INTRODUCTION
Nonlinear problems of the form \( f(w) = 0 \), \( w \in \mathbb{R} \) and \( f: \mathbb{R} \rightarrow \mathbb{R} \) are continually encountered in many branches of science and engineering. This is because, real life situations are modelled into nonlinear equations and in most cases, their solutions are desired to be obtained for further study of the problems. Because of that fact that, most of these nonlinear equations defy the analytic procedures of obtaining their solutions, alternative procedures are being employed. For this reason, numerical analysis have considered the problem of obtaining solution to nonlinear equation as an important problem. In numerical analysis, the iterative procedures are being utilized to deal with the solutions of nonlinear equations. The procedures involves guessing the true solution of the nonlinear equation and then use it to perform a repetitive computation that produces sequence of approximations that eventually converge to the true solution of the nonlinear equation. This procedure is referred to as iterative process. An old and widely used iterative process for obtaining the solution of nonlinear equations is the Newton method (NM) (Traub, 1974) presented as

\[
y_{k+1} = w_k - \frac{f(w_k)}{f'_{(k)}}; \quad k = 0,1,2,...
\]  

(1)

The NM determines the solution of the nonlinear equation with convergence order (CO) two provided \( f'(w_k) \neq 0 \). Its efficiency is measured by the efficiency index due to Traub (1974) and given by \( E = \sqrt[\beta]{\mu} \), where \( \beta \) is number of distinct functions assessment per iteration cycle and \( \mu \) is the iterative scheme CO. Consequently, NM efficiency index is 1.4142, because it require the assessment of two functions \( f(w_k) \) and \( f'(w_k) \) in one cycle of iteration process. Many diverse modifications of the NM with better CO and efficiency are available in literature. They includes the one point and multipoint methods (see Householder,1970; Nazeer et al., 2016; Nadee et al.,2023, Sarima et al.,2020; Obadah,2020; Ogberereiywe and Umar,2023a; Ogberereiywe and Umar,2023b; Ogberereiywe et al.,2023). These modifications were made possible via the use of one or combinations of Taylor expansion, composition, quadrature, variational iteration, weight function or divided difference techniques. The fundamental motivation behind the development of modified NM in most literature, were mainly hinged on providing high CO and efficiency schemes. Consequently, we are motivated to develop a new and optimal family of iterative scheme that is an extension of the NM and the Householder method via the Taylor series expansion technique in this paper. The weight function and function derivative approximation technique was further fused into the developed method with the aim of scaling up its CO and efficiency. This resulted to the development of a new optimal CO eight method.

MATERIALS AND METHODS
The Schemes
Suppose \( w_k \) is the \( k \)th iteration estimation of the zero \( w^* \) of \( f(w) = 0 \) using an iterative scheme, then

\[
w_{k+1} = w_k + E_k
\]

(2)

where \( E_k \) is the \( k \)th iteration error. Consequently,

\[
f(w_k) = 0
\]

(3)

The Taylor’s series expression of \( f(w) \) around \( w_k \) when \( w \) is set as \( w = w_k \), is

\[
f(w^*) = f(w_k) + \sum_{j=1}^{\infty} \left( \frac{w-w_k}{1!} \right)^j f^{(j)}(w_k),
\]

(4)

where \( f^{(j)}(w_k) \) is \( j \)th derivative of the function \( f(w) \) and evaluated at \( w = w_k \). By disposing the higher error terms \( E_k \), \( j \geq 3 \), in (4) and then substituting it in (3), we have

\[
2f(w_k) + 2E_k f'(w_k) + E_k^2 f''(w_k) = 0.
\]

(5)

The solution of the equation in (5) with respect to \( E_k \) is

\[
E_k = \frac{f(w_k)}{f'(w_k)} \left[ 1 + \frac{\eta}{2} + \frac{\eta^2}{2} \right]
\]

(6)

where \( \eta = \frac{f''(w_k) - f'(w_k)^2}{f'(w_k)} \).

Consequent upon the equations in (2) and (6), a scheme for the estimation of the exact solution of nonlinear equation can be obtained by implementing the following scheme.

**Scheme 1**
For an initial guess \( w_0 \) in the neighbourhood of \( w \), where \( w_0, w \in \mathbb{R} \), the \( (k+1) \)th iteration estimation of the solution \( w \) of \( f(w) = 0 \) is obtained using:

\[
w_{k+1} = w_k - \frac{f(w_k)}{f'(w_k)} \left[ 1 + \frac{\eta}{2} + \frac{\eta^2}{2} \right].
\]

(7)
Remark 1: We note that, when both \( \eta = 0 \) and \( \eta^2 = 0 \) in (7), the famous NM is obtained. Again, for \( \eta \) not vanishing and \( \eta^2 \neq 0 \), the scheme in (7) reduces to the CO three Householder scheme (HS) presented in Householder (1970). Consequently, in the case where both \( \eta \neq 0 \) and \( \eta^2 \neq 0 \) in (7), we refer to it as the Extended Householder scheme (EHS) and claim that it retains the HS CO but with better precision. This claim will be substantiated in the proof of its convergence theorem and numerical implementation in the next sections.

The both of HS and EHS have convergence order three and require second derivative evaluation iin which makes it computationally expensive in implementation. Again, the schemes are not optimized in the view of Kung and Traub as reported in Kung and Traub (1974). Kung and Traub posited that an iterative scheme is optimized if it utilizes all of its \( \beta \) number of distinct function evaluation to attain a maximum convergence order of \( 2^\beta - 1 \). The EH and EHS both require three assessment of the distinct functions \( f(w_k), f'(w_k) \) and \( f''(w_k) \) at each iteration cycle. For the HS and EHS to satisfy the Kung and Traub optimal condition, they must be made to attain CO four without requiring additional new function evaluation. Again, not all functions are easily differentiable, let alone two times differentiable. To eliminate the presence of high derivates and optimize the EHS, the following estimations of \( f'(y_k) \) given as:

\[
\frac{f'(y_k)}{f'(w_k)} \approx \frac{f'(w_k + (\theta - 2)f(y_k))}{f'(w_k + \theta f(y_k))} \quad \theta \in \mathbb{R}
\]

was utilized to suggest the next scheme.

Scheme 2
For \( w_0 \) close to \( w_* \), compute \( w_{k+1} \) such that

\[
w_{k+1} = y_k - \frac{G(\phi)}{f'(w_k)} \left[ 1 + \frac{1}{2} \phi + \frac{1}{2} \phi^2 \right]
\]

(9)

\[
\phi = \frac{f'(w_k) - f'(y_k)}{f'(w_k)}
\]

(10)

We claim that under some mild conditions, Scheme 2 can attain CO four and because it requires three distinct functions assessment per iterations cycle, will satisfy the Kung and Traub’s optimality condition.

To scale-up the CO of Scheme 2 from four to eight, the Scheme 2 is composed with an iterative function that involves an operator \( f[y_n, x_0] \) (a divided difference operator at the iteration points \( y_n \) and \( x_0 \)) and two differentiable functions \( f'(\eta) \) and \( R(\mu) \) as described in Scheme 3 next.

Scheme 3
For \( w_0 \) close to \( w_* \), compute \( w_{k+1} \) such that

\[
w_{k+1} = \hat{y}_k - \frac{f(\hat{y}_k)}{f'(\hat{y}_k)} \left[ 1 + \frac{1}{2} \hat{\phi} + \frac{1}{2} \hat{\phi}^2 \right] = f(\hat{y}_k)
\]

(11)

\[
\eta = \hat{y}_k
\]

(12)

The convergence analysis
This section provides information on the developed schemes convergence and their convergence order. An iterative scheme is said to converge if by Taylor’s series expansions of all its functions, we are able to obtain from it an error equation of the form \( E_{k+1} = \Omega E_k^2 + O(E_k^{2s}) \), \( \mu, \Omega \in \mathbb{R} \). In this case, \( \mu \) and \( \Omega \) are referred to as asymptotic error constant and CO respectively, see Ogbereyivwe and Izevbiza (2023) for more details.

Theorem 1
Assume the function \( f(w) \) is at least three times differentiable in the domain \( \Lambda \) such that \( f'(w) \neq 0 \) \( \forall \) \( w \in \Lambda \). Further, let \( w_0 \) be in the neighbourhood of \( w_* \), then by using \( w_0 \) in the Scheme 2, will produce estimations of \( w_* \), that form a sequence that converges to \( w_* \), with CO four so long \( G(0) = 1, \ G'(0) = -1 \) and \( G(0) < \infty \).

Proof
Consider the fourth order Taylor’s series expression of \( f(w) \) about \( w_0 \) given as:

\[
f(w) = f(w_0) + \sum_{i=1}^{n} f^{(i)}(w_0)(w-w_0)^i + O(|w-w_0|^3).
\]

(13)

Using (2) and set \( w = w_k \) in (11), we have

\[
f(w_k) = \sum_{i=1}^{n} (-1)^{i+1} \frac{1}{i} f^{(i)}(w_0) E_k^i + O(|E_k|^3) = f'(w_0) [A_n + \sum_{i=2}^{n} A_i E_k^{i-1} + O(|E_k|^3)]
\]

and

\[
f'(w_k) = f'(w_0) [1 + \sum_{i=2}^{n} i A_i E_k^{i-1} + O(|E_k|^3)].
\]

(14)

where \( A_n = \frac{1}{i} f^{(i)}(w_0)/f'(w_0) \), \( i \geq 2 \).

When the expressions in (12) and (13) are utilized to expand \( y_k \), we have

\[
y_k = w_* + 2A_2 E_k^2 + (2A_2 - 2A_2^2) E_k^4 + (4A_2^2 - 7A_2 A_1 + 3A_2 A_1^2) E_k^6 + O(|E_k|^{10}).
\]

Using (14), the expansion for \( f(y_k) \) was obtained as:

\[
f(y_k) = f'(w_0) [y_k + \sum_{i=2}^{n} A_i E_k^{i-1} + O(|E_k|^3)].
\]

(15)

Using (12), (13) and (15), we have that
\begin{align*}
\Psi[f'(w_k), f'(w_0)] &= f'(w) \begin{bmatrix}
1 + (-A_3 + 2(1 + \theta)A_k^3)E_k^2 - 2((2 + 4\theta + 3\theta^2)A_k^3) \\
-(3 + 4\theta)A_3A_3 + A_4E_k^3 + (2(4 + 13\theta + 7\theta^2 + 3\theta^3)A_k^3) \\
+ \cdots + 4(2 + 3\theta)A_3A_4 - 3A_2E_k^4 + O(E_k^5) 
\end{bmatrix}, \\
&= A_k^3E_k^2 + (2A_3 - 2A_k^3)E_k^4 + \left(3 - 2\theta A_3^2 - 6A_2A_3 + 3A_4\right)E_k^3 + O(E_k^5).
\end{align*}

and
\begin{align*}
\phi &= f'(w) \begin{bmatrix}
2A_3E_k^2 + (4A_3 - 2(3 + \theta)A_k^3)E_k^2 \\
+ 2((8 + 6\theta + 3\theta^2)A_k^3 - 2(5 + 2\theta)A_3A_3 + 3A_4)E_k^3 \\
-((20 + 25\theta + 9\theta^2 + 3\theta^3)A_k^4 - \cdots - 4A_3E_k^4 + O(E_k^5))
\end{bmatrix}.
\end{align*}

Using (18), the next expansion is obtained.
\begin{align*}
1 + \frac{1}{2} \phi + \frac{1}{4} \phi^2 &= f'(w) \begin{bmatrix}
1 + A_3E_k^2 + (2A_3 - (1 + \theta)A_k^3)E_k^2 \\
((-4 + 2\theta + \theta^2)A_k^2 - 2A_3(A_3 + 2\theta A_3) + 3A_4)E_k^3 \\
\left((-30 - 11\theta + 3\theta^2 + \theta^3) + \cdots + 4A_3E_k^4 + O(E_k^5)\right)
\end{bmatrix}.
\end{align*}

The division of (15) by (12) yielded
\begin{align*}
\varphi &= A_k^3E_k^2 + (2A_3 - 3A_k^3)E_k^2 + (8A_k^2 - 10A_2A_3 + 3A_4)E_k^3 + O(E_k^4),
\end{align*}

Applying the expansion in (18) in the Taylor expansion of the weight function \(G(\varphi)\) around the zero, we get
\begin{align*}
G(\varphi) &= G(0) + G'(0)A_k^3E_k^2 + \left(-3G''(0)A_k^2 + \frac{G'''(0)}{2}A_k^2 + 2G''(0)A_k\right)E_k^3 \\
&+ \left(\frac{G''(0)}{2}A_k^2 - 3G''''(0)A_k^2 - 10G'''(0)A_2A_3\right)E_k^4 + O(E_k^5).
\end{align*}

Using equations (15), (17), (19) and (21) in the expansion of \(w_{k+1}\), we have
\begin{align*}
w_{k+1} &= w_k + (1 - G(0))A_k^3E_k^2 + \left((G(0) - G'(0) - 2)A_k^2 - 2(G(0) - 1)A_3\right)E_k^3 \\
&+ \left((4 + 4G'(0) - G''(0) + 3G''(0))A_k^2 + (7 + 2G(0) + 2G(0))A_2A_3 - 3(G(0) - 1)A_4\right)E_k^4 \\
&+ O(E_k^5).
\end{align*}
The expression in (22) is the error equation of the Scheme 2. For the error equation to be reduced to order 4, the coefficients of \(E_k^2\) and \(E_k^3\), must be annihilated. This require finding the values of \(G(0)\) and \(G'(0)\) that satisfies the next set of the equations.
\begin{align*}
1 - G(0) &= 0, \\
(G(0) - G'(0) - 2) &= 0.
\end{align*}
The set of equations in (23) is satisfied when \(G(0) = 1\) and \(G'(0) = -1\). Consequently, (22) will reduce to:
\begin{align*}
w_{k+1} &= w_k - \left(A_k^3G''(0)A_k^2 - 3\theta A_k^2 + A_k^3\right)E_k^2 + \\
&+ \left(6 + 7G''(0) - 17\theta - 3\theta^2)A_k^2 + 2(1 - 3G''(0) + 9\theta)A_k^2A_k^2 - 2A_2A_3\right)E_k^4 \\
&+ O(E_k^5).
\end{align*}
From (24), the error equation is of order four and unperturbed for any value of \(G''(0) < \infty\). This ends the proof.

**Remark 1**

For any function \(G(\varphi) \ni G(0) = 1, G'(0) = -1\) and \(G''(0) < \infty\), and utilized in Scheme 2, a new fourth order iterative scheme can be obtained. In the implementation of Scheme 2, the evaluation of three different functions will be required in each iterative cycle. Consequently, its efficiency index is 1.5874 which is better than the HS and EHS.

**Scheme 2 concrete form**

Given \(G(\varphi) = 1 - \varphi + a\varphi^2, \alpha \in \mathbb{R}\), enabled the suggestion of a new parameterized scheme (S4) given as:
\begin{align*}
y_k &= w_k - f'(w_k)w_{k+1} = y_k - [1 + \varphi + a\varphi^2] \frac{f'(y_k)}{f'(w_k)} \begin{bmatrix}
1 + \frac{1}{2} \phi + \frac{1}{4} \phi^2
\end{bmatrix}.
\end{align*}

**Theorem 2**

Suppose the conditions imposed on \(f(w)\) in Theorem 2.1 holds, then by using \(w_0\) in Scheme 3, will produce estimations of \(w_k\) that form a sequence that converges to \(w\), with CO eight, so long \(P(0) = 1, R'(0) = 0, R''(0) = 2, P'(0) = 2, P''(0) < \infty, G''(0) = 3\theta - 2\).

**Proof**

Note that the expansion for \(w_{k+1}\) in (24) up to order eight is the same as the expansion for \(x_k\). Consequently,
\begin{align*}
f(x_k) &= f'(x_k) \left[ y_k - \left(A_k^3G''(0)A_k^2 - 3\theta A_k^2 + A_k^3\right)E_k^2 + \\
&+ \left(6 + 7G''(0) - 17\theta - 3\theta^2)A_k^2 + 2(1 - 3G''(0) + 9\theta)A_k^2A_k^2 - 2A_2A_3\right)E_k^4 \\
&+ \cdots + \left((-52 + 49\theta + 18\theta^2 + 3\theta^3 + G''''(0)(3\theta - 3))\right)A_k^2 + \cdots - 3A_2(2G''(0))^2 \\
&- 6A_k^2 + A_k^3)E_k^4 + \cdots + \left((-1253 + G''(0)^2 - 427\theta - 164\theta + 3\theta^2)\right)A_k^2 \\
&+ \cdots + 4(5 - 12G''(0) + 36\theta)A_2A_3 - 5A_2)E_k^6 + O(E_k^6)\right] \frac{1}{2} \phi + \frac{1}{4} \phi^2
\end{align*}

Using (11) and (26), we have
\[
\frac{f(x_k)}{f(w_k)} = A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)E_k^2 + ((6 + 8G''(0) - 20\theta - 3\theta^2)A_2^2 + \cdots - 2A_2A_3)E_k^2 \\
+ \cdots + ((-1595 + G''(0)^2 - 314\theta + \cdots + 2(17 - 24G''(0) + 72\theta)A_3A_5 - 5A_2)E_k^2 \\
+ O(|E_k|^3).
\]

(27)

The Taylor’s expansions of the weight functions \(P(\eta)\) and \(R(\mu)\) are
\[
P(\eta) = P(0) - A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)P'(0)E_k^2 + ((6 + 8G''(0) - 20\theta - 3\theta^2)A_2^2 \\
+ 3(1 - 2G''(0) + 6\theta)A_2A_3 - 2A_2^2 - 2A_2A_3)P'(0)E_k^2 + \cdots + ((-1595 + G''(0)^2 \\
- 314\theta - 10\theta^2 + \cdots - 2A_2 - 2A_2A_4))P'(0)E_k^2 + O(|E_k|^3).
\]

(28)

and
\[
R(\mu) = R(0) + A_2R'(0)E_k + \left(2A_2R'(0) + \frac{1}{2}\left(A_2^2(R'(0) - 6R'(0))\right)\right)E_k^2 \\
+ \left(3A_4R'(0) + A_2^2\left,(R'(0) - 6R'(0)) + A_2A_3 + \left(4R''(0) + 10R'(0)\right)\right)E_k^2 \\
+ \cdots + \left(-3A_4A_2R'(0) + 7A_2R'(0) + \cdots + A_2^2(216R''(0) - 228R'(0))\right)E_k^2 \\
+ O(|E_k|^3).
\]

(29)

By substituting (24), (26), (28) and (29) into Scheme 3, we have the next expression
\[
X_{k+1} = w_{k} + A_2(G''(0)A_2^2 - 3\theta A_2^2 + A_3)(P(0)R(0) - 1)E_k^2 + \sum_{j=2}^{n}w_j E_k^2 + O(|E_k|^3),
\]

(30)

where \(w_j = \Psi(G''(0), R(0), R'(0), R''(0), P(0), P'(0), P''(0), A_2, A_3, A_4, \theta)\).

We are required to annihilate the coefficients of \(E_k^2\) \((4 \leq j \leq 7)\) so as the error equation in (30) attain order eight. For the coefficient of \(E_k^2\) to vanish, the next relation must hold.
\[
P(0)R(0) - 1 = 0 \Rightarrow P(0) = \frac{1}{R(0)}
\]

(31)

Using (31) and set \(\Psi = 0\), so as to make vanish the coefficient of \(E_k^2\). This is only achievable when
\[
R'(0) = 0.
\]

(32)

By applying the result in (32) in \(\Psi = 0\), so as to annihilate the coefficient of \(E_k^2\), we obtain the next relation.
\[
R''(0) = 2R(0).
\]

(33)

Consequent upon (33), when \(\Psi = 0\), then the coefficient of \(E_k^2\) will disappear only when
\[
P'(0)R(0) = 2 \quad \text{and} \quad G''(0) = 3\theta - 2.
\]

(34)

Inserting the results in (34) into \(\Psi\), we have
\[
\Psi = A_2^2\left(2A_2^2 - A_2^2\right)\left((6\theta^2 - \theta - 1)A_2 - 4A_2^2A_3 + A_4\right)E_k^2.
\]

(35)

When the expressions in (31), (32), (33), (34) and (35) are substituted into the one in (30), we have
\[
X_{k+1} = w_{k} + A_2^2(2A_2^2 - A_2^2)\left((6\theta^2 - \theta - 1)A_2 - 4A_2^2A_3 + A_4\right)E_k^2 + O(|E_k|^3).
\]

(36)

The error equation in (36), shows that Scheme 2 has convergence order eight. This ends the proof.

**Remark 2**

For any two functions that satisfy the weight functions conditions in Theorem 2, will produce an iterative scheme that has CO eight and since it require the computation of four different functions in an iteration cycle, its efficiency index is 1.6818.

**Scheme 3 concrete form**

Suppose \(P(\eta) = 1 + 2\eta + \eta^2\) and \(R(\mu) = 1 + \phi^2\), where \(\sigma, \phi \in R\). In this case, we suggest a new iterative scheme (88) as:
\[
y_k = y_{k-1} - \frac{f(w_k)}{f'(w_k)};
\]
\[
z_k = y_k - \left[1 - \phi + (3(\theta - 2)\phi^2)\right] \frac{f(y_k)}{\Psi[f'(w_k), f'(y_k)]} \left[1 + \frac{1}{2} + \frac{1}{2} \phi^2\right];
\]
\[
w_{k+1} = \frac{f(y_k)}{f'(y_k)} (1 + 2\eta + \eta^2)(1 + \phi^2).
\]

(37)

**RESULTS AND DISCUSSION**

The implementation of the newly introduced schemes for deciding the solutions of nonlinear equations is presented in this section. The schemes were implemented via a designed computational iteration programs written in mpmath-PYTHON environment for all the schemes. The programs halt criterion is the function residual bound \(|f(w_k)| \leq \varepsilon\), where \(\varepsilon = 10^{-500}\) is error tolerance level. To minimize truncation error, computational outputs were set to 1000 decimal places accuracy. To assess the performance of the schemes, the developed schemes number of iterations required to achieve convergence (NI), residual function of last iteration value \(|f(w_{k+1})|\) and computational order of convergence(\(\mu_{coc}\)) due to Jay (2001) and estimated as
\[
\mu_{coc} = \log_{\left|\frac{|f(w_{k+1})|}{|f'(w_k)|}\right|},
\]

(38)

were compared with that of some existing robust schemes. The compared schemes includes the CO three Householder scheme (HS) in Householder(1970), CO four modified Householder schemes developed in Ogberiyiwe and Umar (2023b) (OS4); Nadeem et al., (2023)(N4), Sarima et al.; (2020) (SM4), Obada (2021) (OB4) and CO eight scheme put forward in Ogberiyiwe et al., (2023).

The nonlinear equations used for the schemes implementation are presented in Table 1.
Table 1: Implementation functions with \( w_o \)

<table>
<thead>
<tr>
<th>( f_i(w) )</th>
<th>( \text{Solution (} w_o \text{)} )</th>
<th>( w_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(w) = \sin 3w + w \cos w )</td>
<td>1.1977 ...</td>
<td>1</td>
</tr>
<tr>
<td>( f_2(w) = \log w - \sqrt{w} + 5 )</td>
<td>8.3094 ...</td>
<td>8</td>
</tr>
<tr>
<td>( f_3(w) = (2 + w) \exp(w) - 1 )</td>
<td>0.4428 ...</td>
<td>0.5</td>
</tr>
<tr>
<td>( f_4(w) = \exp \sin w - w + 1 )</td>
<td>2.6306 ...</td>
<td>2.3</td>
</tr>
<tr>
<td>( f_5(w) = \sin w^2 - 3w + 2 )</td>
<td>0.9137 ...</td>
<td>0.6</td>
</tr>
<tr>
<td>( f_6(w) = e^{-w} + \frac{w}{5} )</td>
<td>4.9651 ...</td>
<td>4.5</td>
</tr>
</tbody>
</table>

In Table 2 and Table 3, the numerical outputs of the developed schemes (EHS, S4, S8) and the compared schemes (HS, N4, OB4, SM4, OS4) when used to obtain the solution of the nonlinear equations in Table 1, are presented.

Table 2: Numerical results comparison

| Schemes | \( f_i(w) \) | Parameters | \( N_I \) | \( |f(w_{k+1})| \) | \( \mu_{cose} \) |
|---------|--------------|------------|--------|----------------|----------------|
| HS      | N/A          | 6          | 4.5E - 449 | 2.99 |
| EHS     | N/A          | 6          | 3.2E - 508 | 3.01 |
| N4      | N/A          | 5          | 9.0E - 880 | 4.00 |
| OB4     | N/A          | 5          | 3.1E - 393 | 4.00 |
| SM4     | N/A          | 5          | 7.3E - 398 | 4.01 |
| OS4     | \( a = \delta = 0.001 \) | 5          | 7.2E - 398 | 4.01 |
| S4      | \( \theta = -1, \alpha = 0 \) | 5          | 1.8E - 398 | 4.01 |
| OS8     | N/A          | 3          | 6.5E - 398 | 8.02 |
| S8      | \( \theta = 2/3, \sigma = 0 \) | 3          | 2.8E - 398 | 8.00 |
|         | \( \theta = -1, \sigma = 6 \) | 3          | 4.5E - 398 | 8.03 |
| HS      | N/A          | 8          | 7.1E - 1439 | 3.00 |
| EHS     | N/A          | 11         | 2.9E - 874 | 3.00 |
| N4      | N/A          | 4          | 3.8E - 254 | 4.03 |
| OB4     | N/A          | 4          | 1.5E - 285 | 4.07 |
| SM4     | N/A          | 4          | 3.7E - 254 | 4.03 |
| OS4     | \( a = \delta = 0.001 \) | 5          | 7.9E - 662 | 4.01 |
| S4      | \( \theta = -1, \alpha = 0 \) | 4          | 5.7E - 246 | 4.03 |
| OS8     | N/A          | 3          | 3.7E - 461 | 8.09 |
| S8      | \( \theta = 2/3, \sigma = 0 \) | 3          | 2.2E - 485 | 8.08 |
|         | \( \theta = -1, \sigma = 6 \) | 3          | 2.1E - 174 | 8.28 |

Table 3: Numerical results comparison

| Schemes | \( f_i(w) \) | Parameters | \( N_I \) | \( |f(w_{k+1})| \) | \( \mu_{cose} \) |
|---------|--------------|------------|--------|----------------|----------------|
| HS      | N/A          | 7          | 6.7E - 579 | 3.00 |
| EHS     | N/A          | 6          | 5.6E - 342 | 3.00 |
| N4      | N/A          | 6          | 3.1E - 942 | 3.99 |
| OB4     | N/A          | 6          | 3.9E - 660 | 4.00 |
| SM4     | N/A          | 5          | 3.0E - 366 | 3.98 |
| OS4     | \( a = \delta = 0.001 \) | 5          | 1.4E - 377 | 4.01 |
| S4      | \( \theta = -1, \alpha = 0 \) | 7          | 4.8E - 685 | 4.01 |
| OS8     | N/A          | 3          | 1.1E - 172 | 7.82 |
| S8      | \( \theta = 2/3, \sigma = 0 \) | 3          | 4.5E - 138 | 8.12 |
|         | \( \theta = -1, \sigma = 6 \) | 3          | 5.2E - 127 | 7.94 |
| HS      | N/A          | 6          | 1.2E - 579 | 3.00 |
| EHS     | N/A          | 6          | 4.5E - 600 | 3.00 |
| N4      | N/A          | 4          | 4.8E - 286 | 4.02 |
| OB4     | N/A          | 4          | 1.9E - 287 | 4.04 |
| SM4     | N/A          | 4          | 1.6E - 287 | 4.04 |
| OS4     | \( a = \delta = 0.001 \) | 4          | 3.1E - 287 | 4.04 |
| S4      | \( \theta = -1, \alpha = 0 \) | 4          | 8.3E - 290 | 4.03 |
| OS8     | N/A          | 3          | 1.7E - 568 | 8.00 |
| S8      | \( \theta = 2/3, \sigma = 0 \) | 3          | 2.2E - 571 | 8.04 |
|         | \( \theta = -1, \sigma = 6 \) | 3          | 1.5E - 559 | 7.99 |
HS
EHS
N4
OB4
SM4
OS4
S4
EHS
N4
OB4
SM4
OS4
S4

From Table 2 and Table 3, the residual function of the last iteration value \( |f(w_{k+1})| \) for each scheme were presented in the form \( M.N.P \) which represents \( M,N \times 10^{-P} \), where \( M,N,P \in R \). Observe that for most of the tested equations, the residual function of the developed schemes are smaller than that of their corresponding compared schemes. This implies that the developed schemes are averagely better in precision when utilized to determine solutions of nonlinear equations.

Furthermore, the computational CO (\( \mu_{\text{comp}} \)) obtained from all the developed schemes are the same with the theoretical CO derived in Theorem 1 and Theorem 2. Finally, the obtained number of iterations (\( N \)) required by the developed schemes to achieve convergence when adopted to solve nonlinear equations, toughly competed with compared schemes.

CONCLUSION
The combination of the Taylor’s expansion with weight function techniques were utilized to developed an extended Householder scheme (EHS) for solving nonlinear equations in this paper. The EHS was then modified with the motivation of increasing its CO, circumvent the Householder scheme major pitfalls and making it optimal in the sense of Kung and Traub (1974). The numerical implementation of the developed schemes reveals that they are very tough competitors to existing schemes.

REFERENCES


