phi = \frac{f'(w_k) - \Psi[f'(w_k), f(w_k)]}{f'(w_k)}, \quad \varphi = \frac{f(y_k)}{f(w_k)}.$$

We claim that under some mild conditions, Scheme 2 can attain CO four and because it requires three distinct functions assessment per iterations cycle, will satisfy the Kung and Traub's optimality condition.
To scale-up the CO of Scheme 2 from four to eight, the Scheme 2 is composed with an iterative function that involves an operator $f[y_n, z_n]$ (a divided difference operator at the iteration points y_n and z_n) and two differentiable functions $P(\eta)$ and $R(\mu)$ as described in Scheme 3 next.

Scheme 3

For w_0 close tow_*, compute w_{k+1} such that

$$z_k = y_k - G(\varphi)\frac{f(y_k)}{\Psi[f'(w_k), f(w_k)]}\left[1 + \frac{1}{2}\phi + \frac{1}{2}\phi^2\right],$$

$$w_{k+1} = z_k - \frac{f(z_k)}{f[y_n, z_n]}[P(\eta) \times R(\varphi)], \tag{10}$$

$$\eta = \frac{f(z_k)}{f(w_k)}.$$

The convergence analysis

This section provides information on the developed schemes convergence and their convergence order. An iterative scheme is said to converge if by Taylor's series expansions of all its functions, we are able to obtain from it an error equation of the form $E_{k+1} = \Omega E_k^{\mu} + O(E_k^{\mu+1}), \quad \mu, \Omega \in R$. In this case, μ and Ω are referred to as asymptotic error constant and CO respectively, see Ogbereyivwe and Izevbizua (2023) for more details.

Theorem 1

Assume the function $f(w), w \in \Lambda$ is at least three times differentiable in the domain Λ such that $f'(w) \neq 0 \ \forall \ w \in \Lambda$. Further, let w_0 be in the neighbourhood of w_*, then by using w_0 in the Scheme 2, will produce estimations of w_* that form a sequence that converges to w_* with CO four so long $G(0) = 1$, $G'(0) = -1$ and $G''(0) < \infty$.

Proof

Consider the fourth order Taylor's series expression of $f(w)$ about w_*given as:

$$f(w) = f(w_*) + \sum_{j=1}^{8} \frac{1}{k!} f^{(j)}(w_*)(w - w_*)^j + O(|w - w_*|^9). \tag{11}$$

Using (2) and set $w = w_k$ in (11), we have

$$f(w_k) = \sum_{j=1}^{8}\left[(-1)^{j+1}\frac{1}{j!}f^{(j)}(w_*)E_k^j\right] + O(|E_k|^9)$$

$$= f'(w_*)\left[A_1 + \sum_{i=2}^{8} A_n E_k^i + O(|E_k|^9)\right] \tag{12}$$

and

$$f'(w_k) = f'(w_*)\left[1 + \sum_{i=2}^{8} iA_n E_k^{i-1} + O(|E_k|^9)\right], \tag{13}$$

where $A_n = \frac{1}{i}\left(f^{(i)}(w_*) \big/ f'(w_*)\right), \ i \geq 2$.

When the expressions in (12) and (13) are utilized to expand y_k, we have

$$y_k = w_* + A_2E_k^2 + (2A_3 - 2A_2^2)E_k^3 + (4A_2^3 - 7A_2A_3 + 3A_4)E_k^4 + O(|E_k|^5). \tag{14}$$

Using (14), the expansion for $f(y_k)$ was obtained as:

$$f(y_k) = f'(w_*)[y_k + \sum_{n=2}^{8} A_n E_k^n + O(|E_k|^9)]. \tag{15}$$

Using (12), (13) and (15), we have that

$$\Psi[f'(w_k), f(w_k)] = f'(w_*)\begin{bmatrix} 1 + (-A_3 + 2(1+\theta)A_2^2)E_k^2 - 2((2 + 4\theta + \theta^2)A_2^3 \\ -(3 + 4\theta)A_2A_3 + A_4)E_k^3 + (2(4 + 13\theta + 7\theta^2 + \theta^3)A_2^4 \\ + \cdots + 4(2 + 3\theta)A_2A_4 - 3A_5)E_k^4 + O(|E_k|^5) \end{bmatrix}, \tag{16}$$

$$\frac{f(y_k)}{\Psi[f'(w_k), f(w_k)]} = A_2^2E_k^2 + (2A_3 - 2A_2^2)E_k^2 + \left((3 - 2\theta)A_2^3 - 6A_2A_3 + 3A_4\right) + O(|E_k|^5), \tag{17}$$

and

$$\phi = f'(w_*)\begin{bmatrix} 2A_2E_k + (4A_3 - 2(3+\theta)A_2^2)E_k^2 \\ +2((8 + 6\theta + \theta^2)A_2^3 - 2(5 + 2\theta)A_2A_3 + 3A_4)E_k^3 \\ -((20 + 25\theta + 9\theta^2 + \theta^3)A_2^4 - \cdots - 4A_5)E_k^4 + O(|E_k|^5) \end{bmatrix}. \tag{18}$$

Using (18), the next expansion is obtained.

$$1 + \frac{1}{2}\phi + \frac{1}{2}\phi^2 = f'(w_*)\begin{bmatrix} 1 + A_2E_k + (2A_3 - (1+\theta)A_2^2)E_k^2 \\ \left((-4 + 2\theta + \theta^2)A_2^3 - 2A_2(A_3 + 2\theta A_3) + 3A_4\right)E_k^3 \\ (-(-30 - 11\theta + 3\theta^2 + \theta^3) + \cdots + 4A_5)E_k^4 + O(|E_k|^5) \end{bmatrix}. \tag{19}$$

The division of (15) by (12) yielded

$$\varphi = A_2E_k + (2A_3 - 3A_2^2)E_k^2 + (8A_2^3 - 10A_2A_3 + 3A_4)E_k^3 + O(|E_k|^4). \tag{20}$$

Applying the expansion in (18) in the Taylor expansion of the weight function $G(\varphi)$ around the zero, we got

$$\begin{aligned} G(\varphi) = G(0) + G'(0)A_1E_k + \left(-3G'(0)A_2^2 + \frac{G''(0)}{2}A_2^2 + 2G'(0)A_3\right)E_k^3 \\ + \begin{pmatrix} 8G'(0)A_2^3 - 3G''(0)A_2^3 - 10G'(0)A_2A_3 \\ +2G''(0)A_2A_3 + 3G'(0)A_4 \end{pmatrix} E_k^4 + O(|E_k|^4). \end{aligned} \tag{21}$$

Using equations (15), (17), (19) and (21) in the expansion of w_{k+1}, we have

$$\begin{aligned} w_{k+1} &= w_* + ((1 - G(0))A_2)E_k^2 + \left((G(0) - G'(0) - 2)A_2^2 - 2(G(0) - 1)A_3\right)E_k^3 \\ &+ \left((4 + 4G'(0) - G''(0) + 3\theta G'(0))A_2^3 + (-7 + 2G(0) + 2G(0))A_2A_3 - 3(G(0) - 1)A_4\right)E_k^4 \\ &+ O(|E_k|^5). \end{aligned} \tag{22}$$

The expression in (22) is the error equation of the Scheme 2. For the error equation to be reduced to order 4, the coefficients of E_k^2 and E_k^3, must be annihilated. This require finding the values of $G(0)$ and $G'(0)$ that satisfies the next set of the equations.

$$\begin{cases} 1 - G(0) = 0 \\ G(0) - G'(0) - 2 = 0. \end{cases} \tag{23}$$

The set of equations in (23) is satisfied when $G(0) = 1$ and $G'(0) = -1$. Consequently, (22) will reduce to:

$$\begin{aligned} w_{k+1} = w_* - \left(A_2\left(G''(0)A_2^2 - 3\theta A_2^2 + A_3\right)\right)E_k^4 + \\ + \left((6 + 7G''(0) - 17\theta - 3\theta^2)A_2^4 + 2(1 - 3G''(0) + 9\theta)A_2^2A_3 - 2A_3^2 - 2A_2A_4\right)E_k^5 \\ + O(|E_k|^6). \end{aligned} \tag{24}$$

From (24), the error equation is of order four and unperturbed for any value of $G''(0) < \infty$. This ends the proof.

Remark 1

For any function $G(\varphi) \ni G(0) = 1$, $G'(0) = -1$ and $G''(0) < \infty$, and utilized in Scheme 2, a new fourth order iterative scheme can be obtained. In the implementation of Scheme 2, the evaluation of three different functions will be required in each iterative cycle. Consequently, its efficiency index is 1.5874 which is better than the HS and EHS.

Scheme 2 concrete form

Given $G(\varphi) = 1 - \varphi + \alpha\varphi^2, \ni \alpha \in \boldsymbol{R}$, enabled the suggestion of a new parameterized scheme (S4) given as:

$$y_k = w_k - \frac{f(w_k)}{f'(w_k)}, w_{k+1} = y_k - [1 + \varphi + \alpha\varphi^2]\frac{f(y_k)}{\Psi[f'(w_k), f(w_k)]}\left[1 + \frac{1}{2}\phi + \frac{1}{2}\phi^2\right]. \tag{25}$$

Theorem 2

Suppose the conditions imposed on $f(w)$ in Theorem 2.1 holds, then by using w_0 in Scheme 3, will produce estimations of w_* that form a sequence that converges to w_* with CO eight, so long$P(0) = R(0) = 1$, $R'(0) = 0$, $R''(0) = 2$,$P'(0) = 2$, $P''(0) < \infty$, $G''(0) = 3\theta - 2$.

Proof

Note that the expansion for w_{k+1} in (24) up to order eight is the same as the expansion for z_k. Consequently,

$$\begin{aligned} f(z_k) = f'(z_k)\ \Big[w_* - \left(A_2\left(G''(0)A_2^2 - 3\theta A_2^2 + A_3\right)\right)E_k^4 + \\ + \left((6 + 7G''(0) - 17\theta - 3\theta^2)A_2^4 + 2(1 - 3G''(0) + 9\theta)A_2^2A_3 - 2A_3^2 - 2A_2A_4\right)E_k^5 \\ + \cdots + \left(\left(-52 + 49\theta + 18\theta^2 + 3\theta^3 + G''(0)(3\theta - 3)\right)\right)A_2^5 + \cdots - 3A_2(2(2G''(0)^2 \\ -6\theta)A_3^2 + A_5))E_k^6 + \cdots + \left((-1253 + G''(0)^2 - 427\theta - 164\theta^2 + \cdots + 3\theta^2)\right)A_2^7 \\ + \cdots + 4(5 - 12G''(0) + 36\theta)A_3A_5 - 5A_7))E_k^8 + O(|E_k|^9). \end{aligned} \tag{26}$$

Using (11) and (26), we have

$$\frac{f(z_k)}{f(w_k)} = A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)E_k^3 + ((6 + 8G''(0) - 20\theta - 3\theta^2)A_2^4 + \cdots - 2A_2A_4)E_k^4$$
$$+ \cdots + ((-1595 + G''(0)^2 - 314\theta + \cdots + 2(17 - 24G''(0) + 72\theta)A_3A_5 - 5A_7))E_k^7$$
$$+O(|E_k|^8). \tag{27}$$

The Taylor's expansions of the weight functions $P(\eta)$ and $R(\mu)$ are

$$P(\eta) = P(0) - A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)P'(0)E_k^3 + ((6 + 8G''(0) - 20\theta - 3\theta^2)A_2^4$$
$$+3(1 - 2G''(0) + 6\theta)A_2^2A_3 - 2A_3^2 - 2A_2A_4)P''(0)E_k^4 + \cdots + (((-1595 + G''(0)^2$$
$$-314\theta - 107\theta^2 + \cdots - 2A_3^2 - 2A_2A_4)))P''(0)E_k^7 + O(|E_k|^8), \tag{28}$$

and

$$R(\mu) = R(0) + A_2R'(0)E_k + \left(2A_3R'(0) + \frac{1}{2}\left(A_2^2\left(R''(0) - 6R'(0)\right)\right)\right)E_k^2$$
$$+ \left(3A_4R'(0) + A_2^3\left(8R'(0) - 6R''(0)\right) + A_2A_3 + \left(4R''(0) + 10R'(0)\right)\right)E_k^3$$
$$+ \cdots + \left(-38A_4A_5R'(0) + 7A_8R'(0) + \cdots + A_3^3\left(216R''(0) - 228R'(0)\right)\right)E_k^8$$
$$+O(|E_k|^9). \tag{29}$$

By substituting (24), (26), (28) and (29) into Scheme 3, we have the next expression

$$X_{k+1} = w_* + A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)(P(0)R(0) - 1)E_k^4 + \textstyle\sum_{j=5}^{8} \Psi_j\, E_k^j + O(|E_k|^9). \tag{30}$$

where $\Psi_j = \Psi(G''(0), R(0), R'(0), R''(0), P(0), P'(0), P''(0), A_2, A_3, A_4, \theta)$.

We are required to annihilate the coefficients of E_k^j $(4 \le j \le 7)$ so as the error equation in (30) attain order eight. For the coefficient of E_k^4 to vanish, the next relation must hold.

$$P(0)R(0) - 1 = 0$$
$$\Longrightarrow \qquad P(0) = \frac{1}{R(0)}. \tag{31}$$

Using (31) and set $\Psi_5 = 0$, so as to make vanish the coefficient of E_k^5. This is only achievable when

$$R'(0) = 0. \tag{32}$$

By applying the result in (32) in $\Psi_6 = 0$, so as to annihilate the coefficient of E_k^6, we obtain the next relation.

$$R''(0) = 2R(0). \tag{33}$$

Consequent upon (33), when $\Psi_7 = 0$, then the coefficient of E_k^7 will disappear only when

$$P'(0)R(0) = 2 \;\text{ and }\; G''(0) = 3\theta - 2. \tag{34}$$

Inserting the results in (34) into Ψ_8, we have

$$\Psi_8 = A_2^2\left(2A_2^2 - A_3\right)\left((6\theta^2 - 8\theta - 1)A_2^3 - 4A_2A_3 + A_4\right)E_k^8. \tag{35}$$

When the expressions in (31), (32), (33), (34) and (35) are substituted into the one in (30), we have

$$X_{k+1} = w_* + A_2^2\left(2A_2^2 - A_3\right)\left((6\theta^2 - 8\theta - 1)A_2^3 - 4A_2A_3 + A_4\right)E_k^8 + O(|E_k|^9). \tag{36}$$

The error equation in (36), shows that Scheme 2 has convergence order eight. This ends the proof.

Remark 2

For any two functions that satisfy the weight functions conditions in Theorem 2, will produce an iterative scheme that has CO eight and since it require the computation of four different functions in an iteration cycle, its efficiency index is 1.6818.

Scheme 3 concrete form

Suppose $P(\eta) = 1 + 2\eta + \sigma\eta^2$ and $R(\mu) = 1 + \varphi^2$, where $\sigma \in R$. In this case, we suggest a new iterative scheme (S8) as:

$$y_k = w_k - \frac{f(w_k)}{f'(w_k)},$$
$$z_k = y_k - [1 - \varphi + (3\theta - 2)\varphi^2]\frac{f(y_k)}{\Psi[f'(w_k), f(w_k)]}\left[1 + \frac{1}{2}\phi + \frac{1}{2}\phi^2\right],$$
$$w_{k+1} = z_k - \frac{f(z_k)}{f[y_n, z_n]}(1 + 2\eta + \sigma\eta^2)(1 + \varphi^2). \tag{37}$$

RESULTS AND DISCUSSION

The implementation of the newly introduced schemes for deciding the solutions of nonlinear equations is presented in this section. The schemes were implemented via a designed computational iteration programs written in mpmath-PYTHON environment for all the schemes. The programs halt criterion is the function residual bound $|f(w_k)| \le \varepsilon$, where $\varepsilon = 10^{-500}$ is error tolerance level. To minimize truncation error, computation outputs were set to 1000 decimal places accuracy. To assess the performance of the schemes, the developed schemes number of iterations required to achieve convergence (NI), residual function of last iteration value $|f(w_{k+1})|$and computational order of convergence(μ_{coc}) due to Jay (2001) and estimated as

$$\mu_{coc} = \frac{\log|f(w_{k+1})|}{\log|f(w_k)|}, \tag{38}$$

were compared with that of some existing robust schemes. The compared schemes includes the CO three Householder scheme (HS) in Householder(1970), CO four modified HouseHolder schemes developed in Ogbereyivwe and Umar (2023b) (OS4); Nadeem *et al.*, (2023)(N4), Sarima *et al.*; (2020) (SM4), Obadah (2021)(OB4) and CO eight scheme put forward in Ogbereyivwe *et al.*, (2023).

The nonlinear equations used for the schemes implementation are presented in Table 1.

Table 1: Implementation functions with w_0

$f_i(w)$	$Solution\ (w_*)$	w_0
$f_1(w) = \sin 3w + w\cos w$	1.1977 …	1
$f_2(w) = \log w - \sqrt{w} + 5$	8.3094 …	8
$f_3(w) = (2 + w)\exp(w) - 1$	0.4428 …	0.5
$f_4(w) = e^{\sin w} - w + 1$	2.6306 …	2.3
$f_5(w) = \sin w^2 - 3w + 2$	0.9137 …	0.6
$f_6(w) = e^{-w} + \frac{w}{5} - 1$	4.9651 …	4.5

In Table 2 and Table 3, the numerical outputs of the developed schemes (EHS, S4, S8) and the compared schemes (HS, N4,OB4, SM4, OS4) when used to obtain the solution of the nonlinear equations in Table 1, are presented.

Table 2: Numerical results comparison

Schemes	$f_i(w)$	Parameters	NI	$\lvert f(w_{k+1})\rvert$	μ_{coc}
HS		N/A	6	$4.5E-449$	2.99
EHS		N/A	6	$3.2E-508$	3.01
N4		N/A	5	$9.0E-880$	4.00
OB4		N/A	5	$3.1E-893$	4.00
SM4		N/A	5	$7.3E-898$	4.01
OS4	$f_1(w)$	$\alpha=\delta=0.001$	5	$7.2E-887$	4.01
S4		$\theta=-1, \alpha=0$	5	$1.8E-918$	4.01
		$\theta=2/3, \alpha=8$	5	$5.5E-953$	4.00
OS8		N/A	3	$6.5E-505$	8.02
S8		$\theta=2/3, \sigma=0$	3	$2.8E-520$	8.00
		$\theta=-1, \sigma=6$	3	$4.5E-474$	8.03
HS		N/A	8	$7.1E-1439$	3.00
EHS		N/A	11	$2.9E-874$	3.00
N4		N/A	4	$3.8E-254$	4.03
OB4		N/A	4	$1.5E-285$	4.07
SM4		N/A	4	$3.7E-254$	4.03
OS4	$f_2(w)$	$\alpha=\delta=0.001$	5	$7.9E-662$	4.01
S4		$\theta=-1, \alpha=0$	4	$5.7E-246$	4.03
		$\theta=2/3, \alpha=8$	4	$9.9E-242$	4.03
OS8		N/A	3	$3.7E-461$	8.09
S8		$\theta=2/3, \sigma=0$	3	$2.2E-485$	8.08
		$\theta=-1, \sigma=6$	3	$2.1E-174$	8.28

Table 3:. Numerical results comparison

Schemes	$f_i(w)$	Parameters	NI	$\lvert f(w_{k+1})\rvert$	μ_{coc}
HS		N/A	7	$6.7E-579$	3.00
EHS		N/A	6	$5.6E-342$	3.00
N4		N/A	6	$3.1E-842$	3.99
OB4		N/A	6	$3.9E-660$	4.00
SM4	$f_3(w)$	N/A	5	$3.0E-366$	3.98
OS4		$\alpha=\delta=0.001$	5	$1.4E-377$	4.01
S4		$\theta=-1, \alpha=0$	7	$4.8E-685$	4.01
		$\theta=2/3, \alpha=8$	6	$4.7E-749$	3.98
OS8		N/A	3	$1.1E-172$	7.82
S8		$\theta=2/3, \sigma=0$	3	$4.5E-138$	8.12
		$\theta=-1, \sigma=6$	3	$5.2E-127$	7.94
HS		N/A	6	$1.2E-579$	3.00
EHS		N/A	6	$4.5E-600$	3.00
N4		N/A	4	$4.8E-286$	4.02
OB4		N/A	4	$1.9E-287$	4.04
SM4		N/A	4	$1.6E-287$	4.04
OS4	$f_4(w)$	$\alpha=\delta=0.001$	4	$3.1E-287$	4.04
S4		$\theta=-1, \alpha=0$	4	$8.3E-290$	4.03
		$\theta=2/3, \alpha=8$	4	$1.6E-292$	4.00
OS8		N/A	3	$1.7E-568$	8.00
S8		$\theta=2/3, \sigma=0$	3	$2.2E-571$	8.04
		$\theta=-1, \sigma=6$	3	$1.5E-559$	7.99

HS		N/A	6	$1.2E-625$	3.00
EHS		N/A	6	$3.3E-604$	3.00
N4		N/A	4	$1.8E-283$	3.99
OB4		N/A	4	$1.9E-281$	4.01
SM4		N/A	4	$1.1E-280$	4.01
OS4		$\alpha=\delta=0.001$	4	$2.5E-282$	4.03
S4	$f_5(w)$	$\theta=-1,\alpha=0$	4	$9.1E-279$	4.04
		$\theta=2/3,\alpha=8$	4	$1.9E-276$	4.00
OS8		N/A	3	$2.8E-621$	8.06
S8		$\theta=2/3,\sigma=0$	3	$2.5E-593$	8.01
		$\theta=-1,\sigma=6$	3	$6.3E-565$	8.07
HS		N/A	5	$2.5E-330$	3.00
EHS		N/A	5	$7.2E-337$	3.00
N4		N/A	4	$2.5E-402$	4.02
OB4		N/A	4	$1.7E-407$	3.99
SM4		N/A	4	$7.0E-408$	4.00
OS4	$f_6(w)$	$\alpha=\delta=0.001$	4	$9.1E-407$	3.99
S4		$\theta=-1,\alpha=0$	4	$9.1E-279$	4.04
		$\theta=2/3,\alpha=8$	4	$3.1E-431$	3.99
OS8		N/A	3	$2.2E-761$	8.01
S8		$\theta=2/3,\sigma=0$	3	$6.8E-750$	7.99
		$\theta=-1,\sigma=6$	3	$6.5E-724$	8.04

From Table 2 and Table 3, the residual function of the last iteration value $|f(w_{k+1})|$ for each schemes were presented in the form $M.NE-P$ which represents $M.N\times 10^{-P}$, where $M,N,P\in R$. Observe that for most of the tested equations, the residual function of the developed schemes are smaller than that of their corresponding compared schemes. This implies that the developed schemes are averagely better in precision when utilized to determine solutions of nonlinear equations.

Furthermore, the computational CO (μ_{coc}) obtained from all the developed schemes are the same with the theoretical CO derived in Theorem 1 and Theorem 2. Finally, the obtained number of iterations (NI) required by the developed schemes to achieve convergence when adopted to solve nonlinear equations, toughly competed with compared schemes.

CONCLUSION

The combination of the Taylor's expansion with weight function techniques were utilized to developed an extended Householder scheme (EHS) for solving nonlinear equations in this paper. The EHS was then modified with the motivation of increasing its CO, circumvent the Householder scheme major pitfalls and making it optimal in the sense of Kung and Traub (1974). The numerical implementation of the developed schemes reveals that they are very tough competitors to existing schemes.

REFERENCES

Householder, A. S. (1970), *The numerical treatment of a single nonlinear equation*. McGraw-Hill, New York.

Jay,L, O. (2001). A note on Q-order of convergence, *BIT Numerical Math.*, vol. 41, pp. 422-429.

Kung, H. and Traub, J. F. (1974). Optimal order of one-point and multi-point iteration, *Applied Mathematics and Computation*, vol. 21, pp. 643-651.

Nazeer, W; Tanveer, M.; Kang, S. M. and Naseem, A. (2016). A new Householder's method free from second derivatives for solving nonlinear equations and polynomiography, *J. of Nonlinear Science and Applications*, vol.9, pp. 998-1007.

Nadeem G. A., Aslam W. and Ali,F. (2023). An optimal fourth-order second derivative freeiterative method for nonlinear scientific equations, *Kuwait J. Sci.*, vol. 50(2A), pp. 1-15,.

Obadah, S. S. and Ishak, H. (2021). Optimal Eight-Order solver for nonlinear equations with applications in chemical engineering, *Intelligent Automation & Soft Computing*, vol. 27, no. 2, pp.379-389.

Ogbereyivwe, O. and Umar, S. S. (2023a). Behind Weerakoon and Fernando's Scheme: Is Weerakoon and Fernado Scheme version computationally better than its power-means variant?, *FUDMA Journal of Sciences*, vol. 7, no. 6, pp.368-371.

Ogbereyivwe, O. and Umar, S. S. (2023b). Optimal Parameterised Families of Modified Householder method with and without Restraint on Function Derivative, *Global Analysis and Discrete Mathematics (In press)*, https://doi.org/10.22128/GADM.2023.686.1093.

Ogbereyivwe, O. and Umar, S. S.and Izevbizua, O. (2023). Some high-Order convergence modifications of the Householder method for Nonlinear Equations, *Communications in Nonlinear Analysis*, vol. 10, pp. 1-11.

Ogbereyivwe, O. and Izevbizua, O. (2023). A three-free-parameter class of power series based iterative method for approximation of nonlinear equations solution, *Iranian Journal of Numerical Analysis and Optimization*, vol. 13, no. 2, 157-169.

Sarima, S. A. and Hashim, I. (2020). New optimal Newton-Householder methods for solving nonlinear equations and their dynamics, *Computers, Materials and Continua*, vol. 65, no. 1, pp. 69-85.