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A NEW SECOND DERIVATIVE METHODS WITH HYBRID PREDICTORS FOR SOLVING STIFF AND NON-STIFF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This study derives new second derivative linear multistep methods with efficient criteria sufficient for the solvability of the stiff initial value problems by means of interpolation and collocation techniques. The hybrid predictors in the procedure are nested. The stiff initial value issues in ordinary differential equations were approximated by using power series as the basis function. The method's stability properties were examined and subsequently provided. The region of absolute stability of the novel schemes was studied using the boundary locus method. Through its combination as a block matrix, the resulting approaches are applied to solve a number of stiff initial value issues. The new techniques produced numerical findings and errors that compared favorably with some existing methods in the literature.

Keywords: Power series collocation, Interpolation, Hybrid multistep method, Nested

INTRODUCTION

Very few differential equations used to model problems in the physical sciences, engineering, biological sciences, and social sciences can be solved analytically. These problems arise in a variety of domains of applications. In situations where an analytical solution is unknown, we attempt to solve the problem by utilizing suitable numerical techniques to get an approximate solution. In this work, we want to determine the approximate solution of the initial value problem (IVP) of the first order differential equation of the form

$y'(x) = f(x, y(x)), y(x_0) = y_0, x \in [a, b]$
$f: R \times R^m \to R^m; y: R \to R^m$ (1)

in a given interval of solution were a and b are finite

Physical quantities are represented by the above. The derivatives show the rates at which a function changes at a particular moment.

There are two types of ordinary differential equations: stiff and non-stiff. Widely different time scales exist in stiff systems, where certain solution components degrade far more quickly than others. Stiff ordinary differential equations are found mostly in fields like chemical kinetics, nuclear reactors, vibrations, chemical reactions, control theory, quantum physics, and engineering, including electrical circuit theory, according to Okoughae and Ikhile (2012) and Babangida and Musa (2016). Since many stiff ordinary differential equations are not analytically solvable, numerical methods are used to solve them

Curtis and Hischfelder (1952) discovered stiffness in differential equation problems for the first time when they created the Backward Differentiation Formula (BDF). According to Ajie (2016) and Abassi et al. (2014), the stiffness problem has been recognized for a while and can be resolved using implicit approaches, which are thought to work better than explicit methods.

Ehiemua and Agbeboh (2019), Adoghe (2021), Abhulimen and Ukpebor (2019), Sabo et al. (2018, 2019) have all employed implicit Runge-Kutta and linear multistep approaches to handle stiff initial value issues. A possibly effective numerical technique for solving stiff ordinary differential equations needs to have a sufficiently wide region of absolute stability and good accuracy, according to Okoughae & Ikhile (2014). The stability property is A-stability, according to Hairer and Wanner.

For the purpose of solving stiff and non-stiff issues, Esuabana and Ekoro (2017) created a family of hybrid linear multistep algorithms using nested hybrid predictors. The second derivative approaches are A-stable. According to Kulikov (2015), layered approaches can provide dense production of integration results of the accuracy as the order without requiring additional costs because they have sufficiently high stage and classical orders. He went on to say that the integration of stiff and non-stiff problems can be done effectively via nested approaches. This study aims to derive and develop methods for solving stiff and non-stiff problems using hybrid predictors in third derivative linear multistep approaches.

MATERIALS AND METHODS

The polynomials to be used to approximate the solution of problem (1) is given as

$y(x) = \sum_{j=0}^{N} a_j x^j$ (2)

$y'(x) = \sum_{j=1}^{N} j a_j x^{j-1} = f(x, y)$ (3)

$y''(x) = \sum_{j=1}^{N} j(j-1) a_j x^{j-2} = f'(x, y)$ (4)

Where N is given as $2k + 2$ and $2k + 3$ respectively

Interpolating the value y with equation (2) at $x = x_n$ and collocating at $x = x_{n+j}, j = 0, 1, \frac{1}{2}, \frac{1}{4}$ to obtain a system of equation of the form

$X = A^{-1}B$ (5)

where

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \dots & x_n^N \\ \dots & 1 & 2x_n & \dots & (2k+2)x_n^{N-2} & (2k+3)x_n^{N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2kx_{n+k} & (2k+1)x_{n+k}^{N-3} & \dots & (2k+3)x_{n+k}^{N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 2 & 2(2k+1)x_{n+k}^{N-4} & 3(2k+2)x_{n+k}^{N-3} \dots . & 4(2k+3)x_{n+k}^{N-2} \end{bmatrix} \quad (6)$$

The second interpolating polynomial to use is given as

$y(x) = \sum_{j=0}^{2k+2} a_j\, x^j$ (7)

After the values of $a_j's, j = 0,2,\ldots 2k+3$ are obtained by solving equation (5) for, they are entered into equation (2). Following a great deal of algebraic simplification, we arrived at the discrete techniques shown below.

$$y_{n+1} = y_n + \frac{1}{10}hf_n + \frac{71}{270}hf_{n+1} + \frac{2}{5}hf_{n+\frac{1}{2}} + \frac{32}{315}hf_{n+\frac{1}{4}} - \frac{1}{45}h^2 f'_{n+1} \quad (8)$$

$$y_{n+\frac{1}{2}} = -\frac{109}{67}y_n + \frac{176}{67}y_{n+\frac{1}{4}} - \frac{115}{536}hf_n + \frac{31}{536}hf_{n+1} - \frac{1}{67}h^2 f'_{n+1} \quad (9)$$

$$y_{n+\frac{1}{4}} = \frac{189}{256}y_n + \frac{67}{256}y_{n+1} + \frac{27}{256}hf_n - \frac{15}{128}hf_{n+1} - \frac{9}{512}h^2 f'_{n+1} \quad (10)$$

Analysis of the properties of the new methods

Order of the methods

In this section we derived the order of the methods in (7--9)

The methods in (7-9) is rewriting in the following form:

$$y_{n+1} - y_n = \frac{1}{10}hf_n + \frac{71}{270}hf_{n+1} + \frac{2}{5}hf_{n+\frac{1}{2}} + \frac{32}{315}hf_{n+\frac{1}{4}} - \frac{1}{45}h^2 f'_{n+1}$$

$$y_{n+\frac{1}{2}} + \frac{109}{67}y_n - \frac{176}{67}y_{n+\frac{1}{4}} = -\frac{115}{536}hf_n + \frac{31}{536}hf_{n+1} - \frac{1}{67}h^2 f'_{n+1}$$

$$y_{n+\frac{1}{4}} - \frac{189}{256}y_n - \frac{67}{256}y_{n+1} = \frac{27}{256}hf_n - \frac{15}{128}hf_{n+1} - \frac{9}{512}h^2 f'_{n+1} \quad (10)$$

Equations (10-12 are put in the matrix form as follows

$$\begin{bmatrix} 0 & 0 & -\frac{189}{256} \\ 0 & 0 & \frac{109}{67} \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & -\frac{67}{256} \\ -\frac{176}{67} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{27}{256} \\ 0 & 0 & -\frac{115}{536} \\ 0 & 0 & \frac{1}{10} \end{bmatrix}\begin{bmatrix} f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{4}} \\ f_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{15}{256} \\ 0 & 0 & \frac{31}{536} \\ \frac{32}{135} & \frac{2}{5} & \frac{71}{270} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} +$$

$$h^2\begin{bmatrix} 0 & 0 & \frac{9}{512} \\ 0 & 0 & -\frac{1}{67} \\ 0 & 0 & -\frac{1}{45} \end{bmatrix}\begin{bmatrix} f'_{n+\frac{1}{4}} \\ f'_{n+\frac{1}{2}} \\ f'_{n+1} \end{bmatrix} \quad (11)$$

From equation (10) we shall define the following parameters

$$\alpha_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} -\frac{189}{256} \\ \frac{109}{67} \\ -1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1 \\ -\frac{176}{67} \\ 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_5 = \begin{bmatrix} -\frac{67}{256} \\ 0 \\ 1 \end{bmatrix}$$

$$\beta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \beta_2 = \begin{bmatrix} \frac{27}{256} \\ -\frac{115}{536} \\ \frac{1}{10} \end{bmatrix}, \beta_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{32}{135} \end{bmatrix}, \beta_5 = \begin{bmatrix} -\frac{15}{128} \\ \frac{31}{536} \\ \frac{71}{270} \end{bmatrix}$$

$$\delta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \delta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \delta_5 = \begin{bmatrix} \frac{9}{512} \\ -\frac{1}{67} \\ -\frac{1}{45} \end{bmatrix}$$

The linear operator L associated with the nested method is given as

$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j\, y(x+jh) - h\beta y'(x+jh) - h^2\delta_j y''(x+jh)]$ (12)

The Taylor series expansion of (12) yield

$$[y(x_n);h] = \sum_{j=0}^{k}\alpha_j\left[y(x) + jhy'(x) + \frac{(jh)^2}{2!}y''(x) + \ldots\right]$$

$$-\sum_{j=0}^{k}h\beta_j\left[y'(x) + jhy''(x) + \frac{(jh)^2}{2!}y'''(x) + \ldots\right] \quad (13)$$

$$-h^2\delta_j\left[y''(x) + jhy'''(x) + \frac{(jh)^2}{2!}y''''(x_n) + \ldots\right]$$

Collecting terms in powers of h. we

$$L[y(x_n);h] = \sum_{j=0}^{k} \alpha_j\, y(x_n) + \sum_{j=0}^{k} j\alpha_j - \beta_j)hy'(x_n)$$
$$+\sum_{j=0}^{k} \frac{j^2}{2!}\alpha_j - j\beta_j - \delta_j)h^2 y''(x_n) \quad (14)$$
$$+\sum_{j=0}^{k} \frac{j^3}{3!}\alpha_j - \frac{j^2}{2!}\beta_j - j\delta_j)h^3 y'''(x_n) + \ldots$$

Thus we have that

$$C_0 = \sum_{j=0}^{k} \alpha_j$$
$$C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j)$$
$$C_2 = \sum_{j=0}^{k} \frac{j^2}{2!}\alpha_j - \sum_{j=0}^{k} j\beta_j - \sum_{j=0}^{k} \delta_j$$
$$C_3 = \sum_{j=0}^{k} \frac{j^3}{3!}\alpha_j - \sum_{j=0}^{k} \frac{j^2}{2!}\beta_j - \sum_{j=0}^{k} j\delta_j$$

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$$C_p = \sum_{j=0}^{k} \frac{j^p}{p!}\alpha_j - \sum_{j=0}^{k} \frac{j^{p-1}}{(p-1)!}\beta_j - \sum_{j=0}^{k} \frac{j^{p-2}}{(p-2)!}\delta_j$$
$$C_{p+1} = \sum_{j=0}^{k} \frac{j^{p+1}}{(p+1)!}\alpha_j - \sum_{j=0}^{k} \frac{j^p}{(p)!}\beta_j - \sum_{j=0}^{k} \frac{\delta_j j^{p-1}}{(p-1)!}$$

The methods has order is as follows

$$y_{n+\frac{1}{4}} - \frac{189}{256}y_n - \frac{67}{256}y_{n+1} = \frac{27}{256}\boldsymbol{h}f_n - \frac{15}{128}\boldsymbol{h}f_{n+1} - \frac{9}{512}\boldsymbol{h}^2 f'_{n+1}$$
$$p = 4, C_{p+1} = \frac{-9}{40960}$$
$$y_{n+\frac{1}{2}} = -\frac{109}{67}y_n + \frac{176}{67}y_{n+\frac{1}{4}} - \frac{115}{536}hf_n + \frac{31}{536}hf_{n+1} - \frac{1}{67}h^2 f'_{n+1}$$
$$p = 4, C_{p+1} = \frac{163}{513560}$$
$$y_{n+1} = y_n + \frac{1}{10}hf_n + \frac{71}{270}hf_{n+1} + \frac{2}{5}hf_{n+\frac{1}{2}} + \frac{32}{315}hf_{n+\frac{1}{4}} - \frac{1}{45}h^2 f'_{n+1}$$
$$p = 5, C_{p+1} = \frac{1}{57600}$$

Consistency of the Method

The method is consistent since it has order $p \geq 1$

Stability of the methods

Zero stability of method

Definition: An LMM is a said to be zero stable if no root of the first polynomial$\rho(\lambda)$ has modulus greater one and if every root with modulus one is simple

Definition: An LMM when applied to the differential equation $y' = \lambda y$ where λa complex constant with negative is real part is said to be A- stable if all the solution of the method tends to zero as $n \to \infty$. That is if $\lambda$is complex, the region of absolute stability is the entire left half of the $h\lambda -$ plane

The zero stability of the method is concerned with stability of the difference equation as $h \to 0$

$$A^{(0)}Y_m = A^{(1)}Y_{m-1} + h(B^{(0)}Y_m + B^{(1)}Y_{m-1}) + h^2[C^{(0)}Y_m] \quad (15)$$

As $h \to 0$ the difference equation becomes

$$A^{(0)}Y_m - A^{(1)}Y_{m-1} = 0 \quad (16)$$

The characteristic polynomial of the above is

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) =$$
$$det\left(R\begin{bmatrix} 1 & 0 & -\frac{67}{256} \\ -\frac{176}{67} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{27}{256} \\ 0 & 0 & -\frac{115}{536} \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \right) = R^2(R-1) = 0$$

Thus $R = 0,0,1$

The method is zero stable since $\rho(R) = 0$ and since we have $|R_i| \leq 1, i = 1,2,3$

Region of Absolute Stability of the Methods

The region of absolute stability of the new method can be determined by substituting the test equations

$y' = \lambda y, y'' = \lambda^2 y$

Thus we obtain

$$(A^{(0)} - \bar{h}B^{(0)} - (\bar{h})^2 C^{(0)}) - (A^{(1)} + \bar{h}B^{(1)}) \quad (17)$$

The stability polynomial $R(t, \bar{h})$ is evaluated by

$$det((A^{(0)} - \bar{h}B^{(0)} - (\bar{h})^2 C^{(0)})t - (A^{(1)} + \bar{h}B^{(1)})) = 0 \quad (18)$$

Where $\bar{h} = h\lambda = z$

Thus we have

$$R(t,\bar{h}) = \frac{3}{20}t^3z^2 - \frac{1039}{670}t^3z + t^3 - \frac{2}{5}t^2z - t^2 - \frac{1}{60}t^3z^3 - \frac{1}{20}t^2z^2 = 0 \quad (19)$$

By setting $z = 0$, we have

$$R(t,\bar{h}) = t^3 - t^2 = 0 \quad (20)$$

Equation (19) can also be solved to obtain zero stability of the method

where

The region of absolute stability of the method is defined as

Where $t = 0,0,1$

The region of absolute stability of the method is shown in the diagram below

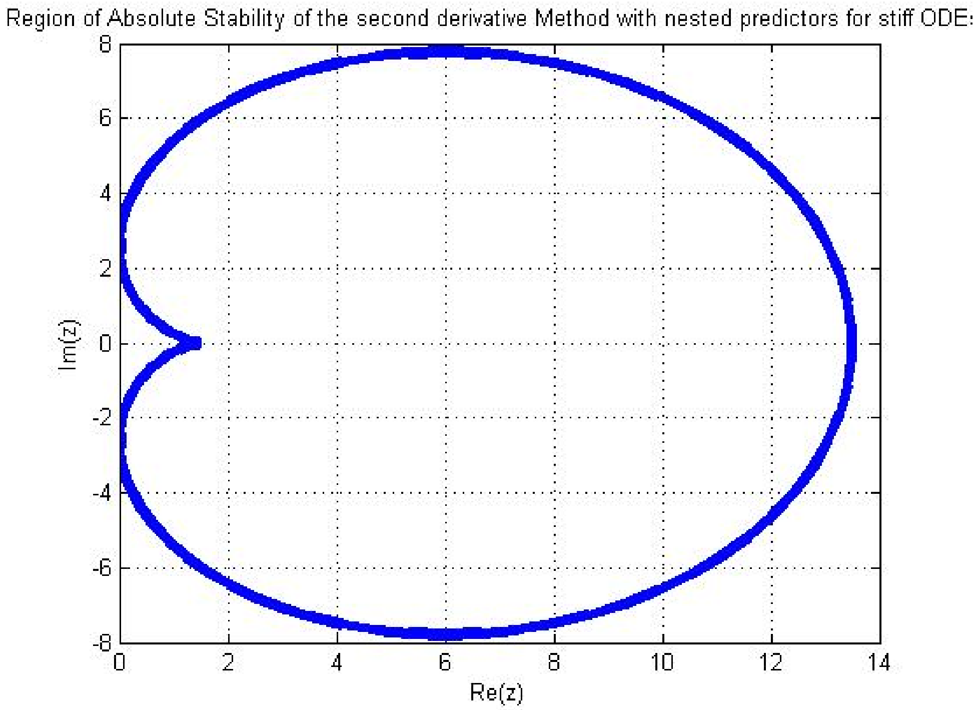

Figure 1: Region of Absolute Stability of the new method

Numerical Experiment

In this section we test the solvability of equation (1) using our method. We shall do so by combiningthe three methods as block matrix . We give following definitions:

Definition: The error in an approximate solution is the difference between the exact solution $y(x_{n+j})$ at $x = x_{n+j}$ and the computed solution y_{n+j} as determined by the numerical method

$error = y(x_{n+j}) - y_{n+j}$

Maximum error = MAXE = $\max_{1\le i\le NS}(error^{(i)})$

Where NS is the total number of steps

The new method is tested on some stiff ODEs and the error in our method compared some existing mrthods:

2BBDF= Fifth order 2-point block BDF method

2OBBDF= 2 point block BDF method with off –step points of order 5

NSDM= New second derivative method with hybrid predictors

Problem 1

$y' = -100y + 9.901z, y(0) = 1$

$z' = 0.1y - z, y(0) = 10 \qquad y(x) = e^{-\frac{99x}{100}}, z(x) = 10e^{-\frac{99x}{100}}$

Table 1: Errors in the use of our method for problem 1

H	N	Exact soln	Computed sol		Error in our method	
			YN	ZN	YN	ZN
0.125	5	0.538617288	0.538617289	5.386172899	1.342e-009	1.360e-009
	10	0.290108584	0.290108585	2.901085850	1.456e-008	1.465e-008
	100	0.000004223	0.000004223	0.000042228	2.131e-013	2.133e-013
0.0625	5	0.733905504	0.733905504	7.339055038	2.187e-010	2.450e-010
	10	0.538617289	0.538617288	5.386172886	1.426e-010	3.595e-010
	100	0.002054958	0.002054958	0.020549577	6.89e-013	1.370e-011
0.03125	5	0.856682849	0.856682849	8.566828489	2.970e-010	6.631e-010
	10	0.733905504	0.733905504	7.339055037	3.113e-010	1.136e-010
	100	0.04533164	0.045331641	0.453316415	8.241e-010	7.020e-010

Problem 2

$y' = 198y + 199z, y(0) = 1$

$z' = -398y - 399z, y(0) = -1 y(x) = e^{-x}, z(x) = -e^{-x}, x \in [0,10]$

Table 2: Numerical Result of problem2

H	NS	Exact soln	Computed sol		MaxError in our method	
			YN	ZN	YN	ZN
0.1	5	0.538617288	0.538617289	5.386172899	1.342e-09	1.360e-09
	10	0.367879441	0.367879438	-0.367879438	3.166e-09	3.341e-09-
	100	0.000045399	0.000045399	-0.000045399	3.71e-12	3.753e-12
	1000	5.549701×10^{-44}	5.549701×10^{-44}	-5.549701e-44	4.498e-50	4.501e-50
0.01	5	0.951229424	0.951229426	-0.951229425	1.608×10^{-09}	1.051e-09
	10	0.904837418	0.904837421	-0.904837421	1.426e-10	3.595e-10
	100	0.367879441	0.367879457	-0.367879457	1.650e-08	1.653e-08
	1000	0.0000453999	0.000045399	-0.000045399	2.065e-11	1.370e-11
0.001	5	0.995012479	0.995012482	-0.995012480	2.924e-09	1.532e-09
	10	0.990049834	0.990049840	-0.9900498385	6.692e09-	4.795e-09
	100	0.904837418	0.904837495	0.90483749251	7.648e-08	7.447e-08

Problem 3:
$y' = -20y - 19z, y(0) = 2$
$z' = -19y - 20z, z(0) = 0 y(x) = e^{-x} + e^{-39x}, z(x) = -e^{-x} + e^{-39x}$, $x \in [0,20]$

Table 3: Numerical Result for problem 3

H	NS	METHOD	MAXE
0.01	2000	2BBDF(5)	8.81087e-2
		2OBBDF	7.00088e-2
		NSDM	9.20950e-12
0.001	**10000**	2BBDF(5)	1.40157e-2
		2OBBDF	1.39480e-2
	20000	**NSDM**	**9.1464e-13**

Problem 4
$y' = -20y + 20\sin x + \cos x \; y(0) = 1$
$y(x) = \sin x + e^{-20x}, x \in [0,2]$,

Table 4: Numerical results of problem 4

H	NS	METHOD	MAXE
0.01	200	2BBDF(5)	8.85478e-2
		2OBBDF	8.05923e-2
		NSDM	9.5389e-011
0.001	**2000**	2BBDF(5)	1.40157e-2
		2OBBDF	1.39480e-2
		NSDM	3.6485e-09
0.0001	**20,000**	2BBDF(5)	1.46428e-3
		2OBBDF	1.46355e-3
		NSDM	1.5144e-08

Problem 5
$y' = -100y + 100x + 1 y(0) = 1$
$y(x) = x + e^{-100x}, x \in [0,10]$

Table 5: Numerical results of problem 5

H	NS	METHOD	MAXE
0.01	**1000**	**2BBDF(5)**	**1.96146e-2**
		2OBBDF	1.95754e-2
		NSDM	**4.75E-12**
0.001	**10000**	2BBDF(5)	**5.6931e-02**
		2OBBDF	**5.5942e-02**
		NSDM	**2.0952e-09**

RESULTS AND DISCUSSION

Programs developed using MAPLE 18 was used to implement our methods.Results obtained are as presented in tables 1 to 5 above. In tables 1 and 2 results were obtained using different step-lengths . The results shows that as the step-length tends to zero the errors arising from using the new become smaller thereby showing the efficiency of the method. The performance of the new scheme are compared with some existing methods in the literature as seen in tables 3-5 and as observed in the tables the new method compared favorably in terms accuracy and error

CONCLUSION

A new second derivatives method with nested hybrid predictors of orders (4,4,5)is formulated in this paper. The developed method is used for solving stiff ODEs with simultaneous to production one solution value with off-step points at each iteration. The method is shown to be A-stable and convergent. Accuracy of the derived method are compared with some existing method in the literature. Our method was found to be competitive in terms of accuracy.

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