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A NEW INVERSE TWO-PARAMETER LINDLEY DISTRIBUTION AND ITS APPLICATION

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ABSTRACT

This paper proposes an inverse two-parameter Lindley distribution by utilizing the one and two parameter Lindley distributions. The key properties of the novel distribution like, its survival function, shape characteristics of the density, entropy measure, hazard rate function, stochastic ordering and stress-strength reliability were examined. Two data sets were employed in the empirical studies. Method of maximum likelihood was used to estimate the parameters. The goodness- of -fit was accessed using the HQIC, BIC, CAIC, and AIC. The proposed distribution was compared with the inverse Lindley and the inverse Akash distributions in order to access its superiority over the two distributions. Ultimately, the new inverse two parameter Lindley distribution was found to be superior by providing a better fit.

Keywords: Moments, Hazard rate, Survival function, Maximum likelihood estimator, Goodness- of -fit

INTRODUCTION

The Lindley distribution is used in variety of fields especially engineering, medicine and biology. Ghitany *et al.* (2011) found the Lindley distribution useful for mortality studies. The distribution was first proposed by Lindley in 1958. Lately, a lot of attention has been given to this distribution because of its power and ability to model complex real lifetime data. After the pioneering work of Lindley, another Lindley distribution with two parameter was derived by (Mishra and Shanker, 2013). Lindley distribution is a mixture of both exponential and gamma distributions and has been reported to be more flexible and provide a decent fit over the exponential distribution. Among the few two parameter Lindley distributions developed in the literature is the two parameter Lindley distribution by Shanker *et. al*. (2013) known for modeling waiting and survival times and the inverse of that distribution was developed by John e*t. al*. (2021).

Ghitany *et al*. (2008) proposed a one parameter lifetime distribution with the following probability density function (pdf) and cumulative distribution function (cdf) given respectively as:

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\quad x>0,\theta>0 \qquad (1)$$

$$F(x;\theta)=1-\frac{(\theta+1)+\theta x}{\theta+1}e^{-\theta x};\quad x>0,\theta>0 \qquad (2)$$

A two-parameter Lindley distribution was also proposed by Shanker *et al.* (2013). He also derived its statistical properties. The details about the distribution are shown below:

$$f(x;\theta,\alpha)=\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \qquad (3)$$

$$F(x;\theta,\alpha)=1-\left[\frac{(\theta+\alpha)+\alpha\theta x}{\theta+\alpha}\right]e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \qquad (4)$$

At $\alpha=1$ both (3) and (4) reduce to the corresponding expressions (1) and (2) of the one parameter Lindley distribution. Mishra and Shanker (2013) proposed another two-parameter Lindley distribution. Its key mathematical and statistical properties were discussed and five (5) data sets were used to check if their newly proposed distribution is superior over Exponential and Lindley distributions been the base distributions. The pdf and cdf of the two-parameter Lindley distribution is given below:

$$f(x;\theta,\beta)=\frac{\theta^2}{\beta\theta+1}(\beta+x)e^{-\theta x};\quad x>0,\theta>0,\beta>0 \qquad (5)$$

$$F(x;\theta,\beta)=1-\left[\frac{1+\beta\theta+\theta x}{\beta\theta+1}\right]e^{-\theta x};\quad x>0,\theta>0,\beta>0 \qquad (6)$$

At $\beta=1$ both (5) and (6) can be reduced to the corresponding expressions (1) and (2) of Lindley distribution. Sharma *et al.* (2015) proposed the inverse of the Lindley distribution (ILD). He derived the properties of the distribution and apply it using data sets. The probability density and cumulative density function of the distribution are:

$$f(x;\theta)=\frac{\theta^2}{(\theta+1)x^3}(1+x)e^{-\theta/x};\quad x>0,\theta>0 \qquad (7)$$

$$F(x;\theta)=1+\frac{\theta}{(\theta+1)x}e^{-\theta/x};\quad x>0,\theta>0 \qquad (8)$$

This paper is motivated to develop the inverse of the two parameter Lindley distribution from the two- parameter lindley by Mishra and Shanker (2013) which is expected to be more flexible and provide a better fit over the two-parameter Lindley distribution and the Inverse Lindley distributions. This study was conducted by dividing the study into sections.

MATERIALS AND METHOD

The two data set shown below were used to show the performance rating of the new inverse two parameter Lindley distributions with the Inverse Lindley distribution (ILD) and Inverse Akash distribution (IAD).

Data set 1: (Times of failure and running times for a sample of 30 devices)

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66

Data set 2: (Remission times (in months) of a random sample of 128 bladder cancer patients)

0.08,2.09,3.48,4.87,6.94,8.66,13.11,23.63,0.20,2.23,3.52,4.98,6.97,9.02,13.29,0.40,2.26,3.57,5.06,7.09,9.22,13.80,25.74,0.50,2.46,3.64,5.09,7.26,9.47,14.24,25.82,0.51,2.54,3.70,5.17,7.28,9.74,14.76,6.1,0.81,2.62,3.82,5.32,7.32,10.06,14.77,32.15,2.64,3.88,5.32,7.39,10.34,14.83,34.26,0.90,2.69,4.18,5.34,7.59,10.66,15.96,36.66,1.05,2.69,4.23,5.41,7.62,10.75,16.62,43.01,1.19,2.75,4.26,5.41,7.63,17.12,46.12,1.26,2.83,4.33

,5.49,7.66,11.25,17.14,79.05,1.35,2.87,5.62,7.87,11.64,17.36,1.40,3.02,4.34,5.71,7.93,11.79,18.10,1.46,4.40,5.85,8.26,11.98,19.13,1.76,3.25,4.50,6.25,8.37,12.02,2.02,3.31,4.51,6.54,8.53,12.03,20.28,2.02,3.36,6.76,12.07,21.73,2.07,3.36,6.93,8.65,12.63,22.69 (Lee and Wang, 2003)

Inverse Two Parameter Lindley Distribution

If a random variable Y has a two-parameter Lindley distribution TPLD (θ), then the inverse of the distribution with scale parameter (θ) and shape parameter β becomes X = $\left({}^{1}\!/_{Y}\right)$ defined by:

$$f(x;\theta,\beta) = \frac{\theta^2}{(\beta\theta+1)x^3}(\beta x+1)e^{-\theta/x}; \quad x>0, \theta>0, \beta>0 \tag{9}$$

$$F(x;\theta,\beta) = \left[1+\frac{\theta}{(\beta\theta+1)x}\right]e^{-\theta/x}; \quad x>0, \theta>0, \beta>0 \tag{10}$$

Since there is a closed form expression in its stress-strength reliability, probability density function and the hazard rate then its applicability in the study of survival analysis cannot be undermined.

Shape Characteristics of the Density

The first derivative from equation (9) is expressed as:

$$\frac{d}{dx}f(x;\theta,\beta) = -\left(\frac{\theta^2}{\beta\theta+1}\right)\frac{e^{-\frac{\theta}{x}}}{x^5}\left(2\beta x^2-(\beta\theta-3)x-\theta\right) \tag{11}$$

And $\left.\frac{d}{dx}f(x;\theta,\beta)\right|_{x=M_0} = M_0$ that is the mode, is mathematically presented as:

$$M_0 = \frac{-(\beta\theta-3)+\sqrt{(\beta\theta-3)^2+8\beta\theta}}{4\beta} \tag{12}$$

Thus, the following graphs in figure 1 and figure 2 respectively depicts the pdf and cdf of the proposed distribution:

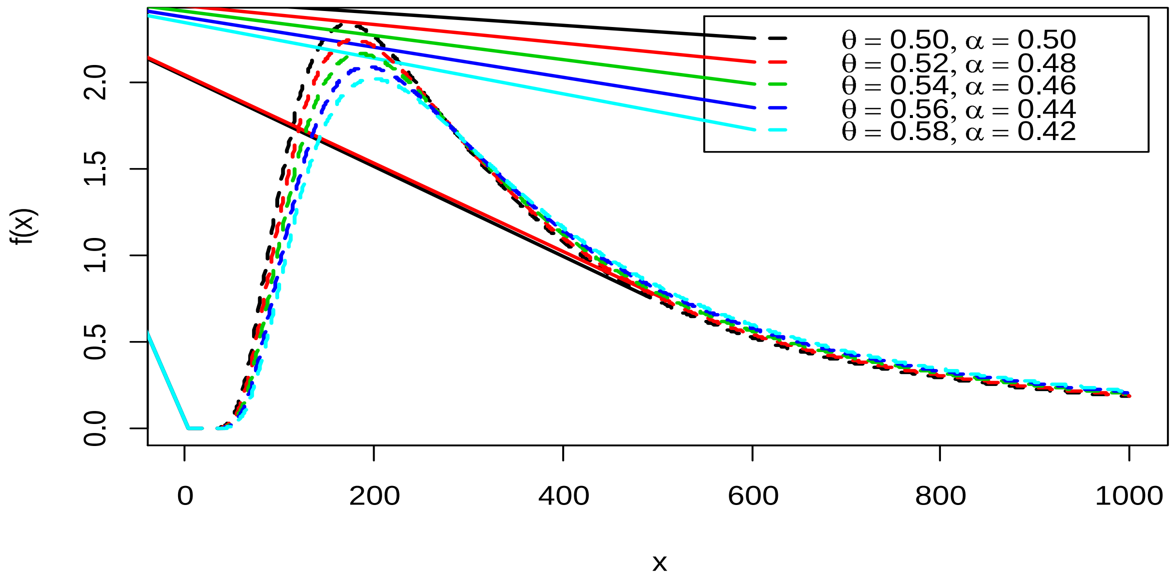

Figure 1. PDF plot of the proposed distribution

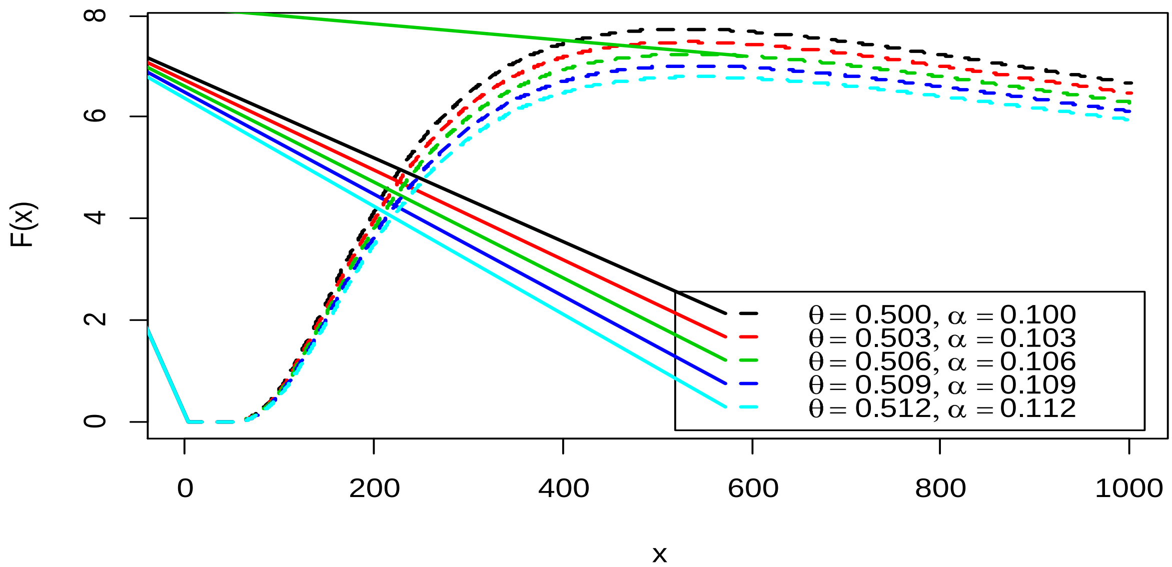

Figure 2: CDF plot of the proposed distribution.

Survival function and hazard rate function

Survival function $S(x;\theta,\beta)$ for the proposed distribution is defined below:

$$S(x;\theta,\beta) = 1 - F(x;\theta,\beta) \quad (13)$$

$$S(x;\theta,\beta) = 1 - \left[1 + \frac{\theta}{(\beta\theta+1)x}\right] e^{-\theta/x} \quad (14)$$

The hazard rate function $h(x;\theta,\beta)$ of the proposed distribution (ITPLD-1) can be defined as:

$$h(x;\theta,\beta) = \frac{f(x;\theta,\beta)}{1-F(x;\theta,\beta)} = \frac{f(x;\theta,\beta)}{S(x;\theta,\beta)} \quad (15)$$

$$h(x;\theta,\beta) = \frac{\theta^2(\beta x+1)}{x^2[x(\beta\theta+1)(e^{\theta/x}-1)-\theta]} \quad (16)$$

Thus, the graphs are shown below:

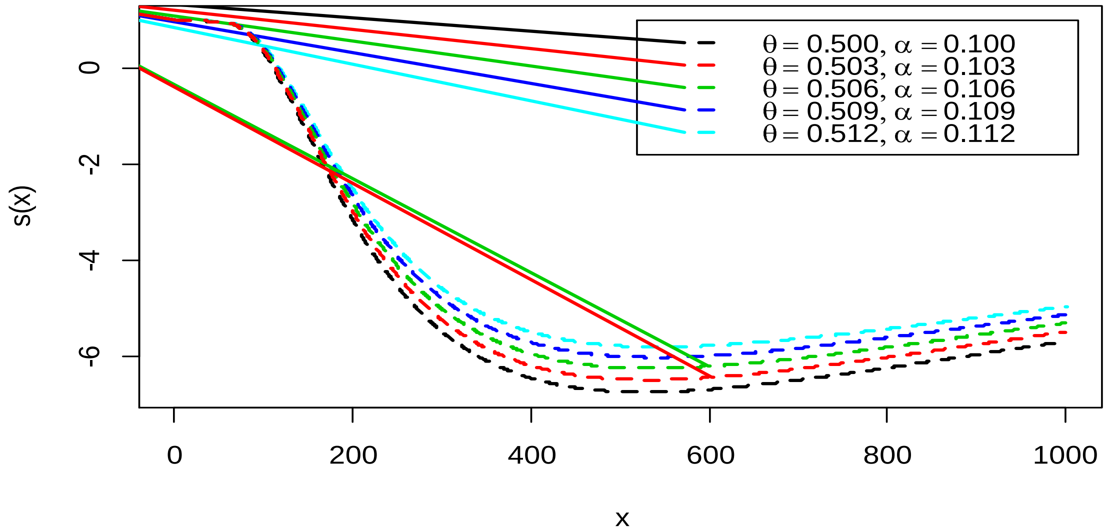

Figure 3: Survival function plot of the proposed distribution ITPLD-1.

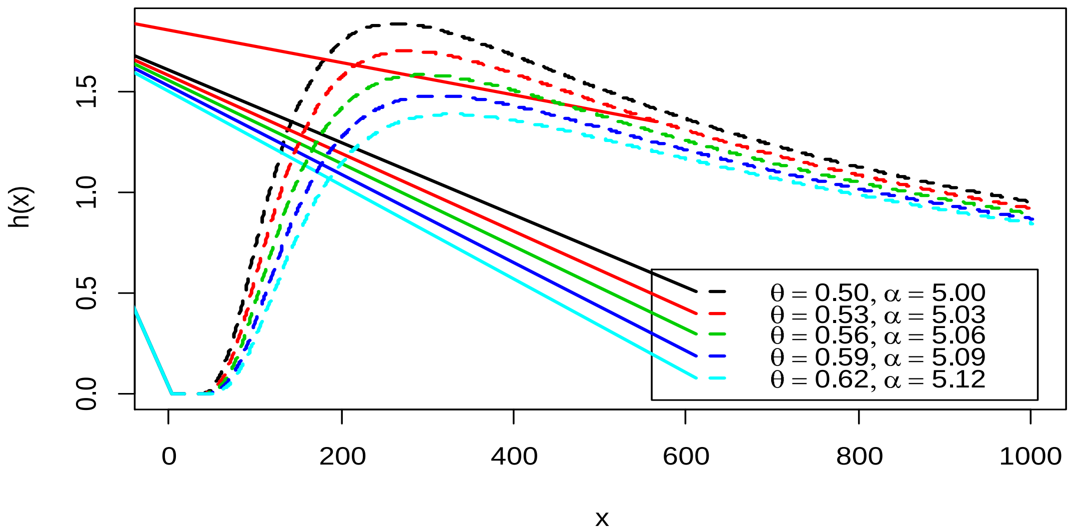

Figure 4: Plot of the hazard rate function of the proposed distribution ITPLD-1

Stochastic Ordering

For accessing the comparative behaviour, the stochastic ordering of positive continuous random variables is a crucial tool. If for any x a random variable X is smaller than a random variable Y in the stochastic order**.**

i. Likelihood ratio order $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

ii. Hazard rate function $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x.

iii. Mean residual life function $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)$ for all x.

iv. Stochastic order $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x.

Shaked and Shanth. (1994) are widely recognized for their studies demonstrating stochastic ordering of distributions:

$$\begin{array}{c} (X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y) \\ \Downarrow \\ (X \leq_{st} Y) \end{array}$$

Theorem. Let X be Inverse two-parameter Lindley (β_1,θ_1) and Y be Inverse two-parameter Lindley distribution (β_2,θ_2). If $\beta_1=\beta_2$ and $\theta_1 \geq \theta_2$ (or if $\theta_1=\theta_2$ and $\beta_1 \geq \beta_2$), then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. To prove this, we have:

$$\frac{f(x;\theta_1,\beta_1)}{f(x;\theta_2,\beta_2)} = \frac{\theta_1^2(\beta_2\theta_2+1)}{\theta_2^2(\beta_1\theta_1+1)}\left(\frac{\beta_1 x+1}{\beta_2 x+1}\right)e^{-(\theta_1-\theta_2)/x}; \quad x>0$$

Now

$$log\frac{f(x;\theta_1,\beta_1)}{f(x;\theta_2,\beta_2)} = log\left[\frac{\theta_1^2(\beta_2\theta_2+1)}{\theta_2^2(\beta_1\theta_1+1)}\right] + log\left(\frac{\beta_1 x+1}{\beta_2 x+1}\right) - (\theta_1-\theta_2)/x$$

This gives

$$\frac{d}{dx}log\frac{f(x;\theta_1,\beta_1)}{f(x;\theta_2,\beta_2)} = \frac{(\beta_1-\beta_2)}{(\beta_1 x+1)(\beta_2 x+1)} + (\theta_1-\theta_2)/x^2$$

Thus, for ($\theta_1 > \theta_2$ and $\beta_1 = \beta_2$) or ($\beta_1 > \beta_2$ and $\theta_1 = \theta_2$)

$$\frac{d}{dx}log\frac{f(x;\theta_1,\beta_1)}{f(x;\theta_2,\beta_2)} > 0$$

This implies that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Entropy Measure

The random variable's entropy X represents the uncertainty variation**.** Renyi entropy developed by Renyi (1961) is a widely used entropy measure. It is defined as follows, If X is a continuous random variable with a probability density function$f(.)$, then Renyi entropy is expressed as:

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\{\int f^{\gamma}(x)\,dx\} \qquad (17)$$

Where $\gamma > 0$ *and* $\gamma \neq 1$

The Renyi entropy of our proposed distribution is given as:

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty \frac{\theta^{2\gamma}}{(\beta\theta+1)^\gamma}\left[\frac{(\beta x+1)^\gamma}{x^{3\gamma}}\right]e^{-\frac{\theta\gamma}{x}}dx$$

We know that $(1+z)^j = \sum_{j=0}^{\infty}\binom{\gamma}{j}z^j$ and $\int_0^\infty e^{-\frac{b}{x}}x^{-a-1}dx = \frac{\Gamma(a)}{b^a}$

$$= \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{(\beta\theta+1)^\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}(\beta)^j\int_0^\infty \frac{e^{-\frac{\theta\gamma}{x}}}{x^{3\gamma-j}}dx\right]$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}\beta^j}{(\beta\theta+1)^\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\Gamma(3\gamma-j-1)}{(\theta\gamma)^{3\gamma-j-1}}\right] \qquad (18)$$

Stress-Strength Reliability

The life of a component with random strength X and random stress Y is described by the stress-strength reliability. The component fails quickly when the tension applied to it surpasses its strength and it will continue to work satisfactorily until X >Y. As a result, $R = P(Y < X)$ is a measure of component reliability and is referred to as stress strength parameter in statistical literature. Let X and Y be independent random variables representing stress and strength respectively that obey the Inverse two-parameter Lindley distribution with parameter θ_1,β_1 *and* θ_2,β_2 respectively. Then, the stress-strength reliability R is defined by taking the formula:

R = P[Y<X] = $\int_0^\infty P[Y < X \mid X = x]\, f_x(x)dx$

$$= \int_0^\infty f(x,\theta_1,\beta_1)\; F(x,\theta_2,\beta_2)dx$$

$$= \int_0^\infty \frac{\theta_1^2}{(\beta_1\theta_1+1)}\left(\frac{\beta_1 x+1}{x^3}\right)e^{\theta_1/x} \times \left[1+\frac{\theta_2}{(\beta_2\theta_2+1)x}\right]e^{\theta_2/x}dx$$

Applying the inverse gamma function, it becomes:

$$R = \theta_1^2\left[\frac{\beta_1(\beta_2\theta_2+1)(\theta_1+\theta_2)^2 + (\beta_2\theta_2+1)(\theta_1+\theta_2) + \beta_1\theta_2(\theta_1+\theta_2) + 2\theta_2}{(\beta_1\theta_1+1)(\beta_2\theta_2+1)(\theta_1+\theta_2)^3}\right]$$

Maximum Likelihood Estimation Method

Let $L(x_1, x_2, \dots, x_n;\, \theta, \beta)$ be a random sample from the Inverse two-parameter Lindley distribution ITPLD (2.1). The likelihood function, L of ITPLD-1 (2.1) is given by

$$L(x_1,x_2,\dots,x_n;\theta,\beta) = \prod_{i=1}^{n}[f(x_i;\theta,\beta)]$$

$$L = \left(\frac{\theta^2}{\beta\theta+1}\right)^n \prod_{i=1}^{n}\frac{(\beta x_i+1)}{x_i^3}e^{\Sigma\,\theta/x_i}$$

To obtain the log likelihood function we get:

$$logL = nlog\theta^2 - nlog(\beta\theta+1) + \sum_{i=1}^{n}log(\beta x_i+1) - \sum_{i=1}^{n}log\left(x_i^3\right) - \theta\sum_{i=1}^{n}(1/x_i)$$

To obtain the estimates $\hat{\theta}$ and $\hat{\beta}$ of θ and β, we obtain

$$\frac{\partial logL}{\partial\theta} = \frac{2n}{\theta} - \frac{n\beta}{\beta\theta+1} - \sum_{i=1}^{n}(1/x_i) \qquad (19)$$

and

$$\frac{\partial logL}{\partial\beta} = -\frac{n\theta}{\beta\theta+1} + \sum_{i=1}^{n}\frac{x_i}{(\beta x_i+1)} \qquad (20)$$

Notably, the analytic solution of (19) and (20) can be found.

RESULTS AND DISCUSSION

Data application

The statistical superiority of the proposed Inverse two-parameter Lindley distribution (ITPLD-1) was investigated by comparing it with the Inverse Lindley distribution (ILD) and Inverse Akash distribution (IAD) were compared on the basis of two emphatical data sets as shown below:

The Bayesian information criterion (BIC), Akaike information criterion (AIC), Standard Error Estimate of the Parameter (SE), Negative Log-Likelihood Function (MLE), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), and the Estimate of the Parameter were used as the performance measures. The result of the comparison is shown below:

Table 1: Performance measure ratings of the three probability distributions applying data Set 1

Model	MLE	Estimates	S.E	HQIC	BIC	CAIC	AIC
ITPLD-1	70.69501	$\hat{\theta} = 0.32077$ $\hat{\beta} = 110.76$	0.058969 204.2214	146.2865	148.1924	145.8345	145.3900
ILD	82.91041	$\hat{\theta} = 0.51791$	0.068897	168.2691	169.2220	167.9637	167.8208
IAD	97.33354	$\hat{\theta} = 0.78856$	0.079868	197.1153	198.0683	196.8099	196.6671

Using the first data set, the ITPLD-1 has the smallest value of MLE = 70.69501, the least AIC = 145.3900, least BIC = 148.1924, the least HQIC = 146.2865, and the least CAIC = 145.8345. We can conclude using the first data set that the ITPLD fits the data better than the ILD and IAD.

Table 2: Performance measure ratings of the three probability distributions applying data Set 2

Model	MLE	Estimates	S.E	HQIC	BIC	CAIC	AIC
ITPLD-1	457.8209	$\hat{\theta} = 2.50504$ $\hat{\alpha} = 37.043$	0.225202 74.51373	921.9595	925.3460	919.7379	919.6419
ILD	465.8056	$\hat{\theta} = 3.08538$	0.228082	934.7699	936.4632	933.6429	933.6111
IAD	461.8305	$\hat{\theta} = 3.26273$	0.2 12089	926.8199	928.5131	925.6928	925.6611

Using the second data set, the ITPLD-1 has the smallest value of MLE =457.8209, the least AIC =919.6419, least BIC =925.3460, the least HQIC = 921.9595, and the least CAIC = 919.7379. We can conclude using the second data set that the ITPLD fits the data better than the two competing distributions.

CONCLUSION

A new "Inverse two-parameter Lindley distribution" has been developed. The mathematical and statistical properties of the proposed distribution have been investigated and discussed. Other key mathematical expression of the proposed distribution such as the stress-strength reliability, maximum likelihood estimation, and the measure of entropy were all derived and discussed.

Lastly, the goodness- of -fit test was conducted using some two sets of data. The study was able to establish the applicability and superiority of the proposed distribution over two competing distributions.

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